On the convergence of an approximate method for solving the problem filtration consolidation with a limiting gradient

Maria F. Pavlova, Elena V. Rung

Kazan Federal University, 35 Kremlyovskaya str., Kazan, 420008, Russian Federation

Abstract
A one-dimensional initial-boundary value problem modelling the process of joint motion of a viscoelastic porous medium and a liquid saturating the medium is considered. In the filtration theory, this process is called filtration consolidation. A finite element in spatial variable and time-implicit difference scheme is constructed. Its solvability is established, the convergence piecewise-constant filling of an approximate solution in the variable \( t \) to generalized solution of problem is proved.

Keywords
filtration, filtration consolidation, difference scheme, finite element method

1. Problem statement
An initial-boundary value problem is considered, which is described by the following system of nonlinear partial differential equations for the unknown functions \( u(x, t) \), \( p(x, t) \):

\[
- \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial t} \right) + \frac{\partial p}{\partial x} = f(x, t), \quad 0 < x < L, \ 0 < t < T,
\]

\[
\frac{\partial^2 u}{\partial x \partial t} - \frac{\partial}{\partial x} \left( g \left( \left| \frac{\partial p}{\partial x} \right| \right) \frac{\partial p}{\partial x} \right) = 0, \quad 0 < x < L, \ 0 < t < T.
\]

We assume that for \( t \in (0, T] \) the following boundary conditions are satisfied

\[
u(0, t) = 0, \quad \frac{\partial u}{\partial x}(L, t) + \frac{\partial^2 u}{\partial x \partial t}(L, t) = 0,
\]

\[
r(0, t) = r(L, t) = 0.
\]

The initial conditions are given as

\[
u(x, 0) = u_0(x), \quad r(x, 0) = r_0(x), \quad 0 \leq x \leq L.
\]

The problem (1)–(5) is of an applied nature: relations (1)–(5) can be used to describe an one-dimensional process of filtration consolidation with a limiting gradient (see, eg, [1]). In this case, \( p \) is liquid pressure in the pores, \( u \) is motion of the skeleton particles, \( f(x, t) \), \( g(\xi) \) are given functions.
The foundations of the theory of filtration consolidation were laid in such works as [2, 3, 4, 5]. In these works mathematical models of filtration consolidation were built, and studies of the models from the standpoint of continuum mechanics were carried out. A rigorous mathematical analysis of problems of filtration consolidation was carried out in [6, 7], where the solvability of these problems in the class of generalized functions is established. The works [8, 9, 10] are devoted to experimental study using numerical methods.

This paper considers the problem of filtration consolidation with limiting gradient in the case when the function \( g(\xi) = \begin{cases} 0, & |\xi| \leq \xi_0 \\ 1, & |\xi| > \xi_0 \end{cases} \)

In what follows, we assume that the functions \( g(\xi), f(x, t) \) satisfy the following conditions:

\( A_1 \). \( g(\xi), \xi \geq 0 \) is an absolutely continuous in \( \xi \), nonnegative, nondecreasing function and there exist \( \xi_0 \geq 0, \eta, \mu > 0 \), such that at \( \xi \geq \xi_0 \) the following inequality holds

\[ \eta (\xi - \xi_0) \leq g(|\xi|) \xi \leq \mu (\xi - \xi_0). \]  

\( A_2 \). The function \( f(x, t) \) is continuous at \((x, t) \in Q_T\), where \( Q_T = (0, L) \times (0, T) \).

Conditions (6) imposed on the function \( g \) mean that the filtration rate will be zero for small values of the gradient modulus.

2. Defining a generalized solution

Let \( \tilde{V} \) be the closure of smooth functions equal to zero at \( x = 0 \) in the norm of the space \( W_2^1(0, L) \), and let \( \tilde{V}_1 \) be the closure of smooth functions equal to zero on the boundary of the interval \([0, L]\), in the norm of the same space.

**Definition.** By a generalized solution to problem (1)–(5), we imply functions \((u, p)\), for which the following conditions hold:

\[ u \in W_2^1(0, T; \tilde{V}), \quad p \in L_2(0, T; \tilde{V}_1), \]

\[ u(x, 0) = u_0(x), \quad p(x, 0) = p_0(x) \quad \text{almost everywhere on} \quad x \in (0, L), \]

and for any functions \( v \in W_2^1(0, T; \tilde{V}), z \in L_2(0, T; \tilde{V}_1) \) the following equality is true:

\[
\int_0^T \int_0^L \left( \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial t} \right) \frac{\partial^2 v}{\partial x \partial t} - p \frac{\partial^2 v}{\partial x \partial t} + \frac{\partial^2 u}{\partial x \partial t} z^+ + g \left( \frac{\partial p}{\partial x} \right) \frac{\partial p}{\partial x} \frac{\partial z}{\partial x} \right) dx dt = \int_0^T \int_0^L f(x, t) \frac{\partial v}{\partial t} dx dt. \tag{7}
\]

3. The discrete problem

The problem (1)–(5) will be solved by the semi-discretization method in combination with the finite element method. Let us construct an arbitrary unequally-spaced grid on the interval \([0, L]\)

\[ \bar{\omega}_h = \{ x_0 = 0 < x_1 < ... < x_n = L \}. \]
Let $V^n$ and $V^n_1$ be finite-dimensional function spaces, continuous on the interval $[0, L]$, satisfying the conditions (3) and (4), respectively, and are polynomials of the first degree on each grid cell $\delta_i = [x_{i-1}, x_i], i = 1, 2, ..., n$. Let also $\omega_\tau = \{t = k\tau, 0 \leq k \leq M, M\tau = T\}$, $\omega_\tau = \omega_\tau \setminus \{0\}$.

**Definition.** By the approximate solution to the problem (1)–(5) constructed by the method of semi-discretization in combination with the finite element method, we imply the functions $(\hat{u}^n(t), \hat{p}^n(t))$ for which the following conditions hold:

$$\hat{u}^n(t) \in V^n, \quad \hat{p}^n(t) \in V^n_1 \quad \forall t \in \omega_\tau,$$

and for any functions $v^n \in V^n, z^n \in V^n_1$ the following equality is true

$$\int_0^L \left\{ \left( \frac{\partial \hat{u}_n}{\partial x} + \frac{\partial u_n}{\partial x} \right) \frac{\partial v^n}{\partial x} - \hat{p}_n \frac{\partial v^n}{\partial x} + \frac{\partial u_n}{\partial x} \frac{\partial z^n}{\partial x} + g \left( \frac{\partial \hat{p}_n}{\partial x} \right) \frac{\partial \hat{p}_n}{\partial x} \frac{\partial z^n}{\partial x} \right\} dx = \int_0^L \hat{f}(x, t) v^n dx. \quad (8)$$

Here $\hat{v} = v(t + \tau), \hat{v}_\tau = \frac{\hat{v} - v}{\tau}$.

**Theorem 1.** Approximate solution of the problem (1)–(5) exists.

**Proof.** Obviously, it suffices to establish the existence of $\hat{p}_n, \hat{u}_n$ satisfying (8), under the assumption that $p^n, u^n$ are known.

Since the choice of the functions $v^n, z^n$ is arbitrary, the equality (8) is equivalent to the following system

$$\int_0^L \left\{ \left( \frac{\partial \hat{u}_n}{\partial x} + \frac{\partial u_n}{\partial x} \right) \frac{\partial \hat{v}_n}{\partial x} - \hat{p}_n \frac{\partial \hat{v}_n}{\partial x} + \frac{\partial u_n}{\partial x} \frac{\partial \hat{z}_n}{\partial x} + g \left( \frac{\partial \hat{p}_n}{\partial x} \right) \frac{\partial \hat{p}_n}{\partial x} \frac{\partial \hat{z}_n}{\partial x} \right\} dx = \int_0^L \hat{f}(x, t) \hat{v}_n dx, \quad (9)$$

$$\int_0^L \left\{ \frac{\partial u_n}{\partial x} \frac{\partial \hat{z}_n}{\partial x} + g \left( \frac{\partial \hat{p}_n}{\partial x} \right) \frac{\partial \hat{p}_n}{\partial x} \frac{\partial \hat{z}_n}{\partial x} \right\} dx = 0. \quad (10)$$

We will look for approximate solutions in the form

$$\hat{u}_n = \sum_{k=0}^n \chi_k^{(1)} \varphi_k, \quad \hat{p}_n = \sum_{k=0}^n \chi_k^{(2)} \psi_k,$$

where $\varphi_k, \psi_k$ are linear on each element, continuous on the interval $[0, L]$ functions satisfying the conditions

$$\varphi_k(x_j) = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases} \quad k = 0, 1, ..., n, \quad \psi_k(x_j) = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases} \quad k = 0, 1, ..., n.$$

The unknown coefficients $\chi_k^{(i)}, k = 1, 2, ..., n, i = 1, 2$ are determined by the following system of equations:

$$\int_0^L \left\{ \left( \frac{\partial \hat{u}_n}{\partial x} + \frac{\partial u_n}{\partial x} \right) \frac{\partial (\varphi_k)_i}{\partial x} - \hat{p}_n \frac{\partial (\varphi_k)_i}{\partial x} \right\} dx = \int_0^L \hat{f}(x, t)(\varphi_k)_i dx, \quad (11)$$
\[ \int_0^L \left\{ \frac{\partial u^n}{\partial x} \psi_k + g \left( \frac{\partial \hat{p}^n}{\partial x} \right) \frac{\partial \hat{p}^n}{\partial x} \frac{\partial \psi_k}{\partial x} \right\} \, dx = 0. \] (12)

Let \( H : R^{2n} \rightarrow R^{2n} \) be a nonlinear operator such that the equation

\[ H(\xi) = 0 \]

is equivalent to system (11)–(12). Let us make sure that \( R^{2n} \) contains a sphere centered at zero of finite radius, on which

\[ (H(\xi), \xi)_{R^{2n}} \geq 0. \] (13)

We have

\[ (H(\xi), \xi)_{R^{2n}} = \int_0^L \left( \frac{\partial \hat{u}^n}{\partial x} + \frac{\partial u^n_i}{\partial x} \right) \frac{\partial u^n_i}{\partial x} \, dx + \int_0^L g \left( \frac{\partial \hat{p}^n}{\partial x} \right) \left( \frac{\partial \hat{p}^n}{\partial x} \right)^2 \, dx - \int_0^L f(x, t) \cdot u^n_i \, dx. \] (14)

The first term on the right-hand side of equality (14) can be transformed to the form:

\[ \int_0^L \left( \frac{\partial \hat{u}^n}{\partial x} + \frac{\partial u^n_i}{\partial x} \right) \frac{\partial u^n_i}{\partial x} \, dx = \left( \frac{1}{\tau} + \frac{1}{\tau^2} \right) \| \hat{u}^n \|_1^2 - \left( \frac{1}{\tau} + \frac{2}{\tau^2} \right) \int_0^L \frac{\partial \hat{u}^n}{\partial x} \frac{\partial u^n_i}{\partial x} \, dx + \frac{1}{\tau^2} \| u^n \|_1^2. \] (15)

Here \( \| v \|_1^2 = \int_0^L \left( \frac{\partial v}{\partial x} \right)^2 \, dx. \)

Using the Cauchy-Bunyakovsky inequality

\[ (x, y) \leq \delta \| x \|^2 + \frac{1}{4\delta} \| y \|^2, \] (16)

from (15) it is easy to obtain the following estimate

\[ \int_0^L \left( \frac{\partial \hat{u}^n}{\partial x} + \frac{\partial u^n_i}{\partial x} \right) \frac{\partial u^n_i}{\partial x} \, dx \geq \left( \frac{1}{\tau} + \frac{1}{\tau^2} - \delta \right) \| \hat{u}^n \|_1^2 - \frac{1}{4\delta} \left( \frac{1}{\tau} + \frac{2}{\tau^2} \right) \| u^n \|_1^2. \] (17)

Using (16) and inequality (6), for the second term in equality (14) we have

\[ \int_0^L g \left( \frac{\partial \hat{p}^n}{\partial x} \right) \left( \frac{\partial \hat{p}^n}{\partial x} \right)^2 \, dx \geq (\eta - \delta) \| \hat{p}^n \|_1^2 - \frac{\eta^2 \xi_0^2 L^2}{4\delta}. \]

To estimate the last term of (14), we use the boundedness of the function \( f(\xi) \), inequality (16), and the Friedrichs inequality. As a result, we obtain

\[ \left| \int_0^L \hat{f}(x, t) \cdot u^n_i \, dx \right| \leq \delta \| \hat{u}^n \|_1^2 + \frac{C^2 \xi_0^2 L^2}{4\delta \tau^2} + \frac{CCF}{\tau} \| u^n \|_1^2, \]
here $C_F$ is a constant of the Friedrichs inequality, $C$ is a constant such that
\[ |f(\xi, \zeta)| \leq C \quad \forall \xi \in [0, L], \forall \zeta \in [0, T]. \]

Substituting the estimates obtained in (14), we have
\[ (H(\xi), \zeta)_{R^2} \geq K(\delta) \left( \| \hat{u}^n \|_1^2 + \| \hat{p}^n \|_1^2 \right) - \overline{K}(\delta), \quad (18) \]
where
\[
\overline{K}(\delta) = \min \left\{ \left( \frac{1}{\tau} + \frac{1}{\tau^2} - 2\delta \right), \eta - \delta \right\},
\]
\[
\overline{K}(\delta) = \left( \frac{C C_F}{\tau} + \frac{1}{4\delta} \left( \frac{1}{\tau} + \frac{2}{\tau^2} \right)^2 \right) \| u^n \|_1^2 + \frac{C^2 C_F^2 L^2}{4\delta \tau^2} + \frac{\eta^2 \xi_0^2 L^2}{4\delta}.
\]

Let $\delta^*$ be a constant such that for all $0 < \delta \leq \delta^*$ the following inequality holds
\[
K(\delta) \geq \beta = \text{const} > 0,
\]
and $S \subset R^2$ be a sphere centered at zero at which the right-hand side of inequality (18) is non-negative.

Then, by the topological lemma ([11], p. 66), there is at least one solution to the system inside this sphere. The proof of Theorem 1 is complete.

**Lemma 1.** For the approximate solution (8), the following a priori estimates are valid
\[ \max_{t'} |u^n(t')|_1^2 \leq C, \quad \sum_{t=0}^{t'} \tau |p^n(t)|_1^2 \leq C, \quad (19) \]
\[ \sum_{t=0}^{t'-\tau} \tau |u^n(t)|_1^2 \leq C, \quad \sum_{t=0}^{t'} \tau \left( \frac{\partial u^n}{\partial x} \right)_{L_2(0, L)}^2 \leq C, \quad (20) \]
\[ \sum_{t=0}^{t'-\tau} \tau \left( \frac{\partial p^n}{\partial x} \right)_{L_2(0, L)}^2 \leq C. \quad (21) \]

**Proof.** Let us assume in (8) $v^n = u^n$, $z^n = p^n$ and obtain
\[ \int_0^L \left( \left( \frac{\partial \hat{u}^n}{\partial x} + \frac{\partial u^n}{\partial x} \right) \frac{\partial u^n}{\partial x} + g \left( \frac{\partial \hat{p}^n}{\partial x} \right) \left( \frac{\partial \hat{p}^n}{\partial x} \right)^2 \right) \, dx = \int_0^L f(x, t) u^n \, dx. \quad (22) \]

Note that
\[
\frac{\partial u^n}{\partial x} \frac{\partial u^n}{\partial x} = \frac{\partial u^n}{\partial x} \frac{\partial}{\partial x} \left( \frac{\hat{u}^n - u^n}{\tau} \right) = \frac{\partial u^n}{\partial x} \frac{1}{\tau} \left( \frac{\partial \hat{u}^n}{\partial x} - \frac{\partial u^n}{\partial x} \right) = \frac{1}{2} \left( \frac{\partial \hat{u}^n}{\partial x} \right)^2 - \frac{1}{2} \left( \frac{\partial u^n}{\partial x} \right)^2 + \frac{\tau^2}{2} \left( \frac{\partial u^n}{\partial x} \right)^2. \quad (23)
\]

We substitute equality (23) into (22), multiply by $\tau$ and sum the resulting relation over $t$ from 0 to $t' - \tau$ and obtain
\[
\frac{1}{2} |u^n(t')|_1^2 - \frac{1}{2} |u^n(0)|_1^2 + \frac{1}{2} \sum_{t=0}^{t'-\tau} \tau^2 |u^n(t)|_1^2 + \sum_{t=0}^{t'-\tau} \tau \left( \frac{\partial u^n}{\partial x} \right)_{L_2(0, L)}^2 +
\]
\[ + \sum_{t=0}^{t'=0} \int_0^L g \left( \left| \frac{\partial \hat{p}}{\partial x} \right| \right) \left( \frac{\partial \hat{p}}{\partial x} \right)^2 dx = \sum_{t=0}^{t'=0} \int_0^L f(x, t) \cdot u_t^i dx. \quad (24) \]

From (24), taking into account inequality (6), we have a priori estimates (19)–(20). Also, considering that

\[ g \left( \left| \frac{\partial \hat{p}^n}{\partial x} \right| \right) \frac{\partial \hat{p}^n}{\partial x} \leq \left| \frac{\partial \hat{p}^n}{\partial x} \right|_{L^2(0,L)} \]

we have estimate (21). The proof of Lemma 1 is complete.

**Lemma 2.** There exist functions \( u \in W^{1,2}_2(0, T; V) \), \( p \in L_2(0, T; \dot{V}_1) \) and sequences \( \{ \tau \}, \{ n \} \) such that at \( \tau \to 0, \ n \to \infty \)

\[ \Pi^+ u^n \to u, \quad \Pi^+ p^n \to p \quad \text{in} \quad L_2(0, T; \dot{V}), \quad \lim_{\tau \to 0, \ n \to \infty} \Pi^+ u^n \to \frac{\partial u}{\partial t} \quad \text{in} \quad L_2(0; L_2(0, L)), \quad \Pi^+ p^n \to p \quad \text{in} \quad L_2(0, T; \dot{V}_1). \quad (25) \]

Here \( \Pi^+ z \) is piecewise-constant filling of \( z \):

\[ \Pi^+ z(t) = \{ z(k\tau) : k\tau \leq t < (k+1)\tau \}. \]

The validity of statements (25)–(27) follows from a priori estimates (19)–(20) and the weak compactness of bounded sets in a reflexive Banach space. The proof of Lemma 2 is complete.

**Theorem 2.** Functions \( u, p \) satisfying relations (25)–(27) are a generalized solution to problem (1)–(5).

**Proof.** Let the functions \( u, p \) satisfy relations (25)–(27), it is required to prove that \( u, p \) satisfy identity (7). To do this, in (8) we put

\[ v^n(x, t) = \frac{1}{\tau} \int_t^{t+\tau} \tilde{v}^n(x, \xi) d\xi, \quad z^n(x, t) = \frac{1}{\tau} \int_t^{t+\tau} \tilde{z}^n(x, \xi) d\xi, \]

where \( \tilde{v}^n, \tilde{z}^n \) are functions from \( C^\infty(0, T; \dot{V}) \) and \( C^\infty(0, T; \dot{V}_1) \) respectively, such that \( \tilde{v}^n(x, T) = \tilde{z}^n(x, T) = 0 \). We multiply (8) by \( \tau \), sum over \( t \) from 0 to \( T - \tau \). The result, using the filling operator \( \Pi^+ \), can be written in the form

\[ \int_0^T \int_0^L \left\{ \left( \frac{\partial \Pi^+ u^n}{\partial x} + \frac{\partial \Pi^+ u^n}{\partial t} \right) \frac{\partial \Pi^+ v^n}{\partial x} - \Pi^+ p^n \frac{\partial \Pi^+ v^n}{\partial x} + \frac{\partial \Pi^+ u^n}{\partial x} \Pi^+ \tilde{z}^n + \right. \]

\[ \left. + g \left( \left| \frac{\partial \Pi^+ \tilde{p}^n}{\partial x} \right| \right) \frac{\partial \Pi^+ \tilde{p}^n}{\partial x} \frac{\partial \Pi^+ \tilde{z}^n}{\partial x} \right\} dx dt = \int_0^T \int_0^L f(x, t) \Pi^+ v^n dx dt. \quad (28) \]
From the boundedness of \( g \) and estimate (21) it follows that there exists a function \( \chi \) from the space \( L^2(0, T; L^2(0, L)) \) such that
\[
\| \frac{\partial \Pi^n \hat{p}}{\partial x} \| \frac{\partial \Pi^n \hat{p}}{\partial x} \to \chi \quad \text{in} \quad L^2(0, T; L^2(0, L)).
\] (29)

Taking into account (25)–(27) and (29) in equality (28), we pass to the limit in \( \tau \to 0 \) and \( n \to \infty \) and obtain
\[
\int_0^T \int_0^L \left\{ \left( \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial t} \right) \frac{\partial^2 v}{\partial x \partial t} - p \frac{\partial^2 v}{\partial x \partial t} + \frac{\partial^2 u}{\partial x \partial t} \frac{\partial z}{\partial x} \right\} dx dt =
= \int_0^T \int_0^L \left\{ g \left( \left( \frac{\partial p}{\partial x} \right) \right) \frac{\partial p}{\partial x} \right\} dx dt. \] (30)

Let us prove that \( \chi = g \left( \left( \frac{\partial p}{\partial x} \right) \right) \frac{\partial p}{\partial x} \). To do this, we use the monotonicity method. We write down the apparent inequality
\[
\sum_{t=0}^{T-\tau} \int_0^L \left( \frac{\partial u^n}{\partial x} - \frac{\partial v^n}{\partial x} \right) \left( \frac{\partial \hat{u}^n}{\partial x} - \frac{\partial \hat{v}^n}{\partial x} \right) dx \geq \frac{1}{2} [u^n(T) - v^n(T)]^2 \hat{u}^n - \frac{1}{2} [u^n(0) - v^n(0)]^2 \hat{v}^n + \frac{1}{2} [u_0 - v^n(x, 0)]^2,
\]
where \( v^n \) is an arbitrary smooth function \( v \in C^\infty(0, T; V^n) \). From this inequality and the monotonicity of the function \( g(\hat{\xi}) \) it follows that
\[
\sum_{t=0}^{T-\tau} \int_0^L \left( \frac{\partial u^n}{\partial x} - \frac{\partial v^n}{\partial x} \right) \left( \frac{\partial \hat{u}^n}{\partial x} - \frac{\partial \hat{v}^n}{\partial x} \right) dx +
+ \sum_{t=0}^{T-\tau} \int_0^L \left\{ g \left( \left( \frac{\partial p^n}{\partial x} \right) \right) \frac{\partial p^n}{\partial x} - g \left( \left( \frac{\partial z^n}{\partial x} \right) \right) \frac{\partial (\hat{p}^n - \hat{z}^n)}{\partial x} \right\} dx \geq -\frac{1}{2} [u_0 - v^n(x, 0)]^2.
\]
The last relation is equivalent to the following integral inequality
\[
I_{\tau, n} = \int_0^T \int_0^L \left( \frac{\partial \Pi^n \hat{u}^n}{\partial x} - \frac{\partial \Pi^n \hat{v}^n}{\partial x} \right) \frac{\partial \Pi^n (\hat{u}^n - \hat{z}^n)}{\partial x} dx dt +
+ \int_0^T \int_0^L g \left( \left( \frac{\partial \Pi^n \hat{p}^n}{\partial x} \right) \right) \frac{\partial \Pi^n \hat{p}^n}{\partial x} \frac{\partial \Pi^n (\hat{p}^n - \hat{z}^n)}{\partial x} dx dt -
- \int_0^T \int_0^L g \left( \left( \frac{\partial \Pi^n \hat{z}^n}{\partial x} \right) \right) \frac{\partial \Pi^n \hat{z}^n}{\partial x} \frac{\partial \Pi^n (\hat{p}^n - \hat{z}^n)}{\partial x} dx dt \geq -\frac{1}{2} [u_0 - v^n(x, 0)]^2. \] (31)
We represent $I_{r,n}$ as the sum $I = I^{(1)}_{r,n} + I^{(2)}_{r,n}$, where

$$I^{(1)}_{r,n} = \int_0^T \int_0^L \left\{ \frac{\partial \Pi^r u^n}{\partial x} \frac{\partial \Pi^r (\dot{u}^n - \dot{v}^n)}{\partial x} + \frac{\partial \Pi^r \dot{p}^n}{\partial x} \frac{\partial \Pi^r (\dot{p}^n - \dot{z}^n)}{\partial x} \right\} dx dt,$$

$$I^{(2)}_{r,n} = -\int_0^T \int_0^L \left\{ \frac{\partial \Pi^r v^n}{\partial x} \frac{\partial \Pi^r (\dot{u}^n - \dot{v}^n)}{\partial x} + \frac{\partial \Pi^r \dot{z}^n}{\partial x} \frac{\partial \Pi^r (\dot{p}^n - \dot{z}^n)}{\partial x} \right\} dx dt.$$

To transform the first relation $I^{(1)}_{r,n}$, we use equality (28) at $v^n = u^n - v^n$, $p^n = p^n - z^n$ and obtain

$$I^{(1)}_{r,n} = \int_0^T \int_0^L \left\{ -\frac{\partial \Pi^r u^n}{\partial x} \frac{\partial \Pi^r (u^n - v^n)}{\partial x} - \frac{\partial \Pi^r \dot{p}^n}{\partial x} \frac{\partial \Pi^r (p^n - z^n)}{\partial x} + \frac{\partial \Pi^r u^n}{\partial x} \frac{\partial \Pi^r \dot{z}^n}{\partial x} - \frac{\partial \Pi^r \dot{p}^n}{\partial x} \frac{\partial \Pi^r (p^n - z^n)}{\partial x} + \frac{\partial \Pi^r u^n}{\partial x} \frac{\partial \Pi^r v^n}{\partial x} \right\} dx dt. \quad (32)$$

In (32), we make the passage to the limit as $\tau \to 0$, $n \to \infty$, taking into account (25)–(27), (29). As a result, we obtain

$$I^{(1)}_{r,n} \to \int_0^T \int_0^L \left\{ -\frac{\partial^2 u}{\partial x \partial t} \frac{\partial^2 (u - v)}{\partial x \partial t} - \frac{\partial u}{\partial x \partial t} \frac{\partial v}{\partial x \partial t} + \frac{\partial^2 u}{\partial x \partial t} \frac{\partial z}{\partial x \partial t} - \frac{\partial u}{\partial x \partial t} \frac{\partial \Pi^r (u^n - v^n)}{\partial x} \frac{\partial \Pi^r \dot{z}^n}{\partial x} \right\} dx dt. \quad (33)$$

Using equality (30), the right-hand side of relation (33) takes the following form

$$I^{(1)}_{r,n} \to \int_0^T \int_0^L \left\{ \frac{\partial^2 u}{\partial x \partial t} \frac{\partial (u - v)}{\partial x} + \frac{\partial u}{\partial x \partial t} \frac{\partial \Pi^r (p - z)}{\partial x} \right\} dx dt. \quad (34)$$

Apparently, from (25)–(27), (29) for $\tau \to 0$, $n \to \infty$ we obtain

$$I^{(2)}_{r,n} \to -\int_0^T \int_0^L \left\{ \frac{\partial^2 v}{\partial x \partial t} \frac{\partial (u - v)}{\partial x} + \frac{\partial v}{\partial x \partial t} \frac{\partial \Pi^r (p - z)}{\partial x} \right\} dx dt. \quad (35)$$

Thus, it follows from the definition of $I_{r,n}$ that

$$\int_0^T \int_0^L \left\{ \frac{\partial^2 (u - v)}{\partial x \partial t} \frac{\partial (u - v)}{\partial x} + \left( \frac{\partial}{\partial x} \left| \frac{\partial z}{\partial x} \right| \frac{\partial (p - z)}{\partial x} \right) \right\} dx dt \geq -\frac{1}{2} \|u_0 - v(x, 0)\|^2. \quad (36)$$

In (36), we choose $v = u + \lambda w$, $z = p + \lambda q$, where $\lambda = \text{const} > 0$, and $w$, $q$ are arbitrary functions from $C^\infty(0, T; C^\infty(0, L))$, where $w(x, 0) = 0$ for $x \in (0, L)$. As a result, we obtain
\[
\lambda \int_0^T \int_0^L \left( \chi - g \left( \frac{|\partial(p + \lambda q)|}{\partial x} \right) \frac{\partial(p + \lambda q)}{\partial x} \right) \frac{\partial q}{\partial x} \, dx \, dt + \\
+ \lambda^2 \int_0^T \int_0^L \frac{\partial^2 w}{\partial x \partial t} \frac{\partial w}{\partial x} \, dx \, dt \geq -\frac{\lambda}{2} || w(x, 0) ||_1^2 = 0. \quad (37)
\]

We divide inequality (37) by \( \lambda \) and pass to the limit as \( \lambda \to 0 \), we obtain
\[
\int_0^T \int_0^L \left( \chi - g \left( \frac{|\partial p|}{\partial x} \right) \frac{\partial p}{\partial x} \right) \frac{\partial q}{\partial x} \, dx \, dt \geq 0. \quad (38)
\]

Since \( q \) is an arbitrary function, the inequality holds at \( q = v \) and \( q = -v \), where \( v \in L_2(0, T; W^2_1(0, L)) \) is an arbitrary function; therefore, we have
\[
\chi = g \left( \frac{|\partial p|}{\partial x} \right) \frac{\partial p}{\partial x}.
\]

The proof of theorem 2 is complete.

References