DECOMPOSITION SCHEMES FOR SYMMETRIC n-ARY BANDS

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ABSTRACT

We extend the classical (strong) semilattice decomposition scheme of certain classes of semigroups to the class of idempotent symmetric n-ary semigroups (i.e. symmetric n-ary bands) where $n \geq 2$ is an integer. More precisely, we show that these semigroups are exactly the strong n-ary semilattices of n-ary extensions of Abelian groups whose exponents divide $n-1$. We then use this main result to obtain necessary and sufficient conditions for a symmetric n-ary band to be reducible to a semigroup.

Keywords Semigroup · idempotency · semilattice decomposition · reducibility

1 Introduction

Semigroups are ubiquitous and have numerous applications both in theoretical and applied mathematics. An extensive study of these structures began in the second half of the 20th century (see the pioneering works [2] and [19], or the textbooks [3, 10, 12, 20, 21] and references therein). In the algebraic analysis of semigroups, it soon became clear that it was useful to obtain a *decomposition scheme* of the semigroup under consideration into subsemigroups that are easier to describe or have additional properties (e.g. being groups), but also to be able to build a semigroup by combining given subsemigroups in a suitable way, that is, to use a *composition scheme* for semigroups.

Several classes of semigroups have the remarkable property to admit such composition/decomposition schemes; see, e.g., Krohn-Rhodes theorem for finite semigroups and finite automata [14]. A noteworthy example of such a scheme is given by strong semilattice decompositions of certain classes of bands[1]. In this paper we generalize these strong semilattice decompositions to structures with higher arities, defined as follows.

An n-ary operation $F\colon X^n \to X$ (where $n \geq 2$ is an integer and X is a non-empty set) is *associative* if

$$F(x_1,\ldots,x_{i-1},F(x_i,\ldots,x_{i+n-1}),x_{i+n},\ldots,x_{2n-1}) = F(x_1,\ldots,x_i,F(x_{i+1},\ldots,x_{i+n}),x_{i+n+1},\ldots,x_{2n-1}), \quad (1)$$

for all $x_1,\ldots,x_{2n-1} \in X$ and all $1 \leq i \leq n-1$. If F is an n-ary associative operation on X, then (X,F) is an *n-ary semigroup*. These n-ary structures, first studied in [9] and [22], have applications in different fields such as automata theory (see, e.g., [11]), coding theory, and cryptology (see, e.g., [16, 17]).

The classical definitions of symmetry and idempotency can also be extended to n-ary operations as follows: F is *idempotent* if $F(x,\ldots,x) = x$ for every $x \in X$ and F is *symmetric* (or *commutative*) if F is invariant under the action of permutations.

Many examples of n-ary semigroups are obtained by extending binary semigroups: if $G\colon X^2 \to X$ is an associative operation, then we can define a sequence of operations inductively by setting $G^1 = G$, and

$$G^m(x_1,\ldots,x_{m+1}) = G^{m-1}(x_1,\ldots,x_{m-1},G(x_m,x_{m+1})), \qquad m \geq 2.$$

Setting $F = G^{n-1}$, it is straightforward to see that the pair (X,F) is indeed an n-ary semigroup. It is said to be the *n-ary extension* of (X,G) and we say that (X,F) is *reducible* to (X,G)[2]. However, not every n-ary semigroup

[1] A band is a semigroup (X,G) where X is a nonempty set and the binary associative operation $G\colon X^2 \to X$ satisfies $G(x,x) = x$ for every $x \in X$; see, e.g., [12] for more details.

[2] We also say that F is the n-ary extension of G or that F is reducible to G or even that G is a binary reduction of F.

is the n-ary extension of a binary semigroup. For instance, the ternary associative operation F defined on $\mathbb{R}^3$ by $F(x_1,x_2,x_3) = x_1 - x_2 + x_3$ is not reducible to any binary associative operation. The problem of reducibility was considered recently in [13, 18] for n-ary semigroups endowed with additional structures and in [1, 4, 6] for the class of quasitrivial n-ary semigroups[3]. These are n-ary semigroups (X,F) that preserve all unary relations, i.e., such that $F(x_1,\ldots,x_n) \in \{x_1,\ldots,x_n\}$ for all $x_1,\ldots,x_n \in X$. It was shown [4] that all quasitrivial n-ary semigroups are reducible. Then in [5], the authors relaxed the quasitriviality condition by considering operations whose restrictions on certain subsets of the domain are quasitrivial. It turns out that these operations are also reducible.

In this work we study the class of symmetric (or commutative) n-ary bands, that is, symmetric idempotent n-ary semigroups. Typical examples of symmetric n-ary bands are given by n-ary extensions of semilattices and n-ary extensions of Abelian groups whose exponents divide $n-1$. Both classes of examples will play a central role in our constructions. However, as shown in the following examples, not every symmetric n-ary band is obtained in this way.

Example 1.1. (a) We consider the set $X = \{1,2,3,4\}$ and we define the symmetric ternary operation $F_1\colon X^3 \to X$ by its level sets given (up to permutations) by $F_1^{-1}(\{1\}) = \{(1,1,1)\}$, $F_1^{-1}(\{2\}) = \{(2,2,2)\}$, $F_1^{-1}(\{3\}) = \{(1,1,2),(1,1,3),(1,2,4),(1,3,4),(2,2,3),(2,3,3),(2,4,4),(3,3,3),(3,4,4)\}$. Then $F_1^{-1}(\{4\})$ is made up of all the remaining elements of X^3. This operation defines a symmetric ternary band and is not reducible to any binary operation.

(b) We consider the set $X = \{1,2,3\}$ and we define the symmetric ternary operation $F_2\colon X^3 \to X$ again by its level sets given (up to permutations) by $F_2^{-1}(\{1\}) = \{(1,1,1)\}$, $F_2^{-1}(\{2\}) = \{(1,1,2),(1,2,2),(1,3,3),(2,2,2),(2,3,3)\}$, and $F_2^{-1}(\{3\}) = \{(1,1,3),(1,2,3),(2,2,3),(3,3,3)\}$. This operation defines a symmetric ternary band. It turns out that it is reducible to a binary operation on X.

In the next section we define the n-ary counterpart of the classical strong semilattice (de)composition for semigroups (namely the strong n-ary semilattice decomposition). We show that it enables us to compose n-ary semigroups: every strong n-ary semilattice of n-ary semigroups is an n-ary semigroup (see Proposition 2.2). Then in Section 3, we provide a constructive description of the class of symmetric n-ary bands, that is, we show that the symmetric n-ary bands are exactly the strong n-ary semilattices of n-ary extensions of Abelian groups whose exponents divide $n-1$ (see Theorem 3.12). In the final section, we give a reducibility criterion for symmetric n-ary bands based on their strong n-ary semilattice decomposition (see Proposition 4.3). Also, Example 1.1 shows how these constructions enable us to build and analyze examples of symmetric n-ary bands. Almost all the definitions and results in this work stem from [7], where the reader may find their proofs and alternative developments as well.

2 Strong n-ary semilattices of n-ary semigroups

Throughout this work, we consider a nonempty set X and an integer $n \geq 2$. Recall that (X,F) is said to be an *n-ary groupoid* whenever $F\colon X^n \to X$ is an n-ary operation. Moreover, if F is associative (i.e., satisfies (1)), then (X,F) is said to be an n-ary semigroup. The concepts of homomorphims and isomorphisms of n-ary groupoids and n-ary semigroups are defined as usual.

Recall that $e \in X$ is said to be a *neutral element* for $F\colon X^n \to X$ if

$$F((k-1)\cdot e, x, (n-k)\cdot e) \;=\; x, \qquad x \in X,\; k \in \{1,\ldots,n\},$$

where, for any $k \in \{0,\ldots,n\}$ and any $x \in X$, the notation $k\cdot x$ stands for the k-tuple $x,\ldots,x$ (for instance $F(3\cdot x, 0\cdot y, 2\cdot z) = F(x,x,x,z,z)$).

In [8, Lemma 1], it was proved that any associative operation $F\colon X^n \to X$ having a neutral element e is reducible to an associative binary operation $G_e\colon X^2 \to X$ defined by

$$G_e(x,y) \;=\; F(x,(n-2)\cdot e, y), \qquad x,y \in X. \tag{2}$$

Finally, recall that an equivalence relation $\sim$ on X is said to be a *congruence* for $F\colon X^n \to X$ (or on (X,F)) if it is compatible with F, that is, if $F(x_1,\ldots,x_n) \sim F(y_1,\ldots,y_n)$ for any $x_1,\ldots,x_n,y_1,\ldots,y_n \in X$ such that $x_i \sim y_i$ for all $i \in \{1,\ldots,n\}$. We denote by $[x]_\sim$ (or $[x]$ when there is no risk of confusion) the equivalence class of x for $\sim$ and by $\tilde{F}$ the map induced by F on $X/{\sim}$ defined by

$$\tilde{F}([x_1]_\sim,\ldots,[x_n]_\sim) = [F(x_1,\ldots,x_n)]_\sim, \quad \forall x_1,\ldots,x_n \in X.$$

[3] For $n = 2$, the quasitrivial semigroups were described by Länger [15].

We say that a congruence $\sim$ on an n-ary groupoid (X,F) is an *n-ary semilattice congruence* if $(X/\sim,\tilde{F})$ is an n-ary semilattice[4].

Now, let us extend the well-known concept of semilattice of semigroups to n-ary semigroups. Let $(Y,\wedge)$ be a semilattice and let $\{(X_\alpha,F_\alpha)\colon \alpha\in Y\}$ be a set of n-ary semigroups such that $X_\alpha\cap X_\beta=\varnothing$ for any $\alpha\neq\beta$. We say that an n-ary groupoid (X,F) is an *n-ary semilattice $(Y,\wedge^{n-1})$ of n-ary semigroups (X_α,F_α)* if $X=\bigcup_{\alpha\in Y}X_\alpha$, $F|_{X_\alpha^n}=F_\alpha$ for every $\alpha\in Y$, and

$$F(X_{\alpha_1}\times\cdots\times X_{\alpha_n})\subseteq X_{\alpha_1\wedge\cdots\wedge\alpha_n},\qquad \alpha_1,\ldots,\alpha_n\in Y. \tag{3}$$

In this case we write $(X,F)=((Y,\wedge^{n-1});(X_\alpha,F_\alpha))$ and we simply say that (X,F) is an n-ary semilattice of n-ary semigroups.

Actually, any decomposition of an n-ary semigroup (X,F) as an n-ary semilattice of n-ary semigroups is associated with an n-ary semilattice congruence on (X,F); see, e.g., [12] for the binary counterpart of this result.

The fact that an n-ary groupoid is an n-ary semilattice of n-ary semigroups is not sufficient to ensure that it is an n-ary semigroup. We need to introduce a generalization of the strong semilattice decomposition. This is done in the following definition.

Definition 2.1. Let $(X,F)=((Y,\wedge^{n-1});(X_\alpha,F_\alpha))$ be an n-ary semilattice of n-ary semigroups. Suppose that for any $\alpha,\beta\in Y$ such that $\alpha\geq\beta$ there is a homomorphism $\varphi_{\alpha,\beta}\colon X_\alpha\to X_\beta$ such that the following conditions hold.

(a) The map $\varphi_{\alpha,\alpha}$ is the identity on X_α.

(b) For any $\alpha,\beta,\gamma\in Y$ such that $\alpha\geq\beta\geq\gamma$ we have $\varphi_{\beta,\gamma}\circ\varphi_{\alpha,\beta}=\varphi_{\alpha,\gamma}$.

(c) For any $(x_1,\ldots,x_n)\in X_{\alpha_1}\times\cdots\times X_{\alpha_n}$ we have
$$F(x_1,\ldots,x_n)=F_{\alpha_1\wedge\cdots\wedge\alpha_n}(\varphi_{\alpha_1,\alpha_1\wedge\cdots\wedge\alpha_n}(x_1),\ldots,\varphi_{\alpha_n,\alpha_1\wedge\cdots\wedge\alpha_n}(x_n)).$$

Then (X,F) is said to be a *strong n-ary semilattice $(Y,\wedge^{n-1})$ of n-ary semigroups (X_α,F_α)*. In this case we write $(X,F)=((Y,\wedge^{n-1});(X_\alpha,F_\alpha);\varphi_{\alpha,\beta})$ and we also say that (X,F) is a strong n-ary semilattice of n-ary semigroups.

This definition enables us to obtain the main result concerning the composition of n-ary semigroups, which is important on its own, but also in the next sections.

Proposition 2.2. *If (X,F) is a strong n-ary semilattice of n-ary semigroups, then it is an n-ary semigroup.*

3 The structure theorem

Throughout this section, we consider a symmetric n-ary band (X,F). We associate with it a family of unary operations and study their most important properties.

Definition 3.1. For every $x\in X$, we define the operation $\ell_x^F\colon X\to X$ by
$$\ell_x^F(y)=F((n-1)\cdot x,y),\qquad y\in X.$$

When there is no risk of confusion, we also denote this operation by ℓ_x. We now study elementary properties of this operation.

Example 3.2. For the structures presented in Example 1.1, these maps are given in the following tables.

y	1	2	3	4
$\ell_1(y)$	1	3	3	4
$\ell_2(y)$	4	2	3	4
$\ell_3(y)$	4	3	3	4
$\ell_4(y)$	4	3	3	4

y	1	2	3
$\ell_1(y)$	1	2	3
$\ell_2(y)$	2	2	3
$\ell_3(y)$	2	2	3

Proposition 3.3. *The pair $(\{\ell_x\colon x\in X\},\circ)$ is a semilattice.*

We also observe that the pair (X,B) where B is defined by $B(x,y)=\ell_x(y)$ for all $x,y\in X$ is a band. For instance, the tables in Example 3.2 are the operation tables of the corresponding binary operations B. We will not elaborate on this in the present work but refer the reader to [7] for more details.

[4]We say that the n-ary extension of a semilattice is an *n-ary semilattice*.

The semilattice defined in Proposition 3.3 can be extended to define a symmetric n-ary band. The following result establishes a tight relation between (X,F) and this n-ary band.

Proposition 3.4. *For every $x_1,\dots,x_n \in X$ we have*

$$\ell_{F(x_1,\dots,x_n)} = \ell_{x_1} \circ \cdots \circ \ell_{x_n},$$

that is, the map $\ell\colon (X,F) \to (\{\ell_x : x \in X\}, \circ^{n-1})$ defined by $\ell(x) = \ell_x$ is a homomorphism.

The map ℓ defined in the previous proposition enables us to characterize the reducibility of a symmetric n-ary band to a symmetric (binary) band, i.e., a semilattice.

Proposition 3.5. *Let (X,F) be a symmetric n-ary band. The following assertions are equivalent.*

(i) The map ℓ is injective.

(ii) The n-ary band (X,F) is isomorphic to $(\{\ell_x : x \in X\}, \circ^{n-1})$.

(iii) The n-ary band (X,F) is an n-ary semilattice.

Also, the map ℓ enables us to characterize those symmetric n-ary bands that are reducible to Abelian groups. Recall that a group $(X,*)$ with neutral element e has *bounded exponent* if there exists an integer $m \geq 1$ such that the m-fold product $x * \cdots * x$ is equal to e for any $x \in X$. In that case, the *exponent* of the group is the smallest integer $m \geq 1$ having this property. Using the characterization of Abelian groups having bounded exponent given by Prüfer and Baer (see [23]), it is straighforward to see that the exponent of an Abelian group divides $n-1$ if and only if the group is a direct sum of cyclic groups whose orders divide $n-1$.

Proposition 3.6. *Let (X,F) be a symmetric n-ary band. The following conditions are equivalent.*

(i) The map ℓ is constant (i.e., ℓ_x is the identity map on X for any $x \in X$).

(ii) The n-ary band (X,F) is the n-ary extension of a group $(X,*)$ (and in particular $(X,*)$ is Abelian and its exponent divides $n-1$).

Now, when the map ℓ associated with F is not injective, it is natural to consider a quotient, and identify the elements of X that have the same image by ℓ. In this context, we have the following result.

Proposition 3.7. *The binary relation $\sim$ on X defined by*

$$x \sim y \quad \Leftrightarrow \quad \ell_x = \ell_y, \qquad x,y \in X,$$

is an n-ary semilattice congruence on (X,F).

Example 3.8. For the structure presented in Example 1.1(a), we have $\ell_3 = \ell_4$ so $[1]_\sim = \{1\}$, $[2]_\sim = \{2\}$, and $[3]_\sim = \{3,4\}$. The binary reduction of $(X/{\sim}, \tilde{F}_1)$ is a semilattice whose Hasse diagram is given in Figure 1 (left). For instance, we have

$$[1]_\sim \wedge [2]_\sim = \tilde{F}_1([1]_\sim,[1]_\sim,[2]_\sim) = [F_1(1,1,2)]_\sim = [3]_\sim.$$

Now, for the structure presented in Example 1.1(b), we only have $\ell_2 = \ell_3$ so $[1]_\sim = \{1\}$ and $[2]_\sim = \{2,3\}$. We see that the binary reduction of $(X/{\sim}, \tilde{F}_2)$ is a semilattice whose Hasse diagram is given in Figure 1 (right).

$$\begin{tikzcd}[1] \arrow[dr, dash] & & [2] \arrow[dl, dash] \\ & [3] & \end{tikzcd} \qquad\qquad \begin{tikzcd}[1] \arrow[d, dash] \\ [2] \end{tikzcd}$$

Figure 1: Hasse diagrams of the binary reductions of $(X/{\sim}, \tilde{F}_1)$ (left) and $(X/{\sim}, \tilde{F}_2)$ (right)

Since $\sim$ is a congruence for F, this operation restricts to each equivalence class. It is then natural to study the most important properties of this restriction. Using Proposition 3.6, we directly obtain the following result.

Proposition 3.9. *For any $x \in X$, $([x]_\sim, F|_{[x]_\sim^n})$ is the n-ary extension of an Abelian group whose exponent divides $n-1$.*

Example 3.10. For both structures presented in Example 1.1, the restrictions of F_1 and F_2 to $[3]_\sim^3$ and $[2]_\sim^3$, respectively, are isomorphic to the ternary extension of $(\mathbb{Z}_2,+)$.

The congruence $\sim$ enabled us to decompose X as a n-ary semilattice of n-ary semigroups. In order to obtain a strong n-ary semilattice decomposition, we still need to define a suitable family of homomorphisms.

Proposition 3.11. *For every $x, y \in X$ such that $[x]_\sim \geq [y]_\sim$, the map $\varphi_{[x]_\sim,[y]_\sim} = \ell_y|_{[x]_\sim}$ is a homomorphism from $([x]_\sim, F|_{[x]_\sim^n})$ to $([y]_\sim, F|_{[y]_\sim^n})$.*

We can now state our main structure theorem for symmetric n-ary bands.

Theorem 3.12. *An n-ary groupoid (X, F) is a symmetric n-ary band if and only if it is a strong n-ary semilattice of n-ary extensions of Abelian groups whose exponents divide $n-1$.*

As a direct application of this theorem, we obtain that a symmetric n-ary band is an n-ary group[5] if and only if it is the n-ary extension of an Abelian group whose exponent divides $n-1$.

In view of the main result, in order to build symmetric n-ary bands, we have to consider Abelian groups whose exponents divide $n-1$, and build homomorphisms between the n-ary extensions of such groups. These homomorphisms are described in the next result.

Proposition 3.13. *Let $(X_1, *_1)$ and $(X_2, *_2)$ be two Abelian groups whose exponents divide $n-1$ and denote by F_1 and F_2 the n-ary extensions of $*_1$ and $*_2$, respectively. For every group homomorphism $\psi\colon X_1 \to X_2$ and every $g_2 \in X_2$, the map $h\colon X_1 \to X_2$ defined by*

$$h(x) = g_2 *_2 \psi(x), \qquad x \in X_1,$$

is a homomorphism of n-ary semigroups.

Conversely, every homomorphism from (X_1, F_1) to (X_2, F_2) is obtained in this way.

4 Reducibility of symmetric n-ary bands

In this section, we use Theorem 3.12 in order to analyze the reducibility problem for symmetric n-ary bands. We thus consider a symmetric n-ary band (X, F).

Proposition 4.1. *If F is reducible to an associative operation $G\colon X^2 \to X$, then the following assertions hold.*

(i) G is surjective and symmetric;

(ii) The n-ary semilattice congruence $\sim$ associated with F is a binary semilattice congruence for G.

It follows from Proposition 4.1 that if F is reducible to G, then G induces an operation $G|_{[x]_\sim^2}$ on every equivalence class $[x]_\sim$ of $X/\sim$. This operation is a binary reduction of $F|_{[x]_\sim^n}$. Therefore, it is natural to study the properties of the reductions of such operations. This is performed in the following result.

Proposition 4.2. *If (X, F) is the n-ary extension of an Abelian group (X, G_1) whose exponent divides $n-1$, then every reduction (X, G_2) of F is a group that is isomorphic to (X, G_1). Moreover, all the reductions of (X, F) are obtained by using (2) with any element e of X.*

We are now able to analyze the reducibility of symmetric n-ary bands.

Proposition 4.3. *A symmetric n-ary band $(X, F) = ((Y, \wedge^{n-1}); (X_\alpha, F_\alpha); \varphi_{\alpha,\beta})$ is reducible to a semigroup if and only if there exists a map $e\colon Y \to X$ such that*

(i) For every $\alpha \in Y$, $e(\alpha) = e_\alpha$ belongs to X_α;

(ii) For every $\alpha, \beta \in Y$ such that $\alpha \geq \beta$, we have $\varphi_{\alpha,\beta}(e_\alpha) = e_\beta$.

Moreover, when (X, F) is reducible to a semigroup, a reduction is given by the semigroup decomposed as $((Y, \wedge^{n-1}); (X_\alpha, G_\alpha); \varphi_{\alpha,\beta})$, where G_α is the reduction of F_α with respect to e_α.

Example 4.4. For the structures of Example 1.1(a) and (b), respectively, the only non obvious homomorphisms are given by $\varphi_{[1]_\sim,[3]_\sim} = \ell_3|_{[1]_\sim}$, $\varphi_{[2]_\sim,[3]_\sim} = \ell_3|_{[2]_\sim}$, and $\varphi_{[1]_\sim,[2]_\sim} = \ell_2|_{[1]_\sim}$, respectively.

- For (X, F_1), we have $\varphi_{[1]_\sim,[3]_\sim}(1) = \ell_3(1) = 4$ and $\varphi_{[2]_\sim,[3]_\sim}(2) = \ell_3(2) = 3$.
- For (X, F_2), we have $\varphi_{[1]_\sim,[2]_\sim}(1) = \ell_2(1) = 2$.

[5] Recall that an n-ary group is an n-ary semigroup (X, F) such that for any $i \in \{1, \ldots, n\}$ and any $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y \in X$ there exists a unique $z \in X$ such that $F(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n) = y$.

By Proposition 3.13 these are homomorphisms from the ternary extension of the trivial group to the ternary extension of $(\mathbb{Z}_2, +)$. It is easy to see that (X, F_1) and (X, F_2), respectively, are the strong ternary semilattices associated with the semilattices whose Hasse diagrams are depicted in Figure 1 (left) and (right), respectively, and the ternary extensions of groups and homomorphisms given here. It follows from Theorem 3.12 that (X, F_1) and (X, F_2) are symmetric ternary bands. Finally, we can use Proposition 4.3 to analyze the reducibility problem for (X, F_1) and (X, F_2).

1. For (X, F_1) we must have $e([1]_\sim) = 1$ and $e([2]_\sim) = 2$. Then we must have $e([3]_\sim) = \varphi_{[1]_\sim,[3]_\sim}(1) = 4$ but also $e([3]_\sim) = \varphi_{[2]_\sim,[3]_\sim}(2) = 3$, a contradiction. So (X, F_1) is not reducible to a semigroup.

2. For (X, F_2) the map e defined by $e([1]_\sim) = 1$ and $e([2]_\sim) = 2$ satisfies the conditions of Proposition 4.3 and so (X, F_2) is reducible to a semigroup (X, G). The operation table of G is given below.

G	1	2	3
1	1	2	3
2	2	2	3
3	3	3	2

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