Two-Dimensional Hardy Operators in Lebesgue Spaces

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Abstract
Characterizations of linear and bilinear Lebesgue norm inequalities involving two-dimensional Hardy integral operators are obtained.

Keywords 1
Hardy integral operator, weighted Lebesgue space, bilinear inequality.

1. Introduction

Let $\mathfrak{M}$ be the set of all Lebesgue measurable functions f on $\mathbb{R}^2_+ := (0,\infty)^2$, and let $\mathfrak{M}^+ \subset \mathfrak{M}$ be the subset of all nonnegative f. If $v \in \mathfrak{M}^+$ and $0 < p \le \infty$ we define the weighted Lebesgue space

$$L^p_v(\mathbb{R}^2) = \left\{f \in \mathfrak{M}: \|f\|_{\mathrm{p,v}} := \left(\int |f(x)|^p v(x)dx\right)^{\frac{1}{\mathrm{p}}} < \infty\right\}, \quad 0 < p < \infty,$$

$$L^\infty_v(\mathbb{R}^2) = \{f \in \mathfrak{M}: \|f\|_{\infty,v} := \operatorname{ess\,sup}_{x\in\mathbb{R}^2} v(x)|f(x)| < \infty\}, \qquad p = \infty.$$

Let $n \in \mathbb{N}$, $0 < q \le \infty$ and $1 \le p_i \le \infty$, $w, v_i \in \mathfrak{M}^+$ for all $i = 1, \dots n$. Define the two-dimensional rectangular Hardy operator

$$I_2 f(x,y) := \int_0^x \int_0^y f(s,t)dsdt, \qquad (x,y) \in \mathbb{R}^2_+, \tag{1}$$

and consider the following multilinear inequality

$$\|(I_2 f_1) \cdot \dots \cdot (I_2 f_n)\|_{q,w} \le C\|f_1\|_{p_1,v_1} \dots \|f_n\|_{p_n,v_n}, \qquad f_i \in \mathfrak{M}^+, \tag{2}$$

where a constant *C>0* is independent of f_i, $i = 1, \dots, n$, and is supposed to be the least possible.

The general problem is to characterize this inequality (2) by establishing a two-sided estimate

$$\alpha\, F(v_1, \dots v_n, w;\ p_1, \dots p_n, q) \le C \ \le\ \beta\, F(v_1, \dots v_n, w;\ p_1, \dots p_n, q)$$

with some irrelevant constants α and β by a functional $F(v_1, \dots v_n, w;\ p_1, \dots p_n, q)$ of an explicit form depending on given weights $v_1, \dots, v_n, w$ and fixed parameters $p_1, \dots, p_n, q$ only.

An operator in the left-hand side of the inequality (2) is n-fold product of two-dimensional Hardy operators (1), it is acting on the product of n Lebesgue spaces. Multi(sub)linear maximal operators, which are related to (1), appeared in connection with multilinear Calderón-Zygmund theory. They were used for the study of multilinear singular integral operators of Calderón-Zygmund type and for building a theory of weights adapted to the multilinear setting [6, 3, 1]. Linear and multi-linear

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inequalities with Hardy operators also play an important role in analysis and its applications [5]. The main purpose of this work is to survey the most recent characterizations of (2) by the authors in linear and bilinear cases. Starting in Section 2 from (quasi)linear case n = 1, we give the results for bilinear inequalities in Section 3. These findings can be similarly extended to any multilinear case.

We use signs := and =: for determining new quantities. For positive functionals F and G we write $F \ll G$ if $F \leq \alpha G$ with some constant $\alpha > 0$ depending, possibly, on irrelevant parameters only. Relations of the type $F \approx G$ mean F ≪ G ≪ F or $F = \alpha G$.

2. Two-dimensional Hardy inequality

Weighted Hardy inequality

$$\|I_2 f\|_{q,w} \leq C\|f\|_{p,v}, \qquad f \in \mathrm{M}^+, \tag{3}$$

with two-dimensional rectangular operator (1) was studied in [4, 8, 11, 12, 24]. In particular, the following criterion for the inequality (3) to hold was obtained by E. Sawyer in [12].

Theorem [12, Theorem 1A]. *Let* $1 < p \leq q < \infty$. *Denote* $p' := p/(p-1)$ *and let* $(I_2^* f)(x,y) := \int_x^\infty \int_y^\infty f(s,t)dsdt$ *be the adjoint to* I_2 *operator. The inequality (3) holds if and only if*

$$A_1 := \sup_{(s,t)\in\mathbb{R}_+^2} [I_2^* w(s,t)]^{1/q} [I_2 v^{1-p'}(s,t)]^{1/p'} < \infty, \tag{4}$$

$$A_2 := \sup_{(s,t)\in\mathbb{R}_+^2} \left(\int_0^s \int_0^t (I_2 v^{1-p'})^q w\right)^{1/q} [I_2 v^{1-p'}(s,t)]^{-1/p} < \infty, \tag{5}$$

$$A_3 := \sup_{(s,t)\in\mathbb{R}_+^2} \left(\int_s^\infty \int_t^\infty (I_2^* w)^{p'} v^{1-p'}\right)^{1/p'} [I_2^* w(s,t)]^{-1/q'} < \infty. \tag{6}$$

Moreover, it holds for the least possible constant C>0 in (3) that $C \approx A_1 + A_2 + A_3$ *with equivalence constants depending of p and q only.*

The one-dimensional analog of the condition (4) is the boundedness of the Muckenhoupt constant [9]. Characteristics (5) and (6) are two-dimensional generalizations of the Tomaselli functional [23, definition (11)] in its direct and dual forms. In one-dimensional case all the conditions (4)-(6) are equivalent to each other (see e.g. [2]), that is $A_1 \approx A_2 \approx A_3$ with equivalence constants depending of *p* and *q*. In two-dimensional case this generally is not true. Moreover, as it was shown in [12, § 4] for *p=q=2* that no two of conditions (4)-(6) guarantee (3). But, it was discovered in the recent work [22] by the authors that the E. Sawyer's theorem is actual for *p=q* only, while for *p<q* the inequality (3) is characterized by only one Muckenhoupt functional $A := A_1$ of the form (4).

Theorem [22, Theorem 2]. *Let* $1 < p < q < \infty$. *Denote* $\gamma := \gamma(p,q) := \frac{p^2(q-1)}{q-p}$, $\gamma' := \gamma(q',p')$ *and*

$$\mathbb{C}_{\gamma,\gamma'} := 3^{3q}\left[\frac{2^{4q}}{3^q}\max\{\gamma, 2q(q')^{q/p'}\}\left(\frac{2^{p-1}}{2^{p-1}-1}\right)^{q/p} + 3^{1/p+1/q'}(\gamma')^{1/p'}\right].$$

The inequality (3) holds if and only if $A < \infty$. *Besides,* $A \leq C \leq \mathbb{C}_{\gamma,\gamma'} A$.

The results of [12, Theorems 1A] and [22, Theorem 2] are valid for any type of weights *v* and *w*.

It was established in [24] that if one of the two weights *v* or *w* is factorizable, that is if

$$v(x_1,x_2) = v_1(x_1)v_2(x_2) \tag{7}$$

or

$$w(x_1,x_2) = w_1(x_1)w_2(x_2), \tag{8}$$

then it is possible to characterize (3) by only one functional for $1 < p \le q < \infty$. This result was extended to all *p,q>1* and generalized to all the types of boundedness constants in [11].

Theorem [11, Theorems 2.1, 2.2]. *Let* $1 < p \le q < \infty$ *and the weight v satisfy the condition (7). Denote* $V_i(x_i) := \int_0^{x_i} v_i^{1-p'}$, $i = 1,2$. *Then the inequality (3) holds for all* $f \ge 0$ *if and only if*

$$A_M := \sup_{(s,t)\in\mathbb{R}_+^2} [I_2^* w(s,t)]^{1/q} [V_1(s)V_2(t)]^{1/p'} < \infty,$$

or if and only if

$$A_T := \sup_{(s,t)\in\mathbb{R}_+^2} \left(\int_0^s \int_0^t [V_1 V_2]^q \, w \right)^{1/q} [V_1(s)V_2(t)]^{-1/p} < \infty.$$

Besides, it holds for the least possible constant C>0 in (3) that $C \approx A_M \approx A_T$ *with equivalence constants depending of p and q only.*

Theorem [11, Theorems 2.4, 2.5]. *Let* $1 < p \le q < \infty$ *and the weight w satisfy the condition (8). Denote* $W_i(x_i) := \int_{x_i}^{\infty} w_i$, $i = 1,2$. *Then the inequality (3) holds for all* $f \ge 0$ *if and only if*

$$A_M^* := \sup_{(s,t)\in\mathbb{R}_+^2} [I_2 v^{1-p'}(s,t)]^{1/p'} [W_1(s)W_2(t)]^{1/q} < \infty,$$

or if and only if

$$A_T^* := \sup_{(s,t)\in\mathbb{R}_+^2} \left(\int_s^{\infty} \int_t^{\infty} [W_1 W_2]^{p'} \, v^{1-p'} \right)^{1/p'} [W_1(s)W_2(t)]^{-1/q'} < \infty.$$

Besides, $C \approx A_M^* \approx A_T^*$ *with equivalence constants depending of p and q only.*

We complete the section by assertions similar to the last two above, but devoted to the case $1 < q < p < \infty$. To state them we put *1/r=1/q-1/p* and define two-dimensional analogs of Maz'ya-Rosin [7, § 1.3.2] and Persson-Stepanov [10, Theorem 3] functionals in their direct and dual forms:

$$B_{MR} := \left(\int [I_2^* w(s,t)]^{r/q} [V_1(s)V_2(t)]^{r/q'} v_1^{1-p'}(s)\, v_2^{1-p'}(t)\, ds\, dt \right)^{1/r},$$

$$B_{PS} := \left(\int \left(\int_0^s \int_0^t [V_1 V_2]^q \, w \right)^{r/q} [V_1(s)V_2(t)]^{-r/q} \, v_1^{1-p'}(s)\, v_2^{1-p'}(t)\, ds\, dt \right)^{1/r},$$

$$B_{MR}^* := \left(\int [I_2 v^{1-p'}(s,t)]^{r/p'} [W_1(s)W_2(t)]^{r/p} w_1(s)\, w_2(t)\, ds\, dt \right)^{1/r},$$

$$B_{PS}^* := \left(\int \left(\int_s^{\infty} \int_t^{\infty} [W_1 W_2]^{p'} \, v^{1-p'} \right)^{r/p'} [W_1(s)W_2(t)]^{-r/p'} \, w_1(s)\, w_2(t)\, ds\, dt \right)^{1/r}.$$

Theorem [11, Theorems 3.1, 3.2]. *Let* $1 < q < p < \infty$. *Suppose that the weight v in (3) satisfies the condition (7) and* $V_1(\infty) = V_2(\infty) = \infty$. *Then the inequality (3) is valid for all* $f \in \mathrm{M}^+$ *if and only if* $B_{MR} < \infty$, *or if and only if* $B_{PS} < \infty$. *Moreover,* $C \approx B_{MR} \approx B_{PS}$.

Theorem [11, Theorems 3.3, 3.4]. *Let* $1 < q < p < \infty$. *Assume that the weight function w in (3) satisfies the condition (8) and* $W_1(0) = W_2(0) = \infty$. *Then the inequality (3) is valid for all* $f \in \mathrm{M}^+$ *if and only if* $B_{MR}^* < \infty$, *or if and only if* $B_{PS}^* < \infty$. *Moreover,* $C \approx B_{MR}^* \approx B_{PS}^*$.

3. Bilinear two-dimensional Hardy inequality

In this section we demonstrate some of the new characteristics from [20] obtained for the inequality

$$\|(I_2 f)(I_2 g)\|_{q,w} \le C\|f\|_{p,v}\|g\|_{s,u}, \qquad f,g \in \mathrm{M}^+. \tag{9}$$

These results are based on statements for the linear two-dimensional Hardy inequality from Section 2.

Distinguish the following zones for the relations between integration parameters $1 < p,s,q < \infty$:

(I) $\quad 1 < \max\{p,s\} \le q < \infty,$

(II) $\quad 1 < \min\{p,s\} \le q < \max\{p,s\} < \infty,$

(III) $\quad 1 < q < \min\{p,s\}.$

The required characteristics for (I), (II) and (III) are given in the assertions below.

Theorem [20, Theorem 4]. *Let $p,s,q \in (I)$. Assume that the weight v in (9) is of product type, that is v satisfies the condition (7). Then the best constant C in the inequality (9) is estimated as*

$$C \approx D_I := \sup_{(x,y)\in\mathbb{R}_+^2}\big(D_1(x,y) + D_2(x,y) + D_3(x,y)\big)[V_1(x)V_2(y)]^{1/p'}, \tag{10}$$

where $V_i(x_i) := \int_0^{x_i} v_i^{1-p'}$, $i = 1,2$, as before and

$$D_1(x,y) := \sup_{(\varrho,\tau)\in\mathbb{R}_+^2}\left[I_2^*\big(w\chi_{(x,\infty)\times(y,\infty)}\big)(\varrho,\tau)\right]^{1/q}[I_2 u^{1-s'}(\varrho,\tau)]^{1/s'},$$

$$D_2(x,y) := \sup_{(\varrho,\tau)\in\mathbb{R}_+^2}\left(\int_0^{\varrho}\int_0^{\tau}(I_2 u^{1-s'})^q w\chi_{(x,\infty)\times(y,\infty)}\right)^{1/q}[I_2 u^{1-s'}(\varrho,\tau)]^{-1/s},$$

$$D_3(x,y) := \sup_{(\varrho,\tau)\in\mathbb{R}_+^2}\left(\int_{\varrho}^{\infty}\int_{\tau}^{\infty}\left(I_2^*\big(w\chi_{(x,\infty)\times(y,\infty)}\big)\right)^{s'}u^{1-s'}\right)^{1/s'}\left[I_2^*\big(w\chi_{(x,\infty)\times(y,\infty)}\big)(\varrho,\tau)\right]^{-1/q'}.$$

Remark [20, Remark 3]. If the weight *u* in (9) is also of product type, that is if

$$u(x_1,x_2) = u_1(x_1)u(x_2), \tag{11}$$

then the expression for the functional D_I in (10) simplifies as follows:

$$D_I := \sup_{(x,y)\in\mathbb{R}_+^2}[I_2^* w(x,y)]^{1/q}[V_1(x)V_2(y)]^{1/p'}[U_1(x)U_2(y)]^{1/p'} < \infty,$$

where $V_i(x_i) := \int_0^{x_i} v_i^{1-p'}$ and $U_i(x_i) := \int_0^{x_i} u_i^{1-s'}$, $i = 1,2$.

Theorem [20, Theorem 5]. *Let $p,s,q \in (II)$. Assume that the weights v and u in (9) are of product type, that is v and u satisfy the conditions (7) and (11), respectively. Then $C \approx D_{II}$, where for $1 < p \le q < s < \infty$, under the condition $U_i(\infty) = \infty, i = 1,2$,*

$$D_{II} := \sup_{(x,y)\in\mathbb{R}_+^2}\left(\int_x^{\infty}\int_y^{\infty}[I_2^* w]^{t/q}[U_1 U_2]^{t/q'}\,u_1^{1-s'}u_2^{1-s'}\right)^{1/t}[V_1(x)V_2(y)]^{1/p'},$$

and for $1 < s \le q < p < \infty$, under the condition $V_i(\infty) = \infty, i = 1,2$,

$$D_{II} := \sup_{(x,y)\in\mathbb{R}_+^2}\left(\int_x^{\infty}\int_y^{\infty}[I_2^* w]^{r/q}[V_1 V_2]^{r/q'}\,v_1^{1-p'}v_2^{1-p'}\right)^{1/r}[U_1(x)U_2(y)]^{1/s'},$$

where 1/r:=1/q-1/p and 1/t=1/q-1/s.

Theorem [20, Theorem 6]. *Let $p, s, q \in (III)$. Assume that all the weights in (9) are of product type, that is v,u and w satisfy the conditions (7), (11) and (8), respectively. Then, under the conditions $V_i(\infty) = \infty, i = 1,2$, and $U_i(\infty) = \infty, i = 1,2$, it holds $C \approx \sum_{i=1}^{4} D_{II}(i)$, where for $1/q \le 1/p + 1/s$*

$$D_{II}(1) := \sup_{(x,y)\in\mathbb{R}_+^2} \left(\int_x^\infty \int_y^\infty [W_1 W_2]^{\frac{t}{q}} [U_1 U_2]^{\frac{t}{q'}} \, dU_1 \, dU_2 \right)^{1/t} [V_1(x) V_2(y)]^{1/p'},$$

$$D_{II}(2) := \sup_{(x,y)\in\mathbb{R}_+^2} \left(\int_x^\infty \int_y^\infty [W_1 W_2]^{\frac{r}{q}} [V_1 V_2]^{\frac{r}{q'}} \, dV_1 \, dV_2 \right)^{1/t} [U_1(x) U_2(y)]^{1/s'},$$

$$D_{II}(3) := \sup_{(x,y)\in\mathbb{R}_+^2} \left(\int_x^\infty [W_1]^{\frac{t}{q}} [U_1]^{\frac{t}{q'}} \, dU_1 \right)^{1/t} \left(\int_y^\infty [W_2]^{\frac{r}{q}} [V_2]^{\frac{r}{q'}} dV_2 \right)^{1/r} [V_1(x)]^{1/p'} [U_2(y)]^{1/s'},$$

$$D_{II}(4) := \sup_{(x,y)\in\mathbb{R}_+^2} \left(\int_x^\infty [W_1]^{\frac{r}{q}} [V_1]^{\frac{r}{q'}} dV_1 \right)^{1/r} \left(\int_y^\infty [W_2]^{\frac{t}{q}} [U_2]^{\frac{t}{q'}} dU_2 \right)^{1/t} [U_1(x)]^{\frac{1}{s'}} [V_2(y)]^{\frac{1}{p'}};$$

and for $1/q > 1/p + 1/s$ with $1/\kappa := 1/q - 1/p - 1/s$

$$D_{II}(1) := \left(\int_0^\infty \int_0^\infty \left(\int_x^\infty \int_y^\infty [W_1 W_2]^{\frac{t}{q}} [U_1 U_2]^{\frac{t}{q'}} \, dU_1 \, dU_2 \right)^{\kappa/t} [V_1(x) V_2(y)]^{\kappa/t'} \, dV_1(x) \, dV_2(y) \right)^{1/\kappa},$$

$$D_{II}(2) := \left(\int_0^\infty \int_0^\infty \left(\int_x^\infty \int_y^\infty [W_1 W_2]^{\frac{r}{q}} [V_1 V_2]^{\frac{r}{q'}} \, dV_1 \, dV_2 \right)^{\kappa/r} [U_1(x) U_2(y)]^{\kappa/r'} \, dU_1(x) \, dU_2(y) \right)^{1/\kappa},$$

$$[D_{II}(3)]^\kappa := \int_0^\infty \int_0^\infty \left(\int_x^\infty [W_1]^{\frac{t}{q}} [U_1]^{\frac{t}{q'}} \, dU_1 \right)^{\kappa/t} \left(\int_y^\infty [W_2]^{\frac{r}{q}} [V_2]^{\frac{r}{q'}} \, dV_2 \right)^{\kappa/s} [V_1(x)]^{\kappa/t'}$$

$$\times [U_2(y)]^{\kappa/s'} [W_2(y)]^{\frac{r}{q}} [V_2(y)]^{\frac{r}{q'}} dV_1(x) \, dV_2(y),$$

$$[D_{II}(4)]^\kappa := \int_0^\infty \int_0^\infty \left(\int_x^\infty [W_1]^{\frac{r}{q}} [V_1]^{\frac{r}{q'}} \, dV_1 \right)^{\kappa/s} \left(\int_y^\infty [W_2]^{\frac{t}{q}} [U_2]^{\frac{t}{q'}} \, dU_2 \right)^{\kappa/t} [U_1(x)]^{\kappa/s'}$$

$$\times [V_2(y)]^{\kappa/t'} [W_1(x)]^{\frac{r}{q}} [V_1(x)]^{\frac{r}{q'}} dV_1(x) \, dV_2(y),$$

where 1/r:=1/q-1/p, 1/t:=1/q-1/s, $V_i(x_i) := \int_0^{x_i} v_i^{1-p'}$, $U_i(x_i) := \int_0^{x_i} u_i^{1-s'}$, $W_i(x_i) := \int_{x_i}^\infty w_i, i = 1,2$.

For some other types of bilinear inequalities with Hardy type operators one can consult [13-19, 21].

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