Numerical Solution of the Crack Problem by the Weighted FEM

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Abstract

In present paper, crack problem in rectangle is considered. Solution of the problem is defined as R_ν-generalized one in special weighted Sobolev set. For calculation of approximate solution, the scheme of the weighted finite element method (FEM) is constructed. Comparison of our method with the classic FEM on model problem is performed. We confirmed experimentally theoretical estimate $O(h)$ of the rate of convergence for constructed method in the norm of the weighted Sobolev space and in weighted energy norm.

Keywords 1

Crack problem, R_ν-generalized solution, angle singularity, weighted FEM

1. Introduction

Two-dimensional Lame equations are usually used as a mathematical model for the crack problem. This problem is one of so-called problems with singularity which arises from presence of reentrant angle of 360° (crack) on the boundary. In this case, stresses in the around crack tip become unbounded, and displacements belong to $W_2^{3/2-\varepsilon}(\Omega)$, when the Dirichlet or Neumann conditions are applied on both sides of crack, and $W_2^{5/4-\varepsilon}(\Omega)$ in case of conditions of different types, ε is an arbitrary small real number. As a result, classic FEM for this problem has convergence rate of only $O(h^{1/2})$ and $O(h^{1/4})$ respectively in the norm of the space $W_2^1(\Omega)$ as well as in energy norm.

In papers [1-5] for elliptic problems with singularity caused by degeneration of input data it was suggested to define the solution as R_ν-generalized one in weighted Sobolev spaces or sets, properties of R_ν-generalized solution and questions about its existence and uniqueness were deeply studied as well as properties of weighted Sobolev spaces and sets. For calculation of approximate solution for such problems, new numerical methods on the base of notion of R_ν-generalized solution were created and investigated [6-8]. In the sequel, idea of R_ν-generalized solution was spread to other singular boundary value problems in electrodynamics, hydrodynamics, theory of elasticity. New numerical methods for these problems built on the notion of R_ν-generalized solution showed high accuracy and convergence rate of approximate solution to the exact one competing with known inflexible and more complicated specialized single-purpose methods [9-13].

For determination of approximate solution to the crack problem with high accuracy and convergence rate $O(h)$ in the norm of the weighted Sobolev space and in weighted energy norm, in [14] it was suggested to define solution as R_ν-generalized one; on this base, the scheme of the weighted FEM was developed and detailed numerical investigation of the model problem was performed.

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In present work we state the crack problem, introduce the notion R_ν -generalized solution, briefly describe the weighted FEM and bring some numerical results for model problem concerning experimental confirmation of the theoretical estimates of the convergence rate and absolute error in the mesh nodes.

2. Problem statement. R_ν -generalized solution

In two-dimensional domain Ω with boundary $\partial\Omega$ and crack $\Gamma_c \in \partial\Omega$ we consider boundary value problem of elasticity stated in displacements $\mathbf{u} = (u_1, u_2)$ (crack problem):

$$-(2\mathbf{div}(\mu\varepsilon(\mathbf{u})) + \nabla(\lambda\,\mathrm{div}\,\mathbf{u})) = \mathbf{f}, \quad x \in \Omega, \tag{1}$$

$$\mathbf{u} = \mathbf{q}, \quad x \in \partial\Omega. \tag{2}$$

Let Ω' be a closure of δ -neighborhood of the point $(0,0)$ in $\bar{\Omega}$: $\Omega' = \{x \in \bar{\Omega} : \sqrt{x_1^2 + x_2^2} \le \delta \ll 1\}$. In Ω' we define weight function $\rho(x)$ as a distance to the point $(0,0)$, and we extend $\rho(x)$ to the rest of $\bar{\Omega}$ as a constant δ. Using weight function $\rho(x)$, we introduce weighted spaces $L_{2,\nu}(\Omega)$, $W^1_{2,\nu}(\Omega)$ and sets $W^1_{2,\alpha}(\Omega,\delta)$, $\overset{0}{W}{}^1_{2,\alpha}(\Omega,\delta)$, $W^{1/2}_{2,\alpha}(\partial\Omega,\delta)$ (see [14]). Vector analogues for these spaces and sets we designate with bold letters.

Now we introduce bilinear and linear forms, respectively:

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega 2\mu\varepsilon(\mathbf{u}) : \varepsilon(\rho^{2\alpha}\mathbf{v}) + \lambda\,\mathrm{div}\,\mathbf{u}\,\mathrm{div}(\rho^{2\alpha}\mathbf{v})dx,$$

$$l(\mathbf{v}) = \int_\Omega \rho^{2\alpha}\mathbf{f}\cdot\mathbf{v}dx.$$

We assume that for some real $\beta \ge 0$ the following inclusions are valid:

$$\mathbf{f} \in L_{2,\beta}(\Omega,\delta), \quad \mathbf{q} \in \mathbf{W}^{1/2}_{2,\beta}(\partial\Omega,\delta). \tag{3}$$

Vector-function $\mathbf{u}_\nu = (u_{\nu,1}, u_{\nu,2}) \in \mathbf{W}^1_{2,\nu}(\Omega,\delta)$ is called R_ν -generalized solution to the problem (1), (2), if almost everywhere on $\partial\Omega$ boundary conditions (2) are met, and for any vector-function $\mathbf{v} = (\mathrm{v}_1, \mathrm{v}_2)$ from $\overset{0}{\mathbf{W}}{}^1_{2,\alpha}(\Omega,\delta)$ the following integral identity is valid:

$$a(\mathbf{u}_\nu, \mathbf{v}) = l(\mathbf{v})$$

where $\nu \ge \beta$ is fixed.

Existence and uniqueness of R_ν -generalized solution to the problem (1), (2) is stated in [15].

Remark. In contrast to definition of weak solution, in the case of R_ν -generalized one bilinear and linear forms contains weight function in some non-negative power as a factor. This function subdues singularity of the solution and provide integrals convergence. This allows us to construct numerical method on meshes without refining toward singularity point with convergence rate $O(h)$ of approximate solution to the exact one in the norm of weighted Sobolev space and in the weighted energy norm.

3. Weighted finite element method

Scheme of the weighted FEM is constructed in [14]. Here we briefly describe this scheme.

Assume that Ω is a rectangle with a crack, $\Omega = (-0.7, 0.3) \times [-1,1] \setminus [0, 0.3] \times \{0\}$, $\Gamma_c = [0, 0.3] \times \{0\}$. In Ω we introduce quasiuniform mesh coordinated with crack and designate mesh parameter as h. Mesh nodes are P_i, $i = 1,...,N$. For each of them we introduce weighted basis function of the form $\psi_i = \rho^{\nu^*}(x)\varphi_i(x)$, $i = 1,...,N$, where $\nu^* \in R$ is a power of weight function, $\varphi_i(x)$ is standard linear basis function of the FEM.

We designate V^h linear span of all basis functions, and $\overset{0}{V}{}^h$ is a linear span of only internal nodes P_i, $i=1,...,n$.

We say that the function $\mathbf{u}_\nu^h=(u_{\nu,1}^h,u_{\nu,2}^h)$, $u_{\nu,i}^h\in V^h$, $i=1,2$ is an approximate R_ν-generalized solution to the problem (1), (2), if $\mathbf{u}_\nu^h$ satisfy boundary conditions (2), and the integral identity

$$a(\mathbf{u}_\nu^h,\mathbf{v}^h)=l(\mathbf{v}^h)$$

holds for any vector-function $\mathbf{v}^h=(\mathrm{v}_1^h,\mathrm{v}_2^h)$, $\mathrm{v}_i^h\in\overset{0}{V}{}^h$, $i=1,2$.

Components of approximate R_ν-generalized solution we expand in basis with unknown coefficients d_i, $i=1,...,2n$:

$$u_{\nu,1}^h=\sum_{i=1}^n d_{2i-1}\psi_i,\ \ u_{\nu,2}^h=\sum_{i=1}^n d_{2i}\psi_i.$$

Coefficients d_i, $i=1,...,2n$, we will find from the system of linear equations

$$\begin{cases} a(\mathbf{u}_\nu^h,(\psi_i,0))=l(\psi_i,0), \\ a(\mathbf{u}_\nu^h,(0,\psi_i))=l(0,\psi_i), \quad i=1,...,n. \end{cases}$$

Remark. In contrast to the classic FEM, in the weighted FEM basis functions contain weight function in some power as a factor. This addition factor allows approximate solution to mimic behavior of exact solution in the neighborhood of singularity point. As a result, we are able to affect the accuracy of approximate solution. Selecting appropriate parameter ν^*, as well as ν and δ, we can gain theoretical convergence rate $O(h)$ for approximate R_ν-generalized solution to the exact one and get absolute error in mesh nodes in 10–100 times less than in the case of approximate generalized solution.

4. Numerical analysis of the model problem

In this section, we describe numerical experiment for model problem using the weighted FEM presented in section 3. We compare derived approximate R_ν-generalized solution and generalized one with respect to convergence rate established in different weighted and non-weighted norms, respectively, and absolute error in mesh nodes.

4.1. Model problem

We consider model problem (1), (2) in domain Ω described in section 3 with following components of exact solution:

$$u_1=\frac{K_I}{\mu\sqrt{2\pi}}\sqrt{r}\cos\left(\frac{\theta}{2}\right)\left(1-\frac{\lambda}{\lambda+\mu}+\sin^2\left(\frac{\theta}{2}\right)\right),$$

$$u_2=\frac{K_I}{\mu\sqrt{2\pi}}\sqrt{r}\sin\left(\frac{\theta}{2}\right)\left(1-\frac{\lambda}{\lambda+\mu}+\sin^2\left(\frac{\theta}{2}\right)\right),$$

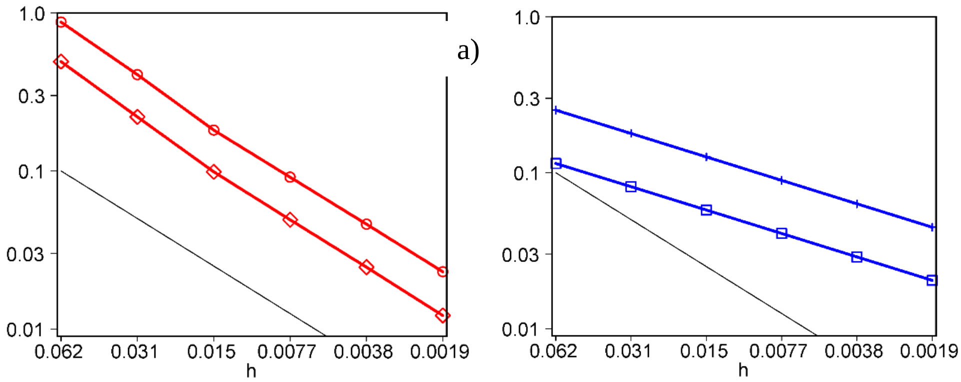
where θ is a polar angle, pole is located in the origin $(0,0)$, and polar axis coincide with positive direction of the Ox axis. Lame parameters are $\lambda=576.923$ and $\mu=384.615$, stress intensity factor $K_I=1.611$.

4.2. Numerical experiment. Optimal parameters

Numerical experiment was carried out on meshes described in section 3 with mesh parameter $h=0.062$, 0.031, 0.0155, 0.0077, 0.0038, 0.0019. Calculations were performed on each mesh with different values of parameters ν^*, ν, δ. Among them, the optimal parameters providing the best accuracy of approximate R_ν-generalized in relative norm of the weighted Sobolev space and weighted energy norm solution were found. More detailed description of this process can be found in [14]. We also calculated approximate generalized solution using our weighted FEM with $\nu = 0$, $\nu^* = 0$, $\rho(x) \equiv 1$ for comparison with approximate R_ν-generalized one.

4.3. Convergence rate investigation

In this section, we briefly describe results concerning convergence rate of approximate R_ν-generalized solution founded on each mesh with appropriate optimal parameters and comparison with convergence rate of approximate generalized solution. For approximate R_ν-generalized solution, convergence was calculated in relative norm η_ν of the weighted Sobolev space and in relative weighted energy norm η_ν^E. For approximate generalized solution, convergence was calculated in relative norm η of the Sobolev space and in relative energy norm η^E. These results are visually presented on figure 1a and 1b respectively. As we can see, appropriate parameters ν^*, as well as ν and δ allowed approximate R_ν-generalized solution to gain theoretical convergence rate $O(h)$, whereas approximate generalized solution converges with only $O(h^{1/2})$ speed.



b)

Figure 1: Convergence rates for approximate R_ν-generalized solution (1a, red rhombic line represents η_ν, red circled line represents η_ν^E) and for approximate generalized solution (1b, blue squared line represents η, blue crossed line represents η^E). Solid black lines represent convergence rate $O(h)$.

4.4. Absolute error investigation

For approximate R_ν-generalized solution and approximate generalized solution absolute error in mesh nodes was also investigated. Here in Table 1 we present percentage of mesh nodes, where absolute errors of first component of R_ν-generalized and generalized solutions do not exceed limit value 10^{-7}.

Table 1
Percent of mesh nodes where absolute errors of first component of R_ν -generalized (n_1^ν) and generalized (n_1) solutions do not exceed limit value 10^{-7}

h	0.062	0.031	0.0155	0.0077	0.0038	0.0019
n_1^ν, %	4.2	10.4	13.1	46.1	73.6	96.9
n_1, %	4.2	9.5	20.4	38.6	60.4	81.3

4.5. Conclusions

Numerical analysis of the model problem allowed us to conclude the following:

1. Introducing of R_ν-generalized solution allowed us to subdue the influence of singularity caused by the crack to the accuracy of approximate solution.
2. For parameters ν^*, ν and δ there are optimal values producing an approximate R_ν-generalized solution with best convergence rate and minimal absolute error.
3. Approximate R_ν-generalized solution with optimal parameters ν^*, ν and δ converges to the exact one with the rate $O(h)$ in norm of the weighted Sobolev space and weighted energy norm, whereas approximate solution converges only with the rate $O(h^{1/2})$.
4. In majority of mesh nodes, approximate R_ν-generalized solution has absolute error in 10-100 times less than approximate generalized solution.
5. Obtained results demonstrate advantages of weighted FEM over classic FEM which required meshes with refining towards singularity point for achieving the same convergence rate. In comparison with known specialized numerical methods for crack problem, weighted FEM is much simpler and do not produce ill-conditioned systems of linear algebraic equations.

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