Weighted FEM operationalized for boundary value problems with singularity and inconsistent degeneracy of input data

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Abstract
On the base of designed computational technologies, several comparative experiments and numerical analysis of the weighted finite element method based on the notion of R_ν-generalized solution and a classical finite element method are carried out.

Keywords 1
computational technologies, boundary value problem with singularity, weighted finite element method

1. Introduction

Boundary value problems for elliptic equations with singularity are divided into two classes: with consistent and inconsistent degeneracy of the input data. In boundary value problems with a consistent degeneration of input data, all components in differential equations have the same asymptotic behaviour in the neighbourhoods of singularity points. It means that the increasing order/degree of singularity arising in the equation terms when the derivative order grows is balanced out by the appropriate behaviour of coefficients. For computational solution of such problems, the concept of R_ν-generalized solution is introduced and the finite element method (FEM) is developed. It allows the authors to find an approximate answer with a rate of $O(h)$ with respect to the norm of a Sobolev weight space [1].

For boundary value problems with inconsistent degeneracy of the original data, all the coefficients of the equation have the same asymptotic order in the neighbourhoods of singularity points, and, it implies that all the terms of the equation have singularities of different order in these neighbourhoods. The simplest example of this problem class is the boundary value problems for differential equations and systems of equations in domains with a boundary containing reentrant angles. In [2], a special weighted set was allocated for such problems, in which it is possible to establish an existence and uniqueness of R_ν-generalized solution. The weighted FEM designed by the authors [3] allowed them to define an approximate R_ν-generalized solution without loss of accuracy and independent of the singularity size. The suggested computational technologies have been modified and developed for the issues of electromagnetism and hydrodynamics. For the system of Maxwell's equations, Stokes and Oseen's laws in domains with reentrant angles on the boundary, the weighted FEM exceeds in accuracy and utilization efficiency both the classical FEM and the FEM with mesh refinement to the singularity points [4-7]. In [8-11], this approach was developed for the problem of the theory of elasticity with singularity. Comparative analysis of many test problems found out that the values of absolute errors in the entire domain and in the neighbourhood of singularity points for the solution

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found by the weighted FEM are in two orders smaller than for the approximate solutions found by both classical FEM and FEM with mesh refinement.

This paper deals with several comparative experiments and numerical analysis of the accurate finding of approximate solution by the weighted FEM and classical FEM for test problems with different types of singularities. They are carried out for boundary value problems with inconsistent degeneracy of input data. Some conclusions are made about the efficient usage of the weighted FEM for finding solutions to boundary value problems with singularity.

2. Principal symbols. Problem statement

Assume that $\Omega \subset R^2$ is a bounded domain with a piecewise-smooth boundary $\partial\Omega$ and closure $\bar{\Omega}$. Denote by τ_i, $i=\overline{1,n}$, the cross points of the continuously differentiable boundary $\partial\Omega$ and $O_i^\delta = \{x : \|x-\tau_i\| \le \delta\}$, and also $O_i^\delta \cap O_j^\delta = \varnothing$, $i \ne j$. Assume, $\Omega' = \bigcup_{i=1}^{n} \Omega_i$, where $\Omega_i = \Omega \cap O_i^\delta$, $i=\overline{1,n}$.

Let $\rho(x)$ be a weight function defined as follows:

$$\rho(x) = \begin{cases} \sqrt{\left(x_1 - x_1^{(i)}\right)^2 + \left(x_2 - x_2^{(i)}\right)^2}, \; x \in \Omega', \\ \delta, \; x \in \Omega \setminus \Omega', \end{cases}$$

where $\left(x_1^{(i)}, x_2^{(i)}\right) = \tau_i$.

We introduce the weighted space $W_{2,\alpha}^k(\Omega)$ with squared norm

$$\|u\|^2_{W_{2,\alpha}^k(\Omega)} = \sum_{|m| \le k} \int_\Omega \rho^{2\alpha} \left|D^m u\right|^2 dx, \tag{1}$$

where k is a nonnegative integer, α is a real nonnegative integer, $D^m = \partial^{|m|} / \partial x_1^{m_1} \partial x_2^{m_2}$, $m=(m_1,m_2)$, $|m| = m_1 + m_2$, $m_j, j=1,2$ are nonnegative integers. If $k=0$, we will represent $W_{2,\alpha}^0(\Omega) = L_{2,\alpha}(\Omega)$.

By $W_{2,\alpha+k-1}^k(\Omega,\delta)$ $(k=1,2,\ \alpha>0)$ denote a set of functions for which $\left|D^m u\right| \le c_1 \left(\delta/\rho(x)\right)^{\alpha+|m|}$, $x \in \Omega'$, where $m=0,1,2$, $c_1>0$ is a constant independent on m, and $\|u\|_{L_{2,\alpha}(\Omega\setminus\Omega')} \ge c_2 > 0$, with a norm defined in (1).

Let $\overset{\circ}{W}{}_{2,\alpha}^k(\Omega,\delta)$ be a subset of functions from the set $W_{2,\alpha}^k(\Omega,\delta)$ which are going to zero almost everywhere on $\partial\Omega$, and let $H_{\infty,-\alpha}^k(\Omega,c_3)$ $(k=0,1)$ be a set of functions with a norm

$$\|u\|_{H_{\infty,-\alpha}^k(\Omega,c_3)} = \max_{|m|\le k} \operatorname*{ess\,sup}_{x\in\Omega} \left|\rho^{-\alpha+|m|} D^m u\right| \le c_3.$$

If $k=0$, then $H_{\infty,-\alpha}^0(\Omega,c_3) = L_{\infty,-\alpha}(\Omega,c_3)$.

The properties of the introduced weighted spaces and sets were studied in [9].

In the domain Ω, we consider the boundary value problem

$$-\sum_{k=1}^{2} \frac{\partial}{\partial x_k}\left(a_{kk}(x)\frac{\partial u}{\partial x_k}\right) + a(x)u = f(x), \; x \in \Omega, \tag{2}$$

$$u(x) = 0, \; x \in \partial\Omega. \tag{3}$$

Definition 1. *The boundary value problem (2), (3) will be called a Dirichlet problem with inconsistent degeneracy of input data, if the coefficients of the equation for some real number β satisfy the requirements*

$$a_{kk}(x) \in H_{\infty,-\beta}^1(\Omega,c_4), \; a(x) \in L_{\infty,-\beta}(\Omega,c_5) \tag{4}$$

$$\sum_{k=1}^{2} a_{kk}(x)\eta_k^2 \ge c_6 \rho^\beta(x) \sum_{k=1}^{2} \eta_k^2, \tag{5}$$

$$a(x) \geq c_7 \rho^{\beta}(x) \tag{6}$$

almost everywhere on Ω, *and the right hand side of the equation for some real nonnegative number* μ *meets the condition*

$$f(x) \in L_{2,\mu}(\Omega, \delta), \tag{7}$$

where c_i $(i = 4,5,6,7)$ *are positive constants that do not depend on* x; $\eta_1, \eta_2 \in R, \eta_1^2 + \eta_2^2 \neq 0.$

Introduce the bilinear and linear forms, respectively:

$$a(u_\nu, v) = \sum_{k=1}^{2} \int_{\Omega} \left[a_{kk} \rho^{2\nu} \frac{\partial u_\nu}{\partial x_k} \frac{\partial v}{\partial x_k} + a_{kk} \frac{\partial \rho^{2\nu}}{\partial x_k} \frac{\partial u_\nu}{\partial x_k} v \right] dx + \int_{\Omega} a \rho^{2\nu} u_\nu v \, dx,$$

$$l(\upsilon) = \int_{\Omega} \rho^{2\nu} f \, v \, dx.$$

Definition 2. *A function* u_ν *from the set* $W^1_{2,\nu+\beta/2}(\Omega,\delta)$ *is called* R_ν*-generalized solution of the problem (2), (3), if the identity* $u_\nu = 0$ *holds almost everywhere on* $\partial\Omega$ *and for all* v *from* $\overset{\circ}{W}{}^1_{2,\nu+\beta/2}(\Omega,\delta)$ *the identity*

$$a(u_\nu, v) = l(v)$$

is valid for any fixed value ν *satisfying the inequality*

$$\nu \geq \mu + \beta / 2. \tag{8}$$

The membership of R_ν-generalized solution to the weighted set $W^{k+2}_{2,\nu+\beta/2+k+1}(\Omega,\delta)$ was studied in [12, 14].

Remark 1. For boundary value problems with singularity caused by the degeneracy of input data (coefficients of a differential equation, right-hand sides of equation and boundary conditions), it is not always possible to determine a generalized solution. To suppress the singularity, a weight function $\rho(x)$ is introduced into the bilinear and linear forms. The degree of it depends on the properties of the problem. The term of R_ν-generalized solution is defined. This allows us to suppress singularity of the solution and provide convergence of integrals in the integral identity.

Remark 2. In [1] a Dirichlet problem with consistent degeneracy of input data is considered. For such a problem, all terms in bilinear form have the same order in neighbourhood of each singularity point. The difference of the problem inconsistent degeneracy of input data investigated in this paper is that the coefficients of equation (2) in the neighbourhood of singularity points $\tau_i, i = \overline{1,n}$ have the same asymptotic behaviour. That is why the additive components in a bilinear form have different order. Such feature of boundary value problems with inconsistent degeneracy of input data necessitates introducing of weighted set $W^1_{2,\nu+\beta/2}(\Omega,\delta)$, since there is a cluster of R_ν-generalized solutions in a weighted space. The selection of a single R_ν-generalized solution in the weighted set can be carried out by adjustment of parameters ν and δ ([14]).

3. Numerical experiments for a Dirichlet problem with inconsistent degeneracy of input data

In [3], a scheme of the weighted FEM is constructed, based on the definition of R_ν-generalized solution of the problem (2), (3), the rate of convergence of the approximate solution of the suggested finite element method to the exact R_ν-generalized solution in the weight set $W^1_{2,\nu+\beta/2+1}(\Omega,\delta)$ is investigated, and estimation of the finite-element approximation is assessed.

In this section, we provide numerical experiments and analysis of the obtained results for two test problems. A differential equation (2) is considered in test boundary value problems. The exact solution $u(x)$ is chosen so that the boundary condition (3) are satisfied. Moreover, the coefficients $a_{kk}(x), k = 1,2, a(x)$ of the differential equation are chosen to satisfy conditions (4) to (7), and then the right hand side of the differential equation (2) is determined.

Numerical experiments implementing the weight FEM described above are carried out using computer program Proba-II and the GMRES method [15]. The optimal values of the parameters ν and δ are determined using the software package [16].

Calculations for each test problem are performed on grids with a different step h. The iteration process of solution of linear algebraic equations stops as soon as the norm of the difference between the approximate solutions on the last two iterations became smaller than 10^{-9}. For each test problem, both the approximate R_ν-generalized solution and the approximate generalized solution u^h $(\nu = 0)$ were calculated. For the found approximate R_ν-generalized solution, the error σ was determined in the norm of the set $W^1_{2,\nu+\beta/2+1}(\Omega,\delta)$

$$\sigma = \left(\sum_{|m|\le 1} \int_\Omega \rho^{2(\nu+\beta/2+1)} \left| D^m \left(u - u^h_\nu \right) \right|^2 dx \right)^{1/2}.$$

In each of the grid nodes P_i, $i = \overline{1, \bar{N}_h}$, the absolute errors were determined for the approximate R_ν-generalized and generalized solutions

$$\Delta_\nu(P_i) = \left| u(P_i) - u^h_\nu(P_i) \right|,\ \Delta(P_i) = \left| u(P_i) - u^h(P_i) \right|,\ i = \overline{1, \bar{N}_h},$$

respectively, and then the values of the largest absolute errors were calculated

$$\Delta_\nu = \max_{i=1\ldots \bar{N}_h} \Delta_\nu(P_i),\ \Delta = \max_{i=1\ldots \bar{N}_h} \Delta(P_i).$$

We introduce the following notation:

n_1 – is the number of several sub-sections along the axes Ox_1 and Ox_2;

$\bar{\sigma}_i$, $i = 1,2$ – is the specified limiting error;

n_2 – is the number of grid nodes where the absolute difference between the values of the exact and approximate generalized or R_ν-generalized solutions exceeds the limiting error $\bar{\sigma}_1$;

n_3 – is the number of grid nodes where the absolute difference between the values of the exact and approximate generalized or R_ν-generalized solutions exceeds the limiting error $\bar{\sigma}_2$ and is less than $\bar{\sigma}_1$;

n_4 – is the number of grid nodes from the δ-neighbourhood of the singularity point;

N_{it} – is the number of iterations required to achieve requires accuracy;

d – is the parameter used to calculate the radius of δ-neighbourhood of the singularity point;

h_{x_1} – is the length of the partition segment along Ox_1 axis;

$\delta = (1 + 0{,}01\mathrm{d}) h_{x_1}$ – is a radius of the neighbourhood of the singularity point;

ν – is the degree of the weight function in R_ν-generalized solution.

Test problem 1. Suppose that

$$\Omega = \{x \colon (x_1, x_2),\ -1 < x_1 < 1,\ -1 < x_2 < 0\}.$$

We choose the following function as an exact solution to test problem 1:

$$u(x) = \left(\sqrt{x_1^2 + x_2^2} \right)^{2/3} \sin\varphi \cos\varphi \left(1 - x_1^2 \right)\left(1 - x_2 \right).$$

The equation coefficients are:

$$a_{11}(x) = a_{22}(x) = a(x) = \frac{1}{\sqrt{x_1^2 + x_2^2}}.$$

Then the right-hand side of equation (2) takes the form:

$$f(x) = \frac{1}{9}\Big(-18x_1^4 - 9x_1^4 x_2 + 9x_1^4 x_2^2 - 41x_1^2 x_2 + 56x_1^2 x_2^2 + 9x_1^2 x_2^4 + 18x_1^2 -$$
$$-9x_1^2 x_2^3 - 63x_2^4 + 63x_2^3 + 38x_2 - 53x_2^2 \Big) x_1 / \left(x_1^2 + x_2^2 \right)^{13/6}.$$

The exact solution $u(x)$ of test problem 1 belongs to the sets $W^1_{2,0}(\Omega,\delta)$ and $W^2_{2,1/3}(\Omega,\delta)$; the coefficients $a_{kk}(x)$, $k = 1,2$, and $a(x)$ of the differential equation (2) belong to the sets $H^1_{\infty,-1}(\Omega, c_8)$

and $L_{\infty,-1}(\Omega,c_9)$, respectively; the right-hand side $f(x)$ of equation (2) belongs to $L_{2,4/3}(\Omega,\delta)$. A weak singularity of the solution of this test problem is due to the degeneracy property of the coefficients $a_{kk}(x)$, $k=1,2$, and $a(x)$ at the origin.

Numerical results for test problem 1 are presented in Tables 1-4 and on Figures 1-4.

Figure 1 shows distribution of the absolute errors of the approximate generalized and R_ν-generalized solutions, respectively.

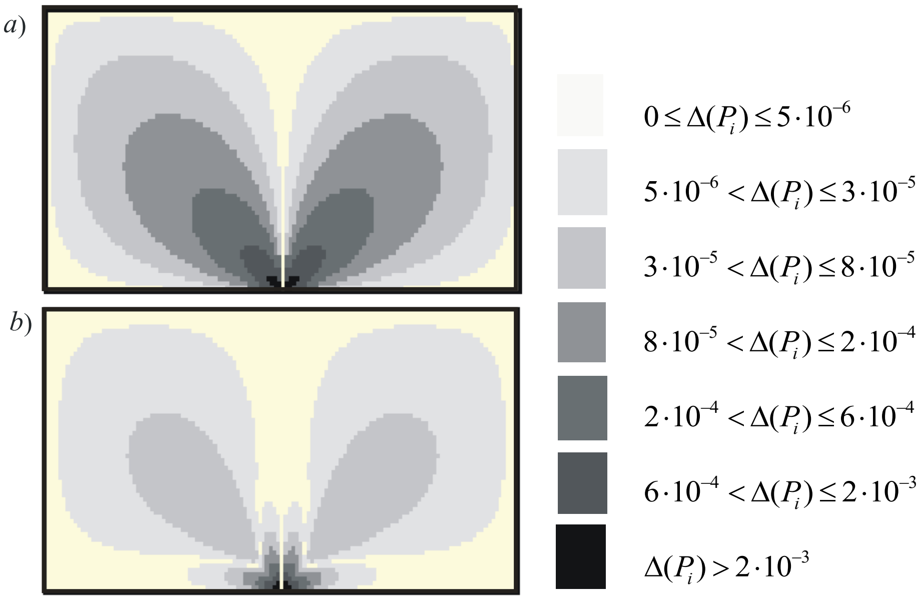

Figure 1: Distribution of absolute errors: a) for approximate generalized solution ($\nu=0$); b) for approximate R_ν-generalized solution ($\nu=2.2$, $\delta=1.71875\cdot 10^{-2}$) in the domain Ω for test problem 1, $n_1=128$.

Table 1 and Figure 2 present the dependence of the accuracy of finding an approximate R_ν-generalized solution on the grid size. The calculations are performed for test problem 1 with the following values $\nu=2{,}2$, $\beta=1$, $\delta=1{,}71875\cdot 10^{-2}$ ($d=10$), $\bar{\sigma}_1=2\cdot 10^{-3}$, $\bar{\sigma}_2=6\cdot 10^{-4}$.

Table 1
Dependence of accuracy of the approximate R_ν-generalized solution on the grid size, test problem 1.

n_1	32	48	64	80	128
σ	4,47·10^{-1}	3,32·10^{-2}	1,34·10^{-2}	1,18·10^{-2}	8,86·10^{-3}
Δ_ν	7,7·10^{-2}	2,9·10^{-2}	8,63·10^{-3}	7,59·10^{-3}	5,71·10^{-3}
n_2	62	161	6	6	6
n_3	8	383	22	18	16
N_{it}	20	59	87	156	524

Reviewing the results presented in Table 1, we can conclude that the value Δ_ν decreases if the parameter decreases h (the parameter n_1 increases).

Table 2 shows dependence of accuracy of the approximate generalized solution on the grid size. Calculations are performed for the test problem 1 with the following values of parameters: $\nu=0$, $\beta=1$, $\bar{\sigma}_1=2\cdot 10^{-3}$, $\bar{\sigma}_2=6\cdot 10^{-4}$.

Table 2
Dependence of accuracy of the approximate generalized solution on the grid size, test problem 1.

n_1	32	48	64	80	128
Δ	$1{,}79\cdot10^{-2}$	$4{,}03\cdot10^{-2}$	$1{,}15\cdot10^{-2}$	$9{,}99\cdot10^{-2}$	$7{,}35\cdot10^{-3}$
n_2	440	274	38	26	18
n_3	158	392	196	178	120
N_{it}	22	65	127	234	839

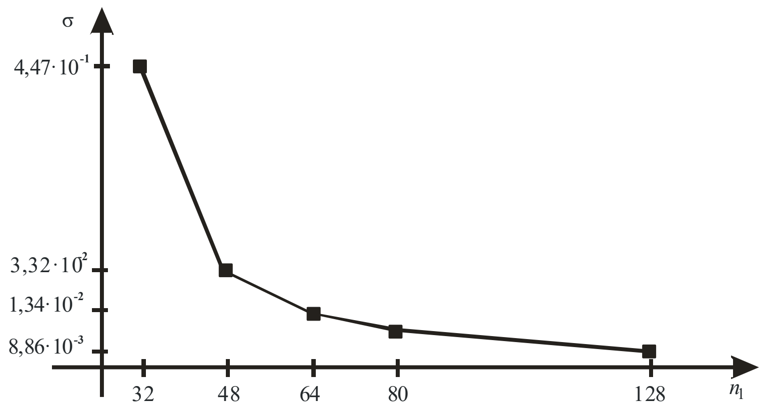

Figure 2: Dependence of accuracy σ on the grid size, test problem 1.

As Table 1 and Figure 2 show, in case of a uniform grid, the error value σ decreases as the parameter h decreases.

Table 1 and 2 also show dependencies of the number of grid nodes n_2 and n_3, as well as the number N_{it} of iterations on the grid size.

Table 3 and Figure 3 present results of influence of parameter δ on the accuracy of the approximate R_ν-generalized solution. Calculations are performed for test problem 1 with the following data: grid 128×64, $\nu = 2{,}2$, $\beta = 1$, $\bar{\sigma}_1 = 2\cdot10^{-3}$, $\bar{\sigma}_2 = 6\cdot10^{-4}$.

Table 3
Influence of parameter δ on accuracy of the approximate R_ν-generalized solution, test problem 1.

d	δ	σ	Δ_ν	n_2	n_3	n_4	N_{it}
-10	$1{,}40625\cdot10^{-2}$	$2{,}53\cdot10^{-3}$	$9{,}84\cdot10^{-4}$	0	4	0	618
0	$1{,}5625\cdot10^{-2}$	$2{,}53\cdot10^{-3}$	$9{,}84\cdot10^{-4}$	0	4	1	831
10	$1{,}71875\cdot10^{-2}$	$8{,}86\cdot10^{-3}$	$5{,}71\cdot10^{-3}$	6	16	1	524
20	$1{,}875\cdot10^{-2}$	$1{,}73\cdot10^{-2}$	$1{,}11\cdot10^{-2}$	12	448	1	65
30	$2{,}03125\cdot10^{-2}$	$2{,}39\cdot10^{-2}$	$1{,}53\cdot10^{-2}$	18	60	1	639
50	$2{,}34375\cdot10^{-2}$	$2{,}58\cdot10^{-2}$	$1{,}69\cdot10^{-2}$	26	122	1	582

Table 4 and Figure 4 present influence of parameter ν on accuracy of the approximate R_ν-generalized solution. Calculations are performed for test problem 1 with the following data: grid 128×64, $\beta = 1$, $\delta = 1{,}71875\cdot10^{-2}$ ($d = 10$), $\bar{\sigma}_1 = 2\cdot10^{-3}$, $\bar{\sigma}_2 = 6\cdot10^{-4}$.

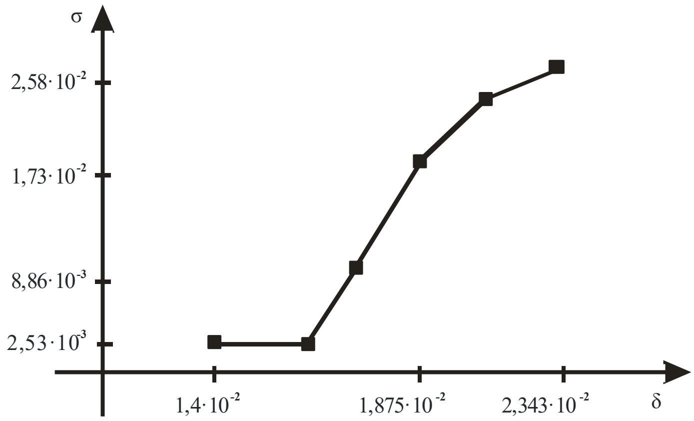

Figure 3: Influence of parameter δ on accuracy of the approximate R_ν-generalized solution, test problem 1.

Table 4

Influence of parameter ν on accuracy of the approximate R_ν-generalized solution, test problem 1.

ν	σ	Δ_ν	n_2	n_3	N_{it}
2	$7{,}96\cdot10^{-3}$	$5{,}13\cdot10^{-3}$	6	12	556
2,1	$8{,}41\cdot10^{-3}$	$5{,}42\cdot10^{-3}$	6	14	593
2,2	$8{,}86\cdot10^{-3}$	$5{,}71\cdot10^{-3}$	6	16	524
2,25	$9{,}09\cdot10^{-3}$	$5{,}85\cdot10^{-3}$	6	18	591
2,7	$1{,}11\cdot10^{-2}$	$7{,}13\cdot10^{-3}$	6	20	660
3	$1{,}24\cdot10^{-2}$	$7{,}95\cdot10^{-3}$	8	24	677

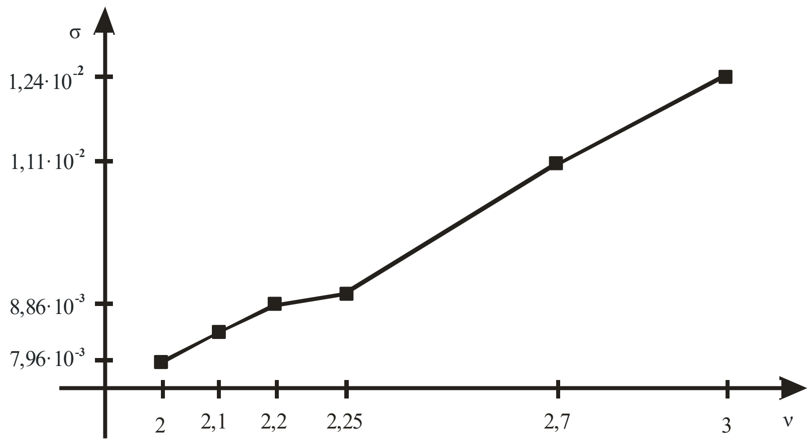

Figure 4: Influence of parameter ν on accuracy of the approximate R_ν-generalized solution, test problem 1.

Test problem 2. Suppose that

$$\Omega = \{x : (x_1, x_2),\ -1 < x_1 < 1,\ -1 < x_2 < 0\}.$$

For test problem 2,

$$u(x) = \left(\sqrt{x_1^2 + x_2^2}\right)^{-1/2} \sin\varphi \cos\varphi \left(1 - x_1^2\right)\left(1 - x_2\right),$$

$$a_{11}(x) = a_{22}(x) = \frac{1}{\sqrt{x_1^2 + x_2^2}},\ a(x) = \frac{1}{x_1^2 + x_2^2},$$

$$f(x) = -\frac{1}{4}\Big(8x_1^4 - 4x_1^2 x_2^2 \sqrt{x_1^2 + x_2^2} - 53x_1^2 x_2^2 - 8x_1^2 + 4x_1^2 x_2 \sqrt{x_1^2 + x_2^2} +$$

$$+37x_1^2x_2 - 13x_2 + 24x_2^4 + 4x_2^2\sqrt{x_1^2+x_2^2} - 24x_2^3 + 29x_2^2 - 4x_2\sqrt{x_1^2+x_2^2}\Big)x_1 / \left(x_1^2+x_2^2\right)^{11/4}.$$

The exact solution $u(x)$ of the test problem 2 belongs to the sets $W^1_{2,1/2}(\Omega,\delta)$ and $W^2_{2,3/2}(\Omega,\delta)$; the coefficients $a_{kk}(x),\ k=1,2$, and $a(x)$ of the differential equation (2) belong to the sets $H^1_{\infty,-1}(\Omega,c_{10})$ and $L_{\infty,-2}(\Omega,c_{11})$, respectively; the right-hand side $f(x)$ of equation (2) belongs to $L_{2,5/2}(\Omega,\delta)$. The strong singularity of the solution of this test problem is due to degeneracy of coefficients $a_{kk}(x),\ k=1,2$, and $a(x)$ at the origin. A generalized solution to this problem does not exist, but R_ν-generalized solution can be defined.

Numerical results for test problem 2 are presented in Tables 5-7 and on Figures 5-8.

Figure 5 shows distribution of the absolute error of the approximate R_ν-generalized solution in the domain.

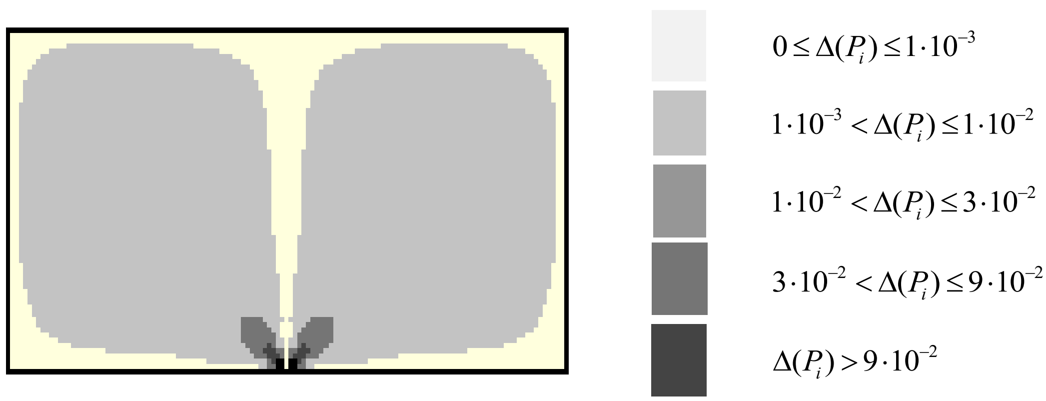

Figure 5: Distribution of absolute error of the approximate R_ν-generalized solution ($\nu = 6{,}5$, $\delta = 2{,}1875\cdot 10^{-2}$) in domain Ω for $n_1 = 128$, test problem 2.

Table 5 and Figure 6 show dependence of accuracy of the approximate R_ν-generalized solution on the grid size. Calculations were performed for test problem 2 with the following values: $\nu = 6{,}5$, $\beta = 2$, $\delta = 2{,}1875\cdot 10^{-2}$ ($d = 10$), $\bar{\sigma}_1 = 7\cdot 10^{-2}$, $\bar{\sigma}_2 = 3\cdot 10^{-2}$. In test problem 2, an approximate generalized solution was not possible to find because of program failure.

Table 5
Dependence of accuracy of the approximate R_ν-generalized solution on the grid size, test problem 2.

n_1	32	64	80	128
σ	$1{,}01\cdot 10^{-4}$	$1{,}38\cdot 10^{-5}$	$7{,}62\cdot 10^{-6}$	$2{,}26\cdot 10^{-6}$
Δ_ν	$7{,}12\cdot 10^{-2}$	$8{,}02\cdot 10^{-2}$	$8{,}5\cdot 10^{-2}$	$9{,}87\cdot 10^{-2}$
n_2	2	2	2	0
n_3	10	12	12	18
N_{it}	15	66	104	232

Remark 3. Numerical results (Table 5) for test problem 2 show that the approximate R_ν-generalized solution can be found with high precision even when the generalized solution cannot be calculated.

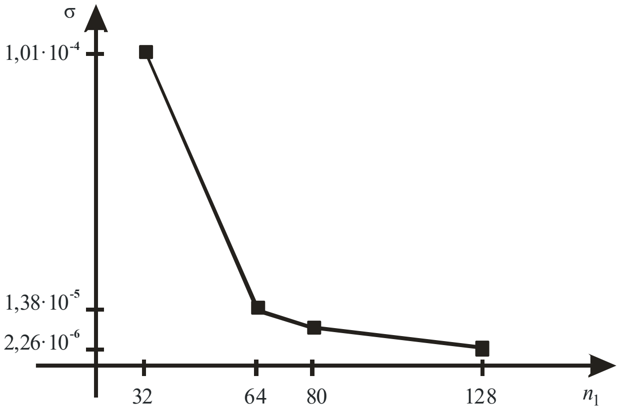

Figure 6: Dependence of accuracy σ on the grid size, test problem 2.

Table 6 and Figure 7 present the results of the radius-neighbourhood δ of the singularity point effect on the accuracy of finding an approximate R_ν-generalized solution. The calculations are performed for test problem 2 with the following data: grid 128×64, $\nu = 6{,}5$, $\beta = 2$, $\bar{\sigma}_1 = 7 \cdot 10^{-2}$, $\bar{\sigma}_2 = 3 \cdot 10^{-2}$.

Table 6
Influence of parameter ν on accuracy of the approximate R_ν-generalized solution, test problem 2.

d	δ	σ	Δ_ν	n_2	n_3	n_4	N_{it}
-10	$1{,}40625 \cdot 10^{-2}$	$5{,}9 \cdot 10^{-6}$	1,3	98	208	0	132
10	$1{,}71875 \cdot 10^{-2}$	$5{,}93 \cdot 10^{-6}$	$7{,}6 \cdot 10^{-1}$	54	114	1	176
20	$1{,}875 \cdot 10^{-2}$	$4{,}03 \cdot 10^{-6}$	$3{,}78 \cdot 10^{-1}$	24	62	1	178
35	$2{,}109375 \cdot 10^{-2}$	$2{,}34 \cdot 10^{-6}$	$1{,}2 \cdot 10^{-1}$	4	26	1	207
40	$2{,}1875 \cdot 10^{-2}$	$2{,}26 \cdot 10^{-6}$	$9{,}87 \cdot 10^{-2}$	0	18	1	232

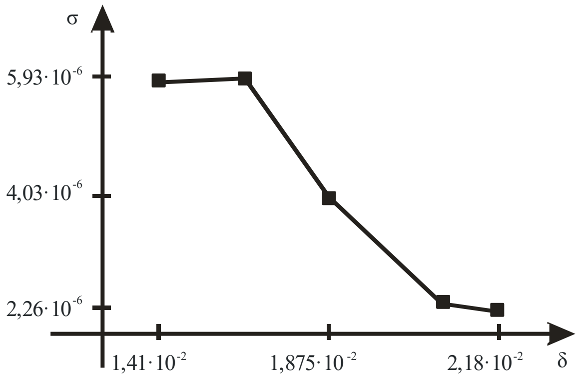

Figure 7: Influence of parameter δ on accuracy of the approximate R_ν-generalized solution, test problem 2.

Table 7 and Figure 8 show influence of parameter ν on the accuracy of the approximate R_ν-generalized solution. The calculations were performed for test problem 2 with the following data: grid 128×64, $\beta = 2$, $\delta = 2{,}1875 \cdot 10^{-2}$ ($d = 40$), $\bar{\sigma}_1 = 7 \cdot 10^{-2}$, $\bar{\sigma}_2 = 3 \cdot 10^{-2}$.

Tables 3, 4, 6 and 7 show that there are ranges for parameters δ and ν within which the convergence rate of the approximate R_ν-generalized solution to the exact one is not less than the theoretical one. For the best values of parameters δ and ν from the discovered ranges, the error values σ and Δ_ν are the smallest ones.

Table 7
Influence of parameter ν on accuracy of the approximate R_ν-generalized solution, test problem 2.

ν	σ	Δ_ν	n_2	n_3	N_{it}
5,5	$3{,}94 \cdot 10^{-6}$	$1{,}92 \cdot 10^{-1}$	14	42	167
6	$2{,}78 \cdot 10^{-6}$	$1{,}27 \cdot 10^{-1}$	10	24	228
6,45	$2{,}29 \cdot 10^{-6}$	$1{,}01 \cdot 10^{-1}$	2	18	232
6,5	$2{,}26 \cdot 10^{-6}$	$9{,}87 \cdot 10^{-2}$	0	18	232
6,55	$2{,}24 \cdot 10^{-6}$	$9{,}63 \cdot 10^{-2}$	0	16	232
6,6	$2{,}23 \cdot 10^{-6}$	$9{,}39 \cdot 10^{-2}$	0	16	230
6,75	$2{,}21 \cdot 10^{-6}$	$8{,}72 \cdot 10^{-2}$	2	18	230
6,85	$2{,}21 \cdot 10^{-6}$	$8{,}3 \cdot 10^{-2}$	2	20	200
7	$2{,}24 \cdot 10^{-6}$	$8{,}02 \cdot 10^{-2}$	4	20	234

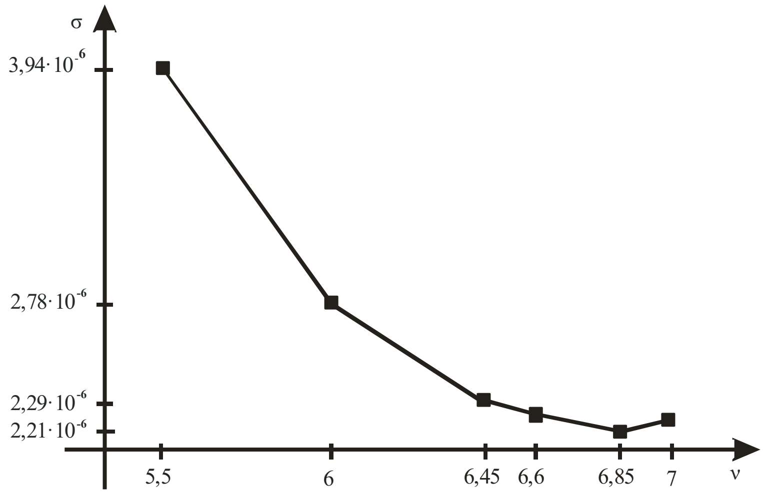

Figure 8: Influence of parameter ν on accuracy of the approximate R_ν-generalized solution, test problem 2.

Reviewing results of the numerical experiment, we can make the following conclusions about approximation properties of the weighted finite element method for boundary value problems with singularity of the solution and inconsistent degeneracy of input data:

1) The approximate R_ν-generalized solution of the problem (2), (3) converges to the exact solution in the norm of the weighted set $W^1_{2,\nu+\beta/2+1}(\Omega,\delta)$ at a rate not less than $O(h)$ (Table 1 and 5), which verifies the theoretical results obtained ([2]).

2) The introduction of the notion of R_ν-generalized solution and application of the weighted finite element method allows us to deal with singularity caused by input data degeneracy (Table 5) and if there is no generalized solution. Although, an approximate R_ν-generalized solution is highly precise even in the neighbourhood of the point of singularity.

3) For the best parameters ν and δ convergence rate of the approximate R_ν-generalized solution to the exact one is the highest.

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