Two-Dimensional Hardy Operators in Lebesgue Spaces

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Abstract
Characterizations of linear and bilinear Lebesgue norm inequalities involving two-dimensional Hardy integral operators are obtained.

Keywords 1
Hardy integral operator, weighted Lebesgue space, bilinear inequality.

1. Introduction
Let $\mathcal{M}$ be the set of all Lebesgue measurable functions $f$ on $\mathbb{R}^2_+ := (0, \infty)^2$, and let $\mathcal{M}^+ \subset \mathcal{M}$ be the subset of all nonnegative $f$. If $\nu \in \mathcal{M}^+$ and $0 < p \leq \infty$ we define the weighted Lebesgue space
\[
L^p_p (\mathbb{R}^2) = \left\{ f \in \mathcal{M} : \|f\|_{p, \nu} := \left( \int_{\mathbb{R}^2} |f(x)|^p \nu(x) dx \right)^{1/p} < \infty \right\}, \quad 0 < p < \infty,
\]
\[
L^\infty (\mathbb{R}^2) = \left\{ f \in \mathcal{M} : \|f\|_{\infty, \nu} := \sup_{x \in \mathbb{R}^2} |f(x)| < \infty \right\}, \quad p = \infty.
\]
Let $n \in \mathbb{N}$, $0 < q \leq \infty$ and $1 \leq p_i \leq \infty$, $v_i \in \mathcal{M}^+$ for all $i = 1, \ldots, n$. Define the two-dimensional rectangular Hardy operator
\[
l_{2f}(x, y) := \int_0^y \int_0^x f(s, t) ds dt, \quad (x, y) \in \mathbb{R}^2_+,
\]
and consider the following multilinear inequality
\[
\|(l_{2f_1}) \cdot \ldots \cdot (l_{2f_n})\|_{q, \nu} \leq C \|f_1\|_{p_1, \nu_1} \ldots \|f_n\|_{p_n, \nu_n}, \quad f_i \in \mathcal{M}^+,
\]
where a constant $C > 0$ is independent of $f_i$, $i = 1, \ldots, n$, and is supposed to be the least possible.

The general problem is to characterize this inequality (2) by establishing a two-sided estimate
\[
\alpha F(v_1, \ldots, v_n, w; p_1, \ldots, p_n, q) \leq C \leq \beta F(v_1, \ldots, v_n, w; p_1, \ldots, p_n, q)
\]
with some irrelevant constants $\alpha$ and $\beta$ by a functional $F(v_1, \ldots, v_n, w; p_1, \ldots, p_n, q)$ of an explicit form depending on given weights $v_1, \ldots, v_n, w$ and fixed parameters $p_1, \ldots, p_n, q$ only.

An operator in the left-hand side of the inequality (2) is $n$-fold product of two-dimensional Hardy operators (1), it is acting on the product of $n$ Lebesgue spaces. Multi(sub)linear maximal operators, which are related to (1), appeared in connection with multilinear Calderón-Zygmund theory. They were used for the study of multilinear singular integral operators of Calderón-Zygmund type and for building a theory of weights adapted to the multilinear setting [6, 3, 1]. Linear and multi-linear
inequalities with Hardy operators also play an important role in analysis and its applications [5]. The main purpose of this work is to survey the most recent characterizations of (2) by the authors in linear and bilinear cases. Starting in Section 2 from (quasi)linear case $n = 1$, we give the results for bilinear inequalities in Section 3. These findings can be similarly extended to any nonlinear case.

We use signs := and =: for determining new quantities. For positive functionals $F$ and $G$ we write $F \ll G$ if $F \leq \alpha G$ with some constant $\alpha > 0$ depending, possibly, on irrelevant parameters only. Relations of the type $F \approx G$ mean $F \ll G \ll F$ or $F = \alpha G$.

2. Two-dimensional Hardy inequality

Weighted Hardy inequality

$$\|I_2f\|_{q,w} \leq C\|f\|_{p,v}, \quad f \in M^+,$$

(3)

with two-dimensional rectangular operator (1) was studied in [4, 8, 11, 12, 24]. In particular, the following criterion for the inequality (3) to hold was obtained by E. Sawyer in [12].

**Theorem** [12, Theorem 1A]. Let $1 < p \leq q < \infty$. Denote $p' := p/(p-1)$ and let $(I_2^p f)(x, y) := \int_x^\infty \int_y^\infty f(s, t)dsdt$ be the adjoint to $I_2$ operator. The inequality (3) holds if and only if

$$A_1 := \sup_{(s,t) \in \mathbb{R}^2_+} \left[I_2^1 w(s,t)\right]^{1/q} \left[I_2^{1-p'}(s,t)\right]^{1/p'} < \infty,$$

(4)

$$A_2 := \sup_{(s,t) \in \mathbb{R}^2_+} \left(\int_0^s \int_0^t \left[I_2^{1-p'}(s,t)\right]^{-1/p} \left[I_2^{1-p'}(s,t)\right]^{1/q} \right) < \infty,$$

(5)

$$A_3 := \sup_{(s,t) \in \mathbb{R}^2_+} \left(\int_0^\infty \int_t^\infty \left[I_2^{1-p'}(s,t)\right]^{-1/q} \left[I_2^{1-p'}(s,t)\right]^{1/p'} \right) < \infty.$$

(6)

Moreover, it holds for the least possible constant $C > 0$ in (3) that $C \approx A_1 + A_2 + A_3$ with equivalence constants depending of $p$ and $q$ only.

The one-dimensional analog of the condition (4) is the boundedness of the Muckenhoupt constant [9]. Characteristics (5) and (6) are two-dimensional generalizations of the Tomaselli functional [23, definition (11)] in its direct and dual forms. In one-dimensional case all the conditions (4)-(6) are equivalent to each other (see e.g. [2]), that is $A_1 \approx A_2 \approx A_3$ with equivalence constants depending of $p$ and $q$. In two-dimensional case this generally is not true. Moreover, as it was shown in [12, § 4] for $p=q=2$ that no two of conditions (4)-(6) guarantee (3). But, it was discovered in the recent work [22] by the authors that the E. Sawyer’s theorem is actual for $p=q$ only, while for $p \neq q$ the inequality (3) is characterized by only one Muckenhoupt functional $A := A_1$ of the form (4).

**Theorem** [22, Theorem 2]. Let $1 < p < q < \infty$. Denote $\gamma := \gamma(p, q) := \frac{p^2(q-1)}{q-p}$, $\gamma' := \gamma(q', p')$ and

$$C_{\gamma, \gamma'} := 3^{3q} \left[2^{4q} \frac{3q}{q-p} \max\{\gamma, 2q(q')^{q'/p'}\} \left(2^{p-1} \frac{q}{p}\right)^{q/p} + 3^{1/p+1/q'} \gamma'(y')^{-1/p'} \right].$$

The inequality (3) holds if and only if $A < \infty$. Besides, $A \leq C \leq C_{\gamma, \gamma'} A$.

The results of [12, Theorems 1A] and [22, Theorem 2] are valid for any type of weights $v$ and $w$.

It was established in [24] that if one of the two weights $v$ or $w$ is factorizable, that is if

$$v(x_1, x_2) = v_1(x_1)v_2(x_2),$$

(7)

or

$$w(x_1, x_2) = w_1(x_1)w_2(x_2),$$

(8)

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then it is possible to characterize (3) by only one functional for $1 < p \le q < \infty$. This result was extended to all $p, q > 1$ and generalized to all the types of boundedness constants in [11].

**Theorem** [11, Theorems 2.1, 2.2]. Let $1 < p \le q < \infty$ and the weight $v$ satisfy the condition (7). Denote $V_i(x_i) := \int_0^{x_i} v_i^{1-p'}$, $i = 1, 2$. Then the inequality (3) holds for all $f \ge 0$ if and only if

$$A_M := \sup_{(s,t) \in \mathbb{R}_+^2} [I_2^2 w(s,t)]^{1/q} [V_2(s)V_2(t)]^{1/p'} < \infty,$$

or if and only if

$$A_T := \sup_{(s,t) \in \mathbb{R}_+^2} \left( \int_0^s \int_0^t [V_2(s)V_2(t)]^{-1/p} \right)^{1/q} \left( \int_0^s \int_0^t [V_2(s)V_2(t)]^{-1/p'} \right)^{1/q} < \infty.$$

Besides, it holds for the least possible constant $C > 0$ in (3) that $C \approx A_M \approx A_T$ with equivalence constants depending of $p$ and $q$ only.

**Theorem** [11, Theorems 2.4, 2.5]. Let $1 < p \le q < \infty$ and the weight $w$ satisfy the condition (8). Denote $W_i(x_i) := \int_0^{x_i} w_i$, $i = 1, 2$. Then the inequality (3) holds for all $f \ge 0$ if and only if

$$A_M^* := \sup_{(s,t) \in \mathbb{R}_+^2} [I_2^2 w^{1-p'}(s,t)]^{1/q} [W_2(s)W_2(t)]^{1/p'} < \infty,$$

or if and only if

$$A_T^* := \sup_{(s,t) \in \mathbb{R}_+^2} \left( \int_s^\infty \int_t^\infty [W_2(s)W_2(t)]^{-1/q'} \right)^{1/p'} \left( \int_s^\infty \int_t^\infty [W_2(s)W_2(t)]^{-1/q} \right)^{1/q'} < \infty.$$

Besides, $C \approx A_M^* \approx A_T^*$ with equivalence constants depending of $p$ and $q$ only.

We complete the section by assertions similar to the last two above, but devoted to the case $1 < q < p < \infty$. To state them we put $1/r = 1/q - 1/p$ and define two-dimensional analoges of Maz'ya-Rosin [7, § 1.3.2] and Persson-Stepanov [10, Theorem 3] functionals in their direct and dual forms:

$$B_{MR} := \left( \int_s^\infty \int_t^\infty [V_2(s)V_2(t)]^r \left( v_1^{1-p'}(s) v_2^{1-p'}(t) \right) ds dt \right)^{1/r},$$

$$B_{PS} := \left( \int_s^\infty \int_t^\infty [W_2(s)W_2(t)]^r \left( w_1^{1-p'}(s) w_2^{1-p'}(t) \right) ds dt \right)^{1/r},$$

$$B_{MR}^* := \left( \int_s^\infty \int_t^\infty [W_2(s)W_2(t)]^{r/p'} w_1^{1-p'}(s) w_2^{1-p'}(t) ds dt \right)^{1/r},$$

$$B_{PS}^* := \left( \int_s^\infty \int_t^\infty [W_2(s)W_2(t)]^{r/p'} w_1^{1-p'}(s) w_2^{1-p'}(t) ds dt \right)^{1/r}.$$

**Theorem** [11, Theorems 3.1, 3.2]. Let $1 < q < p < \infty$. Suppose that the weight $v$ in (3) satisfies the condition (7) and $V_1(\infty) = V_2(\infty) = \infty$. Then the inequality (3) is valid for all $f \in \mathcal{M}^+$ if and only if $B_{MR} < \infty$, or if and only if $B_{PS} < \infty$. Moreover, $C \approx B_{MR} \approx B_{PS}$.

**Theorem** [11, Theorems 3.3, 3.4]. Let $1 < q < p < \infty$. Assume that the weight function $w$ in (3) satisfies the condition (8) and $W_1(0) = W_2(0) = \infty$. Then the inequality (3) is valid for all $f \in \mathcal{M}^+$ if and only if $B_{MR}^* < \infty$, or if and only if $B_{PS}^* < \infty$. Moreover, $C \approx B_{MR}^* \approx B_{PS}^*$. 

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3. Bilinear two-dimensional Hardy inequality

In this section we demonstrate some of the new characteristics from [20] obtained for the inequality

\[ \| (l_2 f)(l_2 g) \|_{q,w} \leq C \| f \|_{p,v} \| g \|_{s,u}, \quad f, g \in M^+. \]  

(9)

These results are based on statements for the linear two-dimensional Hardy inequality from Section 2. Distinguish the following zones for the relations between integration parameters \( 1 < p, s, q < \infty \):

(i) \( 1 < \max\{p, s\} \leq q < \infty \),

(ii) \( 1 < \min\{p, s\} \leq q < \max\{p, s\} < \infty \),

(iii) \( 1 < q < \min\{p, s\} \).

The required characteristics for (I), (II) and (III) are given in the assertions below.

**Theorem** [20, Theorem 4]. Let \( p, s, q \in (I) \). Assume that the weight \( v \) in (9) is of product type, that is \( v \) satisfies the condition (7). Then the best constant \( C \) in the inequality (9) is estimated as

\[ C \approx D_1 := \sup_{(x,y) \in \mathbb{R}_+^2} \left( D_1(x, y) + D_2(x, y) + D_3(x, y) \right) \mathbb{H}_1(V_1(x) V_2(y))^{1/p}, \]  

(10)

where \( V_i(x) := \int_0^{x_i} v_i^{1-p'}, i = 1, 2, \) as before and

\[ D_1(x, y) := \sup_{(x,y) \in \mathbb{R}_+^2} \left( \int_0^y \int_0^x (l_2 w)^{1/s} (l_2 w)^{-1/s} (l_2 w)^{-1/s} \right)^{1/p} \mathbb{H}_1(V_1(x) V_2(y))^{1/p}, \]

\[ D_2(x, y) := \sup_{(x,y) \in \mathbb{R}_+^2} \left( \int_0^y \int_0^x (l_2 w)^{1/s} (l_2 w)^{-1/s} (l_2 w)^{-1/s} \right)^{1/p} \mathbb{H}_1(V_1(x) V_2(y))^{1/p}, \]

\[ D_3(x, y) := \sup_{(x,y) \in \mathbb{R}_+^2} \left( \int_0^y \int_0^x (l_2 w)^{1/s} (l_2 w)^{-1/s} (l_2 w)^{-1/s} \right)^{1/p} \mathbb{H}_1(V_1(x) V_2(y))^{1/p}. \]

**Remark** [20, Remark 3]. If the weight \( u \) in (9) is also of product type, that is if

\[ u(x_1, x_2) = u_1(x_1) u_2(x_2), \]

(11)

then the expression for the functional \( D_1 \) in (10) simplifies as follows:

\[ D_1 := \sup_{(x,y) \in \mathbb{R}_+^2} \left( \int_0^y \int_0^x (l_2 w)^{1/s} (l_2 w)^{-1/s} (l_2 w)^{-1/s} \right)^{1/p} \mathbb{H}_1(V_1(x) V_2(y))^{1/p}, \]

where \( V_i(x) := \int_0^{x_i} v_i^{1-p'} \) and \( U_i(x) := \int_0^{u_i} u_i^{-1/s} \), \( i = 1, 2 \).

**Theorem** [20, Theorem 5]. Let \( p, s, q \in (II) \). Assume that the weights \( v \) and \( u \) in (9) are of product type, that is \( v \) and \( u \) satisfy the conditions (7) and (11), respectively. Then \( C \approx D_{II} \), where for \( 1 < p \leq q < s < \infty \), under the condition \( U_i(\infty) = \infty, i = 1, 2 \),

\[ D_{II} := \sup_{(x,y) \in \mathbb{R}_+^2} \left( \int_0^y \int_0^x (l_2 w)^{1/q} (l_2 w)^{1/q} (l_2 w)^{1/q} \right)^{1/p} \mathbb{H}_1(V_1(x) V_2(y))^{1/p}, \]

and for \( 1 < s \leq q < p < \infty \), under the condition \( V_i(\infty) = \infty, i = 1, 2 \),

\[ D_{II} := \sup_{(x,y) \in \mathbb{R}_+^2} \left( \int_0^y \int_0^x (l_2 w)^{1/q} (l_2 w)^{1/q} (l_2 w)^{1/q} \right)^{1/p} \mathbb{H}_1(V_1(x) V_2(y))^{1/p}, \]

where \( 1/r := 1/q - 1/p \) and \( 1/t = 1/q - 1/s \).
Theorem [20, Theorem 6]. Let $p, s, q \in (1,1)$. Assume that all the weights in (9) are of product type, that is $v, u$ and $w$ satisfy the conditions (7), (11) and (8), respectively. Then, under the conditions $V_i(\infty) = \infty, i = 1,2$, and $U_i(\infty) = \infty, i = 1,2$, it holds $C \approx \sum_{i=1}^{4} D_{H_i}(i)$, where for $1/q \leq 1/p + 1/s$

\[
D_{H_i}(1) := \sup_{(x,y) \in \mathbb{R}^2_+} (\int_{x}^{\infty} \int_{y}^{\infty} [W_i W_2]^\frac{r}{s} [U_i U_2]^\frac{s}{r} dU_1 dU_2)^{1/t} [V_1(x)V_2(y)]^{1/pr},
\]

\[
D_{H_i}(2) := \sup_{(x,y) \in \mathbb{R}^2_+} (\int_{x}^{\infty} \int_{y}^{\infty} [W_i W_2]^\frac{r}{s} [V_i V_2]^\frac{s}{r} dV_1 dV_2)^{1/r} [U_1(x)U_2(y)]^{1/sr},
\]

\[
D_{H_i}(3) := \sup_{(x,y) \in \mathbb{R}^2_+} (\int_{x}^{\infty} \int_{y}^{\infty} [W_i]^\frac{r}{s} [U_i]^\frac{s}{r} dU_1 dU_2)^{1/r} [V_1(x)V_2(y)]^{1/pr},
\]

\[
D_{H_i}(4) := \sup_{(x,y) \in \mathbb{R}^2_+} (\int_{x}^{\infty} \int_{y}^{\infty} [W_i]^\frac{r}{s} [V_i]^\frac{s}{r} dV_1 dV_2)^{1/r} [U_1(x)U_2(y)]^{1/sr},
\]

and for $1/q > 1/p + 1/s$ with $1/\kappa := 1/q - 1/p - 1/s$

\[
D_{H_i}(1) := \left( \int_{0}^{\infty} \int_{0}^{\infty} (\int_{x}^{\infty} \int_{y}^{\infty} [W_i W_2]^\frac{r}{s} [U_i U_2]^\frac{s}{r} dU_1 dU_2)^{\kappa/t} [V_1(x)V_2(y)]^{\kappa/rt} dV_1(x) dV_2(y) \right)^{1/\kappa},
\]

\[
D_{H_i}(2) := \left( \int_{0}^{\infty} \int_{0}^{\infty} (\int_{x}^{\infty} \int_{y}^{\infty} [W_i W_2]^\frac{r}{s} [V_i V_2]^\frac{s}{r} dV_1 dV_2)^{\kappa/r} [U_1(x)U_2(y)]^{\kappa/r} dV_1(x) dV_2(y) \right)^{1/\kappa},
\]

\[
[D_{H_i}(3)]^{\kappa} := \left( \int_{0}^{\infty} \int_{0}^{\infty} (\int_{x}^{\infty} [W_i]^\frac{r}{s} dU_1)^{\kappa/t} (\int_{0}^{\infty} [W_2]^\frac{r}{s} dV_2)^{\kappa/s} [V_1(x)]^{\kappa/rt} \times [U_2(y)]^{\kappa/sr} [V_2(y)]^{\kappa/rt} dV_1(x) dV_2(y),
\]

\[
[D_{H_i}(4)]^{\kappa} := \left( \int_{0}^{\infty} \int_{0}^{\infty} (\int_{x}^{\infty} [W_i]^\frac{r}{s} dV_1)^{\kappa/s} (\int_{0}^{\infty} [W_2]^\frac{r}{s} dU_2)^{\kappa/t} [V_2(y)]^{\kappa/sr} [W_1(x)]^{\kappa/rt} dV_1(x) dV_2(y),
\]

where $1/r := 1/q - 1/p, 1/t := 1/q - 1/s$, $V_i(x_i) := \int_{x_i}^{\infty} v_i^{1-tp}$, $U_i(x_i) := \int_{x_i}^{\infty} u_i^{1-sp}$, $W_i(x_i) := \int_{x_i}^{\infty} w_i, i = 1,2$.

For some other types of bilinear inequalities with Hardy type operators one can consult [13-19, 21].

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5. References
