Analysis of Multilayer Methods of Building Approximate Solutions of Differential Equations in the Context of Solving a Homogeneous Duffing Equation*

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Abstract. Multilayer methods are an alternative approach to building the approximate analytical solution of differential equations. This paper presents the study of the results obtained by the implementation of our modifications of acknowledged implicit and explicit numerical methods. The homogeneous Duffing equation is of practical interest for modeling nonlinear oscillations and considered as a model equation. The accuracy of the obtained solutions is compared. It is shown that moving an initial point can significantly increase the accuracy of the solutions.

Keywords: differential equations, numerical methods, analytical methods, multilayer models, Duffing equation.

1 Introduction

Modeling the behavior of many real objects often reduces to initial or boundary value problems for differential equations. In practice, the analytical solution of differential equations usually cannot be built, therefore, a numerical approximate solution is often sought. But such solutions are not clear enough. It is complicated to use it for studying the effect of changing the parameters of the original problem or adjust it to the behavior of the simulated object using the results of observation. Another well-known approach is building the approximate analytical solution. A lot of different approaches to finding it has been developed. There are various asymptotic methods, series expansion methods, etc. [1]. Classic perturbation methods [2] are quite versatile but, like other nonlinear analytical methods, they have significant limitations and restrictions. The quality of the solution may directly depend on the choice of the parameter by the researcher. In recent decades, new methods have appeared and old ones have been improved [3].

But often the higher-order analytical approximations require analytical solving of systems of complex nonlinear algebraic equations or additional restrictions on the parameters or the function [4-7]. Other approaches imply building a functional approximation in the form of broken lines or splines based on points of the numerical solution. In this article, we consider methods for building multilayer models that allow us to obtain an approximate analytical solution based on classical numerical methods. We compare the solutions built according to recurrence formulas of various numerical methods and investigate the possibility of increasing the accuracy of the obtained solutions on the base of the initial point moving.

2 General Description of Multilayer Methods

The essence of our approach is to apply the well-known recurrence formulas for the numerical integration of differential equations to an interval with a variable right endpoint. [8-12]. The result is an approximate analytical solution in the form of a function of this endpoint.

Consider the Cauchy problem for a system of ordinary differential equations

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases} x \in R, y \in R^m \quad (1)$$

The search for a solution is carried out on the interval $[x_0, x_0 + a]$. According to the main idea of our approach, we use well-known methods for the numerical solution of the problem (1) to an interval with a variable right endpoint $x \in [x_0, x_0 + a]$. We choose a partition $T_n(x)$ of the interval $[x_0, x]$ into n subintervals $x_0 < x_1 < \dots < x_k < \dots < x_n = x, h_k = x_{k+1} - x_k$. By applying the formula

$$y_{k+1} = y_k + F(f, h_k, x_k, y_{k+1}, y_k, y_{k-1}, \dots, y_0) \quad (2)$$

n times, we obtain an approximate solution $y_n(x)$. The operator F defines a specific method we modify as described above. The result is a function defined on the interval $[x_0, x_0 + a]$. We can replace an interval $[x_0, x_0 + a]$ with $[x_0 - b, x_0 + a]$.

3 Multilayer Methods in the Context of Solving the Duffing Equation

As the model task, we consider the homogeneous Duffing equation with constant coefficients [4-5, 13-16].

$$\begin{gathered} y'' + ay + by^3 = 0 \\ y(0) = y_0, y'(0) = y_1 \end{gathered} \quad (3)$$

Higher-order differential equations can always be reduced to the form (1) by increasing the dimension of the system. In our case, it is easy to make a replacement $y = v, y' = u$:

$$\begin{aligned} v' &= u \\ u' &= -av - bv^3 \\ v(0) &= y_0, u(0) = y_1 \end{aligned} \tag{4}$$

The search for a solution is carried out on the interval $[0,3]$, the initial conditions and parameters of the equation are as follows

$$a = 1,\ \ b = 1,\ \ y_0 = 1,\ \ y_1 = 1$$

For simplicity, the partition $T_n(x)$ is considered uniform for each method, namely $h_k = (x - x_0)/n$.

3.1 Explicit Methods

Euler's method. The simplest numerical method for which the operator F has the form

$$F(f, h_k, x_k, y_k) = h_k f(x_k, y_k)$$

Refined Euler's method. Another well-known numerical method in which we used the formula

$$F(f, h_k, x_k, y_k, y_{k-1}) = 2h_k f(x_k, y_k) + y_{k-1} - y_k.$$

In this case, to start the algorithm, the expression

$$y_1 = y_0 + h_1 f\left(x_0 + \frac{h_1}{2}, y_0 + \frac{h_1}{2} f(x_0, y_0)\right)$$

is used.

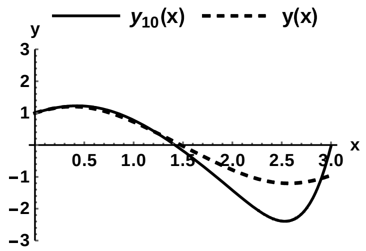

Fig. 1. The plot of the exact solution of problem (3) and the approximate solution built by our modification of the Euler method in the case of n=10

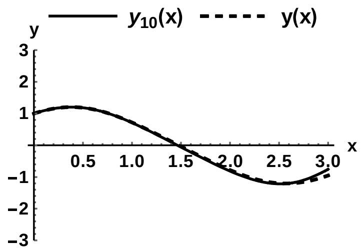

Fig. 2. The plot of the exact solution of problem (3) and the approximate solution built by our modification of the refined Euler method in the case of n=10

The modified Euler method. This method works under the formula

$$F(f, h_k, x_k, y_k) = h_k \left[f(x_k, y_k) + \frac{h_k}{2}\Big(f'_x(x_k, y_k) + f'_y(x_k, y_k) f(x_k, y_k)\Big)\right].$$

Second-order Runge-Kutta method:

$$F(f, h_k, x_k, y_k) = h_k f\left(x_k, y_k + \frac{h_k}{2} f(x_k, y_k)\right).$$

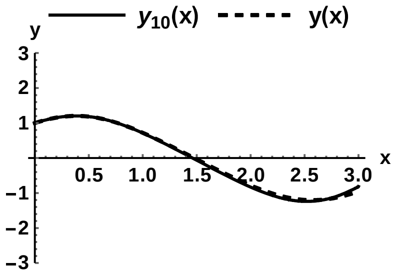

Fig. 3. The plot of the exact solution of problem (3) and the approximate solution built by our modification of the modified Euler method in the case of n=10

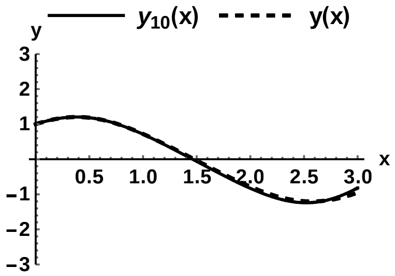

Fig. 4. The plot of the exact solution of problem (3) and the approximate solution built by our modification of the Runge-Kutta method in the case of n=10

Störmer Method. Since initially, the Duffing equation is a second-order differential equation, we can apply the Störmer method. In this case

$$y_{k+1} = 2y_k - y_{k-1} + h_k^2 f(x_k, y_k).$$

This method requires y_1. We calculated it approximately by the Taylor formula

$$y_1 = y_0 + \frac{y'(x_0)}{1!} h_1 + \frac{y''(x_0)}{2!} h_1^2 + \frac{y'''(x_0)}{3!} h_1^3,$$

where $y''(x_0)$ and $y'''(x_0)$ are easily obtained from differential equation (3).

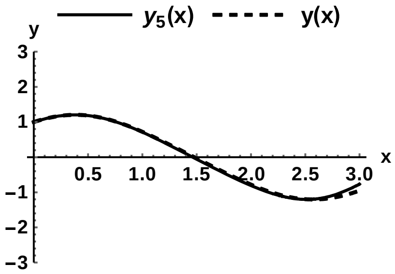

Fig. 5. The plot of the exact solution of problem (3) and the approximate solution built by our modification of the Störmer Method in the case of n=5

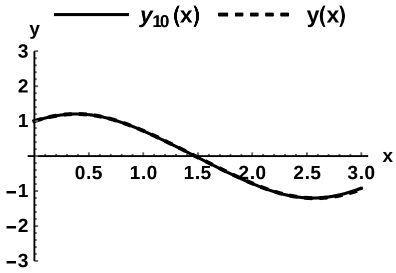

Fig. 6. The plot of the exact solution of problem (3) and the approximate solution built by our modification of the Störmer Method in the case of n=10

3.2 Implicit Methods

Implicit methods are applicable if the equation $y_{k+1} = y_k + F(f, h_k, x_k, y_{k+1}, y_k, y_{k-1}, \dots, y_0)$ is solvable for y_{k+1}. About our problem, this means solving the cubic equation at each step. In some cases, it is advisable not to look for the exact solution of this cubic equation but to use the approximate solving methods. We use one step of the Newton method. The justification of this approach is demonstrated below.

The implicit Euler method. The operator F for an implicit method is as follows

$$F(f, h_k, x_k, y_{k+1}) = h_k f(x_{k+1}, y_{k+1}).$$

Substituting this expression in (2) we obtain the system

$$\begin{cases} v_{k+1} = v_k + h_k u_{k+1} \\ u_{k+1} = u_k + h_k(-a v_{k+1} - b v_{k+1}^3) \end{cases}. \tag{5}$$

This system allows the exact expression v_{k+1}, u_{k+1} in terms of v_k, u_k, h_k:

$$\begin{cases} v_{k+1} = B_1(v_k, u_k, h_k) \\ u_{k+1} = B_2(v_k, u_k, h_k) \end{cases}.$$

Then we can use formula

$$\begin{pmatrix} v_{k+1} \\ u_{k+1} \end{pmatrix} = \begin{pmatrix} v_k \\ u_k \end{pmatrix} + \begin{pmatrix} B_1(v_k, u_k, h_k) \\ B_2(v_k, u_k, h_k) \end{pmatrix}.$$

to perform computations. The result of such solving is presented below. On the other hand, the solution of system (5) can be obtained by using one step of the Newton method. In this case, we obtain another expression,

$$\begin{cases} v_{k+1} = N_1(v_k, u_k, h_k) \\ u_{k+1} = N_2(v_k, u_k, h_k) \end{cases}.$$

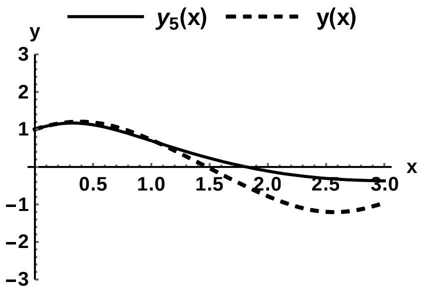

Fig. 7. The plot of the exact solution of problem (3) and the approximate solution built by our modification of the implicit Euler method in the case of the exact solution of (5) and n=5

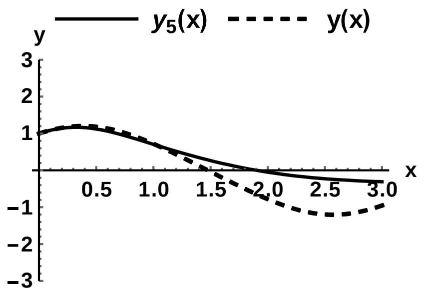

Fig. 8. The plot of the exact solution of problem (3) and the approximate solution built by our modification of the implicit Euler method in the case of the approximate solution of (5) (one step of the Newton method) and n=5

The maximum error in the first case is equal to 0.89163, in the second case it is 0.96993. As we can see, there is no significant change in the plot behavior. But when using the Newton method an expression is easier and therefore the complexity and time of calculations are lower than if we solve system (5) analytically. The solutions below are obtained using the Newton method.

One-step Adams method. Another implicit method, the equation has the form

$$y_{k+1} = y_k + \frac{h_k}{2}\left(f(x_k, y_k) + f(x_{k+1}, y_{k+1})\right).$$

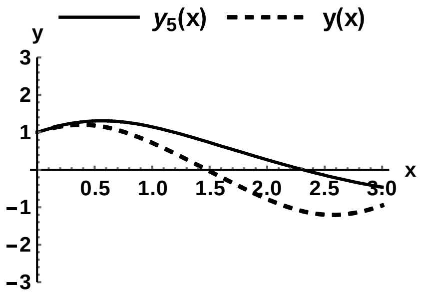

Fig. 9. The plot of the exact solution of problem (3) and the approximate solution built by our modification of the One-step Adams method in the case of the exact solution of the cubic equation and n=5

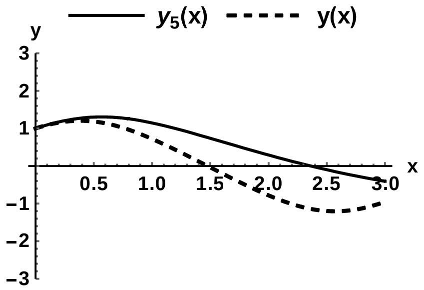

Fig. 20. The plot of the exact solution of problem (3) and the approximate solution built by our modification of the One-step Adams method in the case of the approximate solution of the cubic equation (one step of the Newton method) and n=5

The maximum error in the case of the exact solution according to the cubic equation is equal to 1.11217 and in the case of the approximate solution, it is 1.14777.

3.3 Computational results

The computational results of all methods are presented in Table1. Each method corresponds to its maximum error in the specified interval. We compare the error on the intervals [0,1.5] and [0,3].

Table 1. The maximum errors of the studied methods with the number of layers n=2 and n=5.

Number of iterations n	5		10	
Interval	[0, 1.5]	[0, 3]	[0, 1.5]	[0, 3]
Euler method	0.17474	7.11176	0.14765	1.19331
Refined Euler method	0.10981	11.70914	0.022633	0.19536
Modified Euler method	0.1148	0.90283	0.026272	0.13579
Runge-Kutta method	0.13924	1.90271	0.032397	0.19168
Störmer Method	0.023506	0.18407	0.0058227	0.034254
Implicit Euler method	0.30049	0.96993	0.16892	0.62051
One-step Adams method	0.77332	1.14778	0.19285	0.67787

Implicit methods have no advantages over explicit methods for this task. Of the explicit methods we examined, the most accurate is the Störmer method.

4 Initial Point Moving

To improve the accuracy of the model, we investigated the following approach. Using the methods described above, we build a solution to the Cauchy problem (1) starting from the point $x_1 \in [x_0, x_0 + a]$, other than x_0. The unknown initial condition $y(x_1) = y_1$ in this case is the parameter of the resulting solution $y_n(x, y|1)$. This parameter can be determined from the equation $y_n(x_0, y|1) = y_0$. From the computational results below, it follows that by moving an initial point in this way, it is possible to improve the solution on the interval.

As an example, we consider the Störmer method, as the most accurate method of the previously considered. The table below shows the maximum error of the solution obtained by this method on the interval $[0,1.5]$ in the case of moving the initial point from zero in increments of $\frac{1}{10}$. The number of layers n we took equal to 2 and 5.

Table 2. The maximum errors in the interval [0,1.5] in the case of moving the initial point.

x_1	n = 2	n = 5	x_1	n = 2	n = 5
0	0.11273	0.023506	0.8	0.03919	0.0077468
0.1	0.10936	0.022624	0.9	0.062919	0.0058778
0.2	0.15743	0.024725	1	0.11639	0.012437
0.3	0.20216	0.027498	1.1	0.16437	0.019892
0.4	0.21231	0.028058	1.2	0.20177	0.027324
0.5	0.18527	0.024551	1.3	0.23007	0.034529
0.6	0.13004	0.017213	1.4	0.25373	0.042016
0.7	0.063503	0.0091472	1.5	0.27807	0.050909

As we can see, solutions with moving the initial point have significantly better accuracy on the interval than a conventional solution built starting from zero.

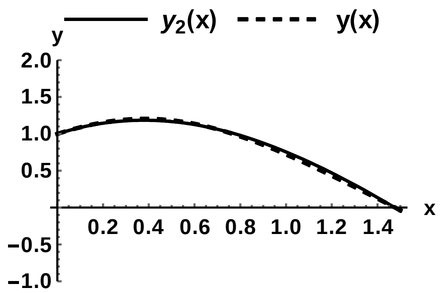

Fig. 11. The plot of the exact solution of problem (3) and the approximate solution built by our modification of the Störmer Method in the case of n=2 and initial point $x_1 = 0.8$

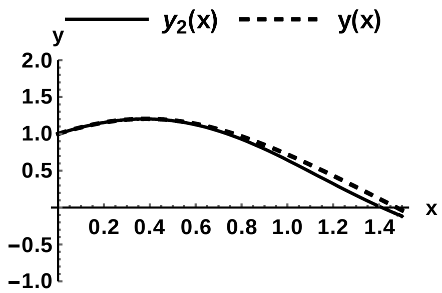

Fig. 32. The plot of the exact solution of problem (3) and the approximate solution built by our modification of the Störmer Method in the case of n=2 and initial point $x_1 = 0$

Next, we select new starting points from the gap between the previous best results in increments of $\frac{1}{100}$. 1/100. From this list of models, the most accurate model can be chosen.

Table 3. The maximum errors in the interval [0,1.5] in the case of moving the initial point in the interval [0.8,0.9]

x_1	n = 2	n = 5
0.8	0.03919	0.0077468
0.81	0.036768	0.007522
0.82	0.034308	0.0072874
0.83	0.031835	0.0070447
0.84	0.034117	0.0067956
0.85	0.038495	0.006542
0.86	0.043071	0.0062857
0.87	0.047822	0.0060285
0.88	0.052729	0.0057722
0.89	0.057768	0.0055186
0.9	0.062919	0.0058778

Thus, the approach with moving an initial point allowed us to significantly reduce the deviation of the solution on the interval. The plots below illustrate the difference between models with a selected initial point and models built starting from zero.

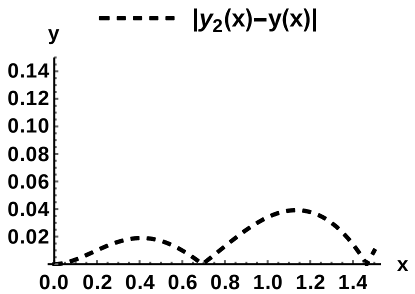

Fig. 43. The plot of the approximate solution error module in the case of $n = 2$ and initial point $x_1 = 0.83$

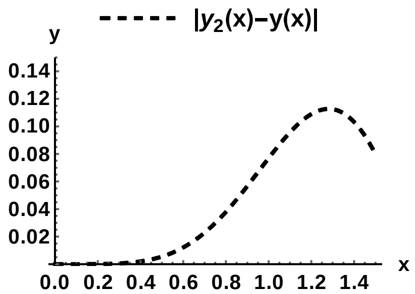

Fig. 54. The plot of the approximate solution error module in the case of $n = 2$ and initial point $x_0 = 0$

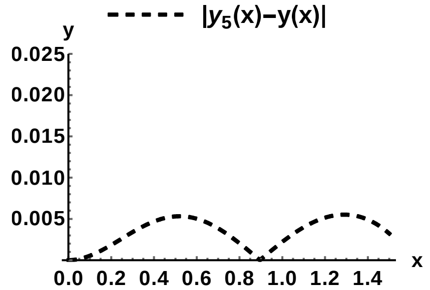

Fig. 15. The plot of the approximate solution error module in the case of $n = 5$ and initial point $x_1 = 0.89$

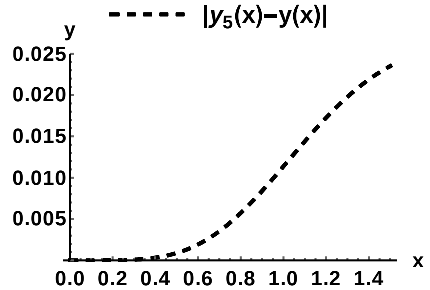

Fig. 16. The plot of the approximate solution error module in the case of $n = 5$ and initial point $x_0 = 0$

5 Conclusions and Discussion

The study allows us to draw the following conclusions:

1. The methods we have proposed allow us to construct an approximate solution of the Duffing equation in the form of a function with the required accuracy.
2. For model equations with parameters considered, implicit methods do not have significant advantages over explicit methods. Implicit methods make sense for those parameters when the task becomes stiff.
3. Moving an initial point lets us obtain an approximate analytical solution of the model task which is several times more accurate than a solution obtained without moving an initial point. Wherein an accuracy increases with the number of layers.
4. Our methods, without requiring additional assumptions, allow us to build parametric approximate analytical solutions [9] concerning the parameters of the original task.

Acknowledgment

This paper is based on research carried out with the financial support of the grant of the Russian Scientific Foundation (project №18-19-00474).

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