Multi-frequency experiment on determining the parameters of a dielectric layer in a waveguide

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Abstract

The inverse problem of reconstructing the real permittivity of a plane-parallel layer in a perfectly conducting rectangular waveguide from experimental data using an explicit expression for the scattering matrix is considered. This problem is ill-posed due to the presence of self-intersection points on the curves of the complex scattering coefficients. It is shown that the traditional multi-frequency measurement method used in vector network analyzers can be justified by the fact that the algorithm for processing the measurement results by the least squares method becomes stable if the number of frequencies is large enough.

Keywords 1

waveguide, well-posedness of inverse problem, convergence conditions for the least squares method, multi-frequency measurements

1. Introduction

Application of materials in various fields of science and technology such as material science, microwave engineering, aerospace, microelectronics, and communication industries requires the exact knowledge of material parameters such as permittivity and permeability. The most widely used methods for measuring dielectric materials parameters are cavity perturbation techniques and free-space and transmission line of waveguide methods. Each approach has its own advantages and limitations.

To implement the free-space methods it is necessary to have sufficiently large sheets of the studied material. The accuracy of this method is low due to mismatch between the experimental setup and the simple mathematical model having an exact solution. The cavity perturbation techniques are more accurate, but they are applicable only over a narrow frequency band in which the resonance effect is observed.

In our work we study the widely used method based on transmission line in the form of a rectangular metal (perfectly conducting) waveguide. If the material sample used in these measurements is a plane-parallel layer, then the mathematical model of the experiment is the simplest. In this case the exact explicit expression for the permittivity and permeability as a function of the S-parameters (the coefficient of transmission of the electromagnetic field through the inclusion and the reflection coefficient) is known. This procedure named Nicolson-Ross-Weir method (NRW) [1, 2]. Its disadvantage is the solution phase ambiguity for low-loss materials, except for those samples which width is less than one-half wavelength.

In the general case of inclusion of an arbitrary shape numerical simulation of the experiment is necessary. The desired value of the dielectric constant of the material can be found as a solution to the inverse problem by comparing the experimental data and theoretical values of the transmission coefficient.

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We study the well-posedness condition for the inverse problem of determining the layer parameters from experimental data; namely, the existence and uniqueness of solution and its continuous dependence on the input data. Unfortunately this algorithm is improperly posed. In fact, (a) the range of the function specifying the transmission coefficient is a curve on the complex plane; therefore the probability that the experimental data belongs this curve is equal to zero; and (b) the parametric curve of the function on the complex plane has self-intersection points which means that the solution may not be unique.

It is shown that the traditional multi-frequency method of measurements applied in vector network analyzers can be used to formulate a well-posed problem. For a noiseless experiment that perfectly matches the mathematical model the non-uniqueness of the solution can be eliminated if we consider the vector formulation of the problem determined by a set of frequencies in the range of single-mode waveguide. The vector function of the transmission coefficient becomes a one-to-one function of the inclusion dielectric constant if the frequency resolution is sufficiently small.

For an actual physical experiment, the least squares method (LSM) can be applied for the solution of the inverse problem under study. The LSM solution converges to the desired value of the layer permittivity if the quality of the experiment (determined both by noise and defects of the measurement setup and material samples) is improved. The convergence rate is enhanced if the number of frequencies used in experiment is taken large enough.

2. The transmission coefficient of the principal mode

The waveguide of rectangular cross section and perfectly conducting walls shown in Fig. 1 contains a dielectric parallel-plane diaphragm (layer). $\Omega_1=(0,a)\times(0,b)\times(z_-^{(layer)},z_+^{(layer)})$ with a dielectric inclusion Ω_2, $\Omega_2\subset\Omega_1\subset\Omega$.

The relative permittivity in the waveguide $\varepsilon(\mathbf{r})/\varepsilon_0=\begin{cases}1,\ \mathbf{r}\in\Omega_0\setminus\Omega_1,\\ \varepsilon_{teflon},\ \mathbf{r}\in\Omega_1\setminus\Omega_2,\\ \varepsilon_{cube},\ \mathbf{r}\in\Omega_2,\end{cases}$

where $\varepsilon_{teflon}\approx 2.04$, $\varepsilon_{cube}\approx 3.8$ (quartz), 10.0 (ruby), and ε_0 is permittivity of vacuum.

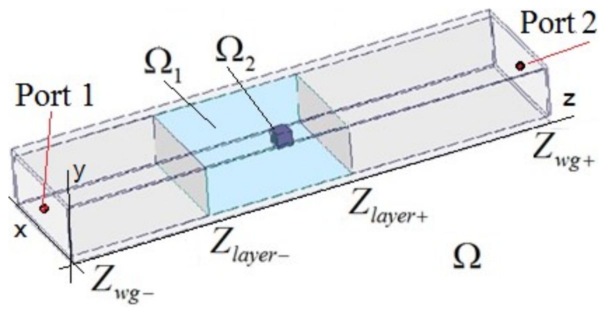

Figure 1: Waveguide with a diaphragm and inclusion.

We develop algorithms of reconstructing permittivity of the diaphragm (a test problem) and of a small inclusion inside the diaphragm by comparing the results of multi-frequency series of the field measurements on the waveguide flanges and the data obtained from the mathematical model corresponding to the experimental setup. The scheme of measurements using Vector Network Analyzer (VNA) are performed at the points (ports) shown in Fig. 1.

Define $f_{(1,0)}=c/(2a)$, $f_{(2,0)}=2f_{(1,0)}$, $f_{(1,0)}$, $f_{(2,0)}$ are the cutoff frequency for $TE_{(1,0)}$, $TE_{(2,0)}$ modes, $f_{(1,0)}\approx 6.52$ (GHz).

It is known that in the empty parts of the waveguide the steady-state solution $\mathbf{E}(\mathbf{r},t)=\hat{\mathbf{E}}(\mathbf{r})e^{-i\omega t}$ of Maxwell's equations is represented as a sum of harmonic and evanescent waves. If calculations are performed at the frequency f : $f_{(1,0)}<f<f_{(2,0)}$, when only one (principal) waveguide mode

propagates in empty parts of the waveguide; the higher-order modes are evanescent (standing) waves decaying exponentially on both sides of the diaphragm.

At the output (for $z = z_+^{(wg)}$) of the waveguide with inclusions in the diaphragm, the transmitted field has the form

$$\hat{E}_y^{(trans)}(x, y, z_+^{(wg)}) = \hat{E}_{y,(1,0)}^{(trans)}(x, y, z_+^{(wg)}) + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(1-\delta_{(n,1)}\delta_{(m,0)})\hat{E}_{y,(n,m)}^{(trans)}(x, y, z_+^{(wg)}), \tag{1}$$

where

$$\hat{E}_{y,(1,0)}^{(trans)}(x, y, z_+^{(wg)}) = R_{(1,0)}^{+} e^{ik_{0,(1,0)}^{(z)} z_+^{(wg)}}, \quad \hat{E}_{y,(n,m)}^{(trans)}(x, y, z_+^{(wg)}) = R_{(n,m)}^{+} X_{(n)}(x) Y_{(m)}(y) e^{-|k_{0,(n,m)}^{(z)}| z_+^{(wg)}},$$

$x \in (0, a), \;\; y \in (0, b)$.

If the waveguide is long enough then we can assume that

$$\hat{E}_y^{(trans)}(x, y, z_+^{(wg)}) \approx \hat{E}_{y,(1,0)}^{(trans)}(x, y, z_+^{(wg)}).$$

Here

$$R_{(n,m)}^{+} = \int_0^b\int_0^a X_{(n)}(x) Y_{(m)}(y) \hat{E}_y^{(trans)}(x, y, z_+^{(wg)})\, dxdy,$$

$X_{(n)}(x) = (2/a)^{1/2}\sin(k_{(n)}^{(x)}x)$, $Y_{(m)}(y) = ((2-\delta_{(m,0)})/b)^{1/2}\cos(k_{(m)}^{(y)}y)$ are the basis functions, $k_{(n)}^{(x)} = \pi n/a$, $k_{(m)}^{(y)} = \pi m/b$, $k_{0,(n,m)}^{(z)} = (k_0{}^2 - (k_{(n)}^{(x)})^2 - (k_{(m)}^{(y)})^2)^{1/2}$, $k_0^2 = \omega^2\varepsilon_0\mu_0$, $n \in \mathbb{N}$, $m \in \mathbb{N}_0$, and $\delta_{(n,m)} = \operatorname{sgn}|n-m|$.

For the sample under test situated between two antennas (ports) the measured quantities are the complex scattering S-matrix coefficients (transmission and reflection), where

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{11} \end{pmatrix},$$

S_{12} is the transmission coefficient from port 2 to port 1, calculated as the ratio of the field measured at the exit port in the presence of an inclusion to the source field measured at the input port.

The input and the output (at the ports with z= $z_-^{(wg)}$, z= $z_+^{(wg)}$ correspondingly) measurement data give the following elements of the scattering matrix describing the transmitted wave for the empty waveguide and the waveguide containing the diaphragm, respectively:

$$S_{12}^{(wg)} = \frac{\hat{E}_y^{(inc)}(x, y, z_+^{(wg)})}{\hat{E}_y^{(inc)}(x, y, z_-^{(wg)})} = e^{ik_0^{(z)}d^{(wg)}},$$
$$S_{12}^{(wg,layer)} = \frac{\hat{E}_y^{(trans)}(x, y, z_+^{(wg)})}{\hat{E}_y^{(inc)}(x, y, z_-^{(wg)})}. \tag{2}$$

If measurements are taken at the layer boundary, then

$$S_{12}^{(layer)} = \frac{\hat{E}_y^{(trans)}(x, y, z_+^{(layer)})}{\hat{E}_y^{(inc)}(x, y, z_-^{(layer)})}, \quad S_{12}^{(0,layer)} = \frac{\hat{E}_y^{(inc)}(x, y, z_+^{(layer)})}{\hat{E}_y^{(inc)}(x, y, z_-^{(layer)})} = e^{ik_0^{(z)}d^{(layer)}},$$

$d^{(wg)} = z_+^{(wg)} - z_-^{(wg)}$ is the waveguide length, $d^{(layer)} = z_+^{(layer)} - z_-^{(layer)}$ is the layer width.

Define the transmission coefficient of the principal waveguide mode as

$$F_{(1,0)} = \frac{\hat{E}_{y,(1,0)}^{(trans)}(x, y, z_+^{(wg)})}{\hat{E}_{y,(1,0)}^{(inc)}(x, y, z_+^{(wg)})} = R_{(1,0)}^{+}/A. \tag{3}$$

The waveguide transmission coefficient $F_{1,0}$ defined according to (1), (2), and (3) can be represented as

$$F_{(1,0)} \approx \frac{\hat{E}_y^{(trans)}(x, y, z_+^{(wg)})}{\hat{E}_y^{(inc)}(x, y, z_+^{(wg)})} = \frac{S_{12}^{(wg,layer)}}{S_{12}^{(wg)}}.$$

Thus the transmission coefficient of the principal mode through a layer in a waveguide is calculated as the ratio of the field measured at the output port far enough from inclusion to the field measured at the output port in the empty waveguide.

Problem for the Maxwell's equations in the unbounded waveguide containing a dielectric layer with permittivity ε admits the closed-form solution for transmission coefficient of the principal mode $TE_{(1,0)}$:

$$F(\varepsilon,f)=\frac{4t_\varepsilon(f)e^{-ik_0^{(z)}d^{(layer)}}}{(1+t_\varepsilon(f))^2e^{-ik_\varepsilon^{(z)}d^{(layer)}}-(1-t_\varepsilon(f))^2e^{ik_\varepsilon^{(z)}d^{(layer)}}},$$

or in another form [3]

$$F(\varepsilon,f)=\frac{1}{S_{12}^{(0,layer)}(f)}\frac{1}{g(\varepsilon,f)}, \tag{4}$$

where

$$g(\varepsilon,f)=c_\varepsilon(f)-i\,H\big(t_\varepsilon(f)\big)\,s_\varepsilon(f), \tag{5}$$

$$s_\varepsilon(f)=\sin\left(k_\varepsilon^{(z)}(f)d^{(layer)}\right),\;\; c_\varepsilon(f)=\cos\left(k_\varepsilon^{(z)}(f)d^{(layer)}\right), \tag{6}$$

$$t_\varepsilon(f)=\frac{k_\varepsilon^{(z)}(f)}{k_0^{(z)}(f)},\;\; k_\varepsilon^{(z)}(f)=k_{\varepsilon,(1,0)}^{(z)}(f)=\left(k_\varepsilon^2(f)-(k^{(x)})^2\right)^{1/2},\;\; k_0^{(z)}(f)=k_{0,(1,0)}^{(z)}(f)=\left(k_0{}^2(f)-(k^{(x)})^2\right)^{1/2},$$

$$k_\varepsilon(f)=\varepsilon^{1/2}k_0(f),\;\; k_0(f)=\omega(\varepsilon_0\mu_0)^{1/2},\;\; \lambda_\varepsilon(f)=2\pi/k_\varepsilon(f),\;\; \mu=\mu_0,$$

$$H(x)=0.5\left(x+\frac{1}{x}\right),$$

with $x>0$.

3. Algorithms of experimental data processing

Let us assume that the relative dielectric constant of the sample is real and does not depend on the frequency of oscillations of the electromagnetic field in a certain range within the single-mode range of the waveguide. If the material properties are *a priori* unknown, we will look for the effective dielectric constant of the inclusion for the selected bandwidth.

Introduce the vectors

$$\mathbf{f}=(f_1,\dots,f_{N^{(\exp)}})\in\mathbb{R}^{N^{(\exp)}},\;\; \mathbf{F}^{(\exp)}=(F_1^{(\exp)},\dots,F_{N^{(\exp)}}^{(\exp)})\in\mathbb{C}^{N^{(\exp)}}$$

of the frequency and complex-valued measurement data of $N^{(\exp)}$ experiments. Consider the equation

$$\mathbf{g}\left(\varepsilon^{(layer)},\mathbf{f}\right)=\mathbf{g}^{(\exp)}, \tag{7}$$

for the (unknown) dielectric constant of the layer $\varepsilon^{(layer)}\geq 1$, where

$$\mathbf{g}(\varepsilon,\mathbf{f})=\left(g(\varepsilon,f_1),\dots,g(\varepsilon,f_{N^{(\exp)}})\right)\in\mathbb{C}^{N^{(\exp)}}$$

with g defined in (5), (6),

$$\mathbf{g}^{(\exp)}=(g_1^{(\exp)},\dots,g_{N_{\exp}}^{(\exp)})\in\mathbb{C}^{N^{(\exp)}},$$

$$g_n^{(\exp)}=\frac{1}{Z_0^{(layer)}(f_n)}\frac{1}{F_n^{(\exp)}},$$

$n=1,\dots,N^{(\exp)}$.

We formulate inverse problems that constitute different permittivity reconstruction scenarios of the layer in the waveguide. To this end, let

$$\Omega^{(\varepsilon)}=\{\varepsilon:\varepsilon\geq 1\},\; \Omega_{\mathrm{E}}^{(\varepsilon)}=\{\varepsilon:1\leq\varepsilon\leq\mathrm{E}\},$$

$\mathrm{E}>1$, and by

$$G\left(\mathbf{f},\Omega^{(\varepsilon)}\right)=\left\{\;\mathbf{g}(\varepsilon,\mathbf{f})\in\mathbb{C}^{N^{(\exp)}},\;\varepsilon\in\Omega^{(\varepsilon)}\;\right\}$$

denote the set of values of function $\mathbf{g}(\varepsilon,\mathbf{f})$ for the fixed frequency vector $\mathbf{f}$ (it is a curve in $N^{(\exp)}$-dimensional complex space).

Formulate two inverse problems considered in the study.

Problem 1

Find a real $\varepsilon^{(layer)} \in \Omega^{(\varepsilon)}$ satisfying relation (7) for a given complex vector $\mathbf{g}^{(\exp)} \in G\left(\mathbf{f},\Omega^{(\varepsilon)}\right)$ with the fixed frequency vector $\mathbf{f}$.

Problem 2

Find a real $\varepsilon^{(layer)} \in \Omega^{(\varepsilon)}$ satisfying relation (7) for a given complex vector $\mathbf{g}^{(\exp)} \in \mathbb{C}^{N^{(\exp)}}$ with the fixed frequency vector $\mathbf{f}$.

Check the fulfillment of the well-posedness condition for these problems; namely, the existence and uniqueness of solution and its continuous dependence on the input data.

Problem 1 describes a perfect experiment exactly corresponding to the mathematical model, it is solvable by the definition of the set $G\left(\mathbf{f},\Omega^{(\varepsilon)}\right)$. However, its uniqueness may be violated. In fact, if $N^{(\exp)}=1$ for any chosen frequency the solution is not unique due to the existence of a countable set $\{\varepsilon_m\}_{m=1,\ldots}$, satisfying the relation $\sin(k_{\varepsilon_m}(f)\,d^{(layer)})=0$ that specifies self-intersections points of curve $G(f,\Omega^{(\varepsilon)})$ (Fig. 2).

Using *a priori* information about $\varepsilon^{(layer)}$ we can achieve the uniqueness by adjusting domain $\Omega_{\mathrm{E}}^{(\varepsilon)}$ and a frequency range $[f_1, f_{N^{(\exp)}}]$. However, the formally well-posed problem may be ill-conditioned in the vicinity of the intersection points mentioned above where the parameter values are such that the quantity $\sin(k_\varepsilon(f)d^{(layer)})$ in the denominator virtually vanishes.

Proposition 1 below demonstrates that for $N^{(\exp)}>1$ the solution to Problem 1 is unique if the frequency resolution is sufficiently small. In fact, $\mathbf{g}(\varepsilon,\mathbf{f})$ becomes a one-to-one vector function of real variable ε for a fixed set of frequency values $\mathbf{f}$.

Problem 2 simulates the processing of noisy experimental data. This problem is also ill-posed since it may be unsolvable: in actual experiments, it is typical that $\mathbf{g}^{(\exp)} \notin G(\mathbf{f},\Omega^{(\varepsilon)})$ because the set (a curve) has the zero measure on the complex plane. We will replace Problem 2 with an LSM problem such that its solution approximates the sought solution of perfect Problem 1 when the defects of the experimental setup and measurement error decrease.

4. One-to-one correspondence

Let us consider the problem of determining the value of the dielectric constant of a plane-parallel non-magnetic layer without losses. One can show that the transition from a single-frequency experiment to a multi-frequency experiment improves the properties of the inverse problem, ensuring its unique solvability. We present without proof the following statement [4].

Proposition 1

Set $N^{(\exp)}=1$. For any $\mathrm{E}>1$ there is one-to-one correspondence between $\Omega_{\mathrm{E}}^{(\varepsilon)}$ and $G(f,\Omega_{\mathrm{E}}^{(\varepsilon)})$ for the fixed frequency f if

$$\frac{d^{(layer)}}{0.5\lambda_E(f)}<1. \tag{8}$$

Assume that $N^{(\exp)}\geq 2$. For any $\mathrm{E}>1$ there is one-to-one correspondence between $\Omega_{\mathrm{E}}^{(\varepsilon)}$ and $G(\mathbf{f},\Omega_{\mathrm{E}}^{(\varepsilon)})$ for the fixed frequency vector $\mathbf{f}$ if the following condition is satisfied in at least one of the two equivalent forms:

$$\frac{d^{(layer)}}{0.5\lambda_E(f_{n+1})}-\frac{d^{(layer)}}{0.5\lambda_E(f_n)}<1, \tag{9}$$

$n=1,\ldots,N^{(\exp)}-1$, or

$$h^{(f)} < h_E^{(f)} = \frac{c}{2d^{(layer)}} \frac{1}{E^{1/2}}. \qquad (10)$$

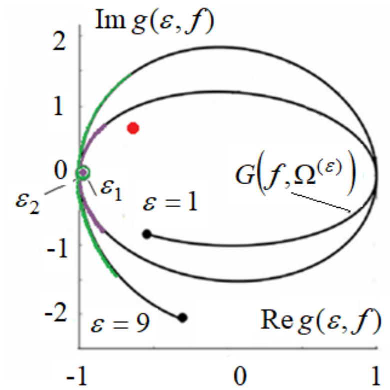

Figure 2: The branches of the curve $G\left(f,\Omega^{(\varepsilon)}\right)$, $f = 9.25\,\text{GHz}$, $\Omega^{(\varepsilon)} = \{\varepsilon : 1.0 \le \varepsilon \le 9.0\}$, $\Omega_1^{(\varepsilon)} = (2.05, 2.13)$, $\Omega_2^{(\varepsilon)} = (3.06, 3.12)$, their intersection point $g(\varepsilon_1, f) = g(\varepsilon_2, f)$, $\varepsilon_1 = 2.09 \in \Omega_1^{(\varepsilon)}$, $\varepsilon_2 = 3.12 \in \Omega_2^{(\varepsilon)}$, red point is experimental value $g^{(\exp)}$.

5. Multi-frequency least squares method

Let us show that the solution of ill-posed Problem 2 can be found approximately with an accuracy determined by the quality of the experiment. Here along with the problem for determining the real dielectric constant of a low-loss sample we formulate a problem for determining the complex parameter of a loss sample without discussing its properties; the results of the numerical solution of both problems are shown in Fig. 3.

Problem 3 (4) (LSM)

Find $\varepsilon^{(LS,1)} \in \mathbb{R}^{N^{(\exp)}}$ ($\varepsilon^{(LS,2)} \in \mathbb{C}^{N^{(\exp)}}$) satisfying the condition

$$\left\|\mathbf{g}(\varepsilon^{(LS)}, \mathbf{f}) - \mathbf{g}^{(\exp)}\right\| = \min\left(\left\|\mathbf{g}(\varepsilon, \mathbf{f}) - \mathbf{g}^{(\exp)}\right\|,\ \varepsilon \in \Omega_E^{(\varepsilon)}\right)$$

for a given vector $\mathbf{g}^{(\exp)} \in \mathbb{C}^{N^{(\exp)}}$ with the fixed frequency vector $\mathbf{f}$.

We present without proof the following statements [5].

Proposition 2

The Problems 3 is solvable.

Proposition 3

If the conditions of Proposition 1 are satisfied and

$$\left\|\mathbf{g}(\varepsilon^{(layer)}, \mathbf{f}) - \mathbf{g}^{(\exp)}\right\|^{(\exp)} \to 0, \qquad (11)$$

then

$$\varepsilon^{(LS,1)} \to \varepsilon^{(layer)}. \qquad (12)$$

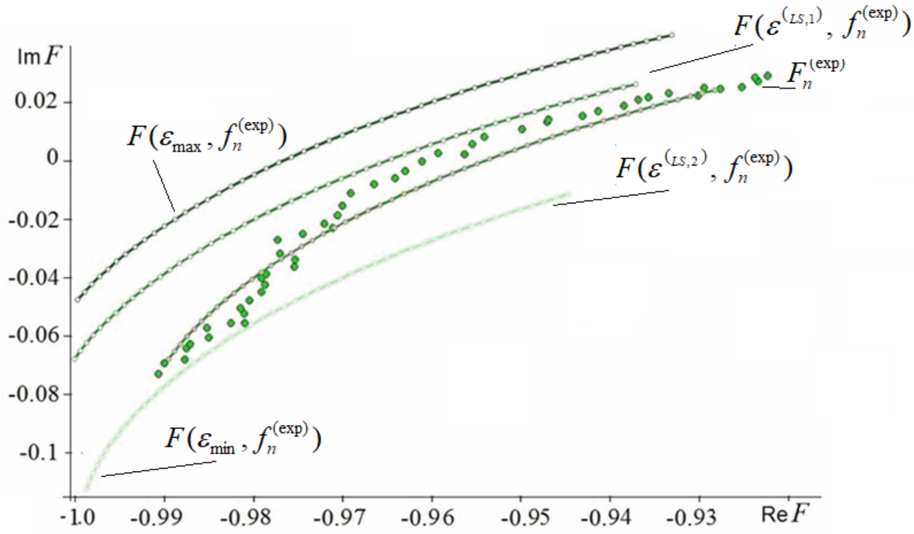

Figure 3: Application of least squares to the determination of permittivity $\varepsilon_{\min} \le \varepsilon \le \varepsilon_{\max}$ from transmission coefficient experimental data $\{F_n^{(\exp)}\}_{n=1,\ldots N_{\exp}}$ for frequency set $\{f_n^{(\exp)}\}_{n=1,\ldots N_{\exp}}$; $F(\varepsilon_{\min}, f_n^{(\exp)})_{n=1,\ldots N_{\exp}}$, $F(\varepsilon_{\max}, f_n^{(\exp)})_{n=1,\ldots N_{\exp}}$ are test points; $\left\{F(\varepsilon^{(LS,1)}, f_n^{(\exp)}\right\}_{n=1,\ldots N_{\exp}}$, $\left\{F(\varepsilon^{(LS,2)}, f_n^{(\exp)}\right\}_{n=1,\ldots N_{\exp}}$ are theoretical transmission coefficient values for least squares solutions.

6. Conclusion

Our findings have demonstrated that conducting an experiment in a multi-frequency mode makes it possible to turn a technical possibility into a mathematical achievement. The inverse problem of determining the sample parameter for one perfect experiment fully corresponding to the mathematical model may have a non-unique solution. Considering the results of several experiments as a vector, we arrive at an inverse problem which is well-posed if the frequency step is small enough. For a non-perfect multi-frequency experiment it is shown that the solution can be found by the least squares method. It converges to the solution of the perfect problem if the experimental data approach the theoretically predicted values for the unknown dielectric constant of the inclusion.

7. References

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