Two-Dimensional Mathematical Model of Pipelines with a Complex Intersected Profile

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Abstract

The boundary value problem of the pipeline statics with a branch is formulated. On intersection line, conjugation conditions are set in the assumption that branch is small compared to a large pipe. The original three-dimensional boundary value problem is projected onto the symmetry plane of mechanical systems and is represented in Cartesian coordinate system. A reduced two-dimensional mathematical model of intersecting elastic cylindrical shells is obtained. Boundary conditions are set on all the edges of the plane domain. Numerical analysis is performed, which shows that the replacement of initial conjugation conditions with conditions of bushing coupling type introduces an error in the solution of boundary value problem that is small in comparison with the error of shell theory.

Keywords 1

Intersecting shells, reduced mathematical model, numerical calculations

1. Introduction

Intersecting cylindrical shells are widely used in modern pipeline systems. A detailed analysis of studies of shell structures containing intersections can be found in [1]. In contrast to curved pipelines [2, 3], in pipelines with insets, a stress concentration occurs at the shells junction, as shown in [4]. Special weighted finite element methods exist for calculating problems in domains with singularity [5, 6, 7, 8]. Obtaining a numerical solution by this method will make it possible to predict the stress-strain state of complex pipeline systems, which is an actual engineering problem [9].

The aim of this paper is to formulate the boundary value problem of membrane cylindrical shells having a complex intersection of the profile, and to reduce the original problem into a two-dimensional form in Cartesian coordinates.

The following tasks were solved:

- construction of a mathematical model of thin elastic cylindrical shells intersecting at right angles;
- projection of the original problem onto the symmetry plane of the mechanical system;
- setting boundary conditions;
- justification on a numerical example of the permissibility of replacing the coupling conditions with bushing connections.

2. Problem statement for thin elastic cylindrical shells intersecting at right angles

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We consider, based on the membrane theory, two cylindrical shells intersecting at right angles, for which the ratio of thickness to radius is satisfied $<1/20$. We write the equilibrium equations [10] for these shells in displacements:

$$\begin{cases} \dfrac{\partial^2 u^{(1)}}{\partial x^2} + \dfrac{1-\nu}{2R^2}\dfrac{\partial^2 u^{(1)}}{\partial \theta^2} + \dfrac{1+\nu}{2R}\dfrac{\partial^2 v^{(1)}}{\partial x \partial \theta} + \dfrac{\nu}{R}\dfrac{\partial w^{(1)}}{\partial x} = 0, \\ \dfrac{1+\nu}{2R}\dfrac{\partial^2 u^{(1)}}{\partial x \partial \theta} + \dfrac{1-\nu}{2}\dfrac{\partial^2 v^{(1)}}{\partial x^2} + \dfrac{1}{R^2}\dfrac{\partial^2 v^{(1)}}{\partial \theta^2} + \dfrac{1}{R^2}\dfrac{\partial w^{(1)}}{\partial \theta} = 0, \\ \dfrac{\nu}{R}\dfrac{\partial u^{(1)}}{\partial x} + \dfrac{1}{R^2}\dfrac{\partial v^{(1)}}{\partial \theta} + \dfrac{w^{(1)}}{R^2} = \dfrac{1-\nu^2}{EH}p, \\ \dfrac{\partial^2 u^{(2)}}{\partial z^2} + \dfrac{1-\nu}{2r^2}\dfrac{\partial^2 u^{(2)}}{\partial \varphi^2} + \dfrac{1+\nu}{2r}\dfrac{\partial^2 v^{(2)}}{\partial z \partial \varphi} + \dfrac{\nu}{r}\dfrac{\partial w^{(2)}}{\partial z} = 0, \\ \dfrac{1+\nu}{2r}\dfrac{\partial^2 u^{(2)}}{\partial z \partial \varphi} + \dfrac{1-\nu}{2}\dfrac{\partial^2 v^{(2)}}{\partial z^2} + \dfrac{1}{r^2}\dfrac{\partial^2 v^{(2)}}{\partial \varphi^2} + \dfrac{1}{r^2}\dfrac{\partial w^{(2)}}{\partial \varphi} = 0, \\ \dfrac{\nu}{r}\dfrac{\partial u^{(2)}}{\partial z} + \dfrac{1}{r^2}\dfrac{\partial v^{(2)}}{\partial \varphi} + \dfrac{w^{(2)}}{r^2} = \dfrac{1-\nu^2}{Eh}p, \end{cases} \tag{1}$$

where the values of the radii of curvature and the Lame coefficients are taken into account $R_1^{(1)} = R_1^{(2)} = \infty$, $R_2^{(1)} = A_2^{(1)} = R$, $A_1^{(1)} = A_1^{(2)} = 1$, $R_2^{(2)} = A_2^{(2)} = r$ for the entered cylindrical coordinates (x, θ, ρ_1) and (z, φ, ρ_2) for the large and small cylinders. Notation: ν – Poisson ratio, E – modulus of elasticity, H, h – thickness of large and small cylinders; $u^{(1)}, v^{(1)}, w^{(1)}$ and $u^{(2)}, v^{(2)}, w^{(2)}$ – components of the displacement vector, where the index (1) denotes belonging to a large cylinder, and index (2) denotes belonging to a small cylinder, p – uniform internal pressure.

Boundary conditions at the ends of the large and small cylinders:

$$\begin{aligned} &x = -L/2: \quad u^{(1)} = 0, \ \frac{1}{R}\frac{\partial u^{(1)}}{\partial \theta} + \frac{\partial v^{(1)}}{\partial x} = 0, \\ &x = L/2: \quad u^{(1)} = 0, \ \frac{1}{R}\frac{\partial u^{(1)}}{\partial \theta} + \frac{\partial v^{(1)}}{\partial x} = 0; \\ &z = R + l: \quad v^{(2)} = 0, \ \frac{\partial u^{(2)}}{\partial z} + \frac{\nu}{r}\frac{\partial v^{(2)}}{\partial \varphi} + \frac{\nu}{r}w^{(2)} = 0. \end{aligned} \tag{2}$$

We complete the problem statement for thin elastic cylindrical shells intersecting at right angles in displacements by conjugation conditions:

$$\begin{aligned} &-u^{(1)}\sin\varphi - v^{(1)}\sin\theta\cos\varphi + w^{(1)}\frac{r}{R}\cos\theta\cos\varphi = v^{(2)}, \\ &u^{(1)}\cos\varphi - v^{(1)}\sin\theta\sin\varphi + w^{(1)}\cos\theta\sin\varphi = w^{(2)}, \\ &v^{(1)}\cos\theta + w^{(1)}\sin\theta = u^{(2)}, \\ &\frac{H}{R}\frac{\partial u^{(1)}}{\partial \theta} + H\frac{\partial v^{(1)}}{\partial x} = -\frac{h}{r}\frac{\partial u^{(2)}}{\partial \varphi} - h\frac{\partial v^{(2)}}{\partial z}. \end{aligned} \tag{3}$$

It is assumed that the relation is satisfied $r/R < 1/5$, so the intersection line can be approximated by a circle. The first three kinematic conditions are obtained based on [11], the fourth (force) condition is equality of shear forces $S^{(1)} = -S^{(2)}$.

3. Reduced two-dimensional mathematical model

The original boundary value problem (1)–(3) is transformed in order to reduce the number of required functions in the system (1). To do this, we express from the third and sixth equations of displacement $w^{(1)}$ and $w^{(2)}$. We perform the substitution of these displacements in the first, second, fourth and fifth equations of the system (1), in boundary conditions and conjugation conditions. Thus, we obtain an alternative formulation of the original problem.

We transform the alternative problem into a Cartesian coordinate system to obtain a single displacement vector for the entire domain. To do this, we use expression of Cartesian coordinates through of cylindrical coordinates and formulas for replacing independent variables [12] for large and small pipes, respectively. By performing such a transformation with equations, boundary conditions, and conjugation conditions, we obtain a three-dimensional boundary value problem of pipeline equilibrium in Cartesian coordinates.

We project resulting three-dimensional problem on the symmetry plane xOz, using equation of large $y^2 + z^2 = R^2$ and small $y^2 + x^2 = r^2$ cylinders, taking into account $y > 0$, and replacing independent variables:

$$
\begin{array}{ll}
y = \sqrt{R^2 - z^2}; & y = \sqrt{r^2 - x^2}; \\
\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial x}, \quad \dfrac{\partial}{\partial y} = -\dfrac{\sqrt{R^2 - z^2}}{z}\dfrac{\partial}{\partial z}, & \dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial x}, \quad \dfrac{\partial}{\partial y} = -\dfrac{\sqrt{r^2 - x^2}}{x}\dfrac{\partial}{\partial x}, \\
\dfrac{\partial}{\partial z} = \dfrac{\partial}{\partial z}, \quad \dfrac{\partial^2}{\partial x \partial y} = -\dfrac{\sqrt{R^2 - z^2}}{z}\dfrac{\partial^2}{\partial x \partial z}, & \dfrac{\partial}{\partial z} = \dfrac{\partial}{\partial z}, \quad \dfrac{\partial^2}{\partial y \partial z} = -\dfrac{\sqrt{r^2 - x^2}}{x}\dfrac{\partial^2}{\partial x \partial z}, \\
\dfrac{\partial^2}{\partial x^2} = \dfrac{\partial^2}{\partial x^2}, \quad \dfrac{\partial^2}{\partial x \partial z} = \dfrac{\partial^2}{\partial x \partial z}, & \dfrac{\partial^2}{\partial x^2} = \dfrac{\partial^2}{\partial x^2}, \quad \dfrac{\partial^2}{\partial x \partial z} = \dfrac{\partial^2}{\partial x \partial z}, \\
\dfrac{\partial^2}{\partial y^2} = \dfrac{R^2 - z^2}{z^2}\dfrac{\partial^2}{\partial z^2} - \dfrac{R^2}{z^3}\dfrac{\partial}{\partial z}, \quad \dfrac{\partial^2}{\partial z^2} = \dfrac{\partial^2}{\partial z^2}. & \dfrac{\partial^2}{\partial y^2} = \dfrac{r^2 - x^2}{x^2}\dfrac{\partial^2}{\partial x^2} - \dfrac{r^2}{x^3}\dfrac{\partial}{\partial x}, \quad \dfrac{\partial^2}{\partial z^2} = \dfrac{\partial^2}{\partial z^2}.
\end{array} \tag{4}
$$

The result of the performed transformations will be two-dimensional equations:

$$
\begin{cases}
2R^2(1+\nu)\dfrac{\partial^2 u_x}{\partial x^2} + 4\left(R^2 - z^2\right)\dfrac{\partial^2 u_x}{\partial z^2} - 2z\dfrac{\partial u_x}{\partial z} - 2z\sqrt{R^2 - z^2}\,\dfrac{\partial^2 v_y}{\partial x \partial z} - \\
\qquad - \sqrt{R^2 - z^2}\,\dfrac{\partial v_y}{\partial x} + 2\left(R^2 - z^2\right)\dfrac{\partial^2 w_z}{\partial x \partial z} - z\dfrac{\partial w_z}{\partial x} = 0, \\
2\sqrt{R^2 - z^2}\,\dfrac{\partial^2 u_x}{\partial x \partial z} - z\dfrac{\partial^2 v_y}{\partial x^2} + \sqrt{R^2 - z^2}\,\dfrac{\partial^2 w_z}{\partial x^2} = 0, \\
R\nu\dfrac{\partial u_x}{\partial x} - 2z\dfrac{\sqrt{R^2 - z^2}}{R}\dfrac{\partial v_y}{\partial z} + 2\dfrac{R^2 - z^2}{R}\dfrac{\partial w_z}{\partial z} = \dfrac{1 - \nu^2}{EH}pR^2.
\end{cases} \tag{5}
$$

$$
\begin{cases}
2\left(r^2 - x^2\right)\dfrac{\partial^2 u_x}{\partial x \partial z} - x\dfrac{\partial u_x}{\partial z} - x\sqrt{r^2 - x^2}\,\dfrac{\partial^2 v_y}{\partial x \partial z} + x^2\dfrac{\partial^2 v_y}{\partial z^2} - \sqrt{r^2 - x^2}\,\dfrac{\partial v_y}{\partial z} + \\
\qquad + 4\left(r^2 - x^2\right)\dfrac{\partial^2 w_z}{\partial x^2} + 2r^2(1+\nu)\dfrac{\partial^2 w_z}{\partial z^2} - 2x\dfrac{\partial w_z}{\partial x} = 0, \\
\sqrt{r^2 - x^2}\,\dfrac{\partial^2 u_x}{\partial z^2} + \sqrt{r^2 - x^2}\,\dfrac{\partial^2 v_y}{\partial x \partial z} + 2\sqrt{r^2 - x^2}\,\dfrac{\partial^2 w_z}{\partial x \partial z} = 0, \\
2\dfrac{r^2 - x^2}{r}\dfrac{\partial u_x}{\partial x} - 2x\dfrac{\sqrt{r^2 - x^2}}{r}\dfrac{\partial v_y}{\partial x} + r\nu\dfrac{\partial w_z}{\partial z} = \dfrac{1 - \nu^2}{Eh}pr^2.
\end{cases} \tag{6}
$$

Note that equations (5) are projected on the symmetry plane of the large cylinder, and equations (6) on the symmetry plane of the small cylinder.

Then we transform the boundary conditions and the conjugation conditions in the same way. When moving to the two-dimensional formulation, we will need additional boundary conditions on the sections 2, 3, 5, 6, 8, see Figure 1. From the expression of the peripheral force in a large $N_\theta = pR$ and small $N_\varphi = pr$ pipe, we get one of the necessary conditions. Since the nature of the distribution of displacement v_y and the application of load is symmetric with respect to the plane xOz, we equate $v_y = 0$ it on the boundaries under consideration.

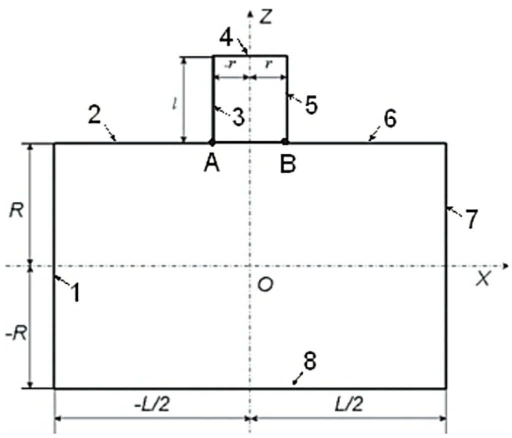

Figure 1: The middle surface projection of the cylinders on the symmetry plane

We divide the two rectangular domains of Figure 1 along the line AB. We will consider this line as the border between two rectangles. We will set boundary conditions on it by transforming the conjugation conditions. After substituting formulas (4), given that on the line AB $z = R$, the first conjugation condition is satisfied identically, and the left part of the second condition and the right part of the third condition is also zero. Thus, we get one boundary condition each for the small and large cylinders. The right and left parts of the fourth condition are equal to zero, since, based on the Vekua bushing connections [13, 14], the tangential stresses at the boundary become zero, and the fourth condition is a force condition.

Thus, in a rectangular domain, the following boundary conditions are obtained for a large cylinder:

$$\begin{aligned}
&(1) x = -L/2, -R \le z < R: && u_x = 0, && 2\sqrt{R^2 - z^2}\frac{\partial u_x}{\partial z} - z\frac{\partial v_y}{\partial x} + \sqrt{R^2 - z^2}\frac{\partial w_z}{\partial x} = 0;\\
&(2) z = R, -L/2 < x \le -r: && v_y = 0, && \frac{\partial u_x}{\partial x} = \frac{1}{\nu}\frac{1-\nu^2}{EH} pR;\\
&(AB) z = R, -r < x \le r: && \frac{\partial v_y}{\partial x} = 0, && \frac{\partial u_x}{\partial x} = \frac{1}{\nu}\frac{1-\nu^2}{EH} pR;\\
&(6) z = R, r \le x < L/2: && v_y = 0, && \frac{\partial u_x}{\partial x} = \frac{1}{\nu}\frac{1-\nu^2}{EH} pR;\\
&(7) x = L/2, -R < z < R: && u_x = 0, && 2\sqrt{R^2 - z^2}\frac{\partial u_x}{\partial z} - z\frac{\partial v_y}{\partial x} + \sqrt{R^2 - z^2}\frac{\partial w_z}{\partial x} = 0;\\
&(8) z = -R, -L/2 < x < L/2: && v_y = 0, && \frac{\partial u_x}{\partial x} = \frac{1}{\nu}\frac{1-\nu^2}{EH} pR,
\end{aligned} \tag{7}$$

for a small cylinder:

$$\begin{aligned}
&(3) x = -r, R \le z < R + l: && v_y = 0, && \frac{\partial w_z}{\partial z} = \frac{1}{\nu}\frac{1-\nu^2}{Eh} pr;\\
&(4) z = R + l, -r < x < r: && -\sqrt{r^2 - x^2}u_x + xv_y = 0, && \frac{\partial w_z}{\partial z} = -\frac{\nu}{Eh} pr;\\
&(5) x = r, R < z < R + l: && v_y = 0, && \frac{\partial w_z}{\partial z} = \frac{1}{\nu}\frac{1-\nu^2}{Eh} pr;\\
&(AB) z = R, -r \le x < r: && \sqrt{r^2 - x^2}\frac{\partial u_x}{\partial z} - x\frac{\partial v_y}{\partial z} + 2\sqrt{r^2 - x^2}\frac{\partial w_z}{\partial x} = 0, && \frac{\partial w_z}{\partial z} = \frac{1}{\nu}\frac{1-\nu^2}{Eh} pr.
\end{aligned} \tag{8}$$

4. Numerical example

We illustrate by a numerical example the permissibility of replacing the conjugation conditions with bushing connections. In the application package FreeCAD, two models of cylinders intersecting at right angles are constructed with the following parameters: $R = 37.5\,mm$, $r = 8.5\,mm$, $H = 5\,mm$, $h = 3\,mm$, $L = 100\,mm$, $l = 72\,mm$, steel S335JO, $\nu = 0.3$, $E = 2.1 \cdot 10^5\,MPa$, $\rho = 7800\,kg/m^3$, $[\sigma] = 510\,MPa$, $p = 10\,MPa$. The first model is built as a single system. The second model consists of two parts: a large cylinder with a hole and a small cylinder inserted into the large cylinder as a bushing. The results of numerical calculation for both models are presented in

Figure **2** in the form of the Mises stress distribution over mesh nodes and histograms. It can be seen that nature of this distribution is identical in both cases. The maximum stress value for the first model is 275.87 MPa. For the second model, this value is 268.35 MPa. The relative error of maximum stresses is 2.7%, which is less than simplifications that form the basis of the shell theory.

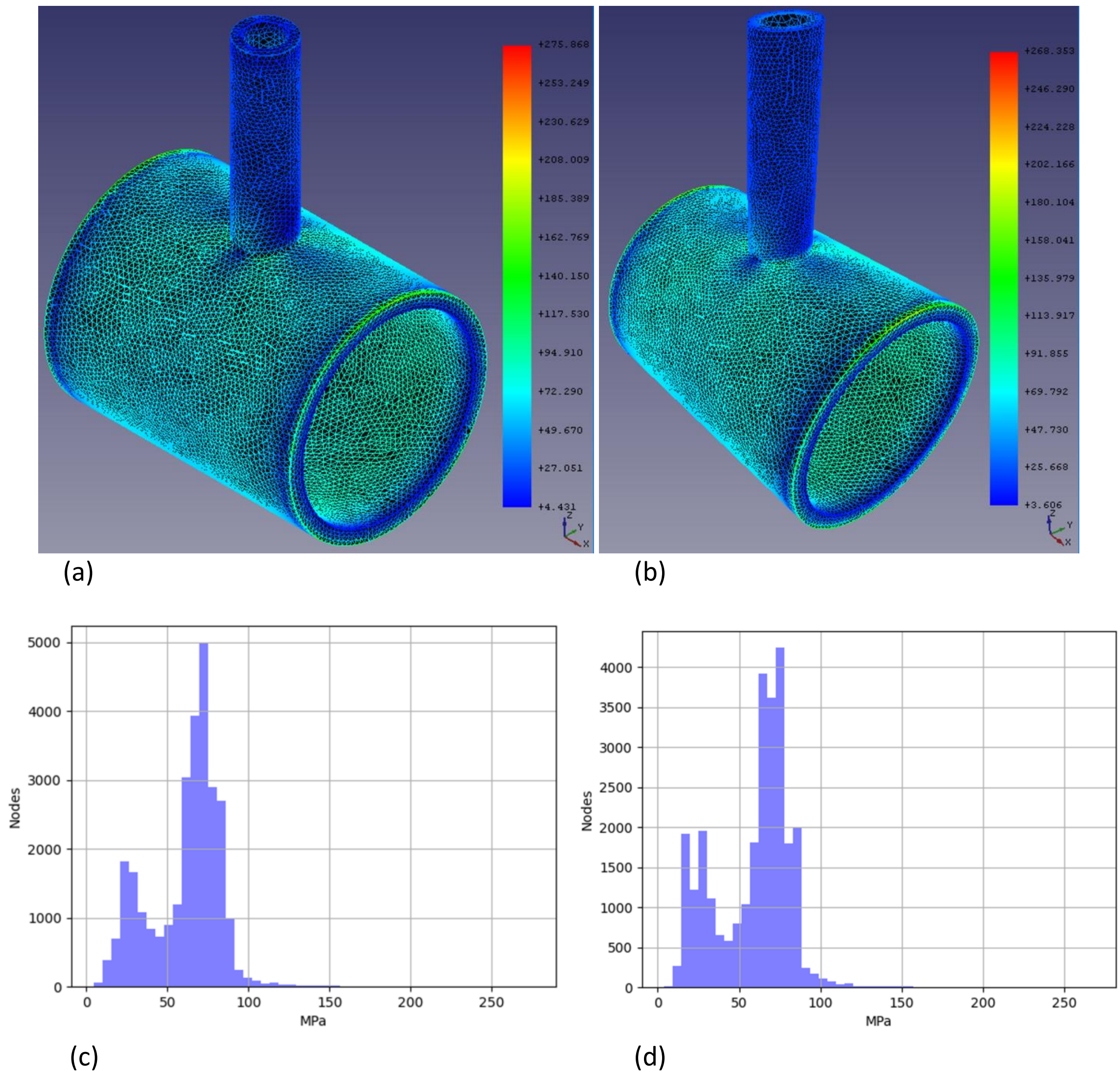

Figure 2: The stress distribution on the nodes of the mesh: (a) – for one-piece model of T-shaped pipe intersection, (b) – for the bushing model; histogram of the stress distribution over mesh nodes: (c) – for one-piece model of T-shaped pipe intersection, (d) – for the bushing model

5. Conclusion

The boundary value problem in displacements for two normally intersecting cylinders is formulated. A constructive algorithm for projecting the original three-dimensional problem onto the symmetry plane of a mechanical system is created. A reduced mathematical model in Cartesian coordinates is constructed. The scope of the model is limited by the ratio of the radii of the median surface of the shells $r/R < 1/5$. The final boundary value problem is divided into two problems: in the rectangular projection of large cylinder (5), (7) and in the rectangular projection of small cylinder (6), (8). In this case, conjugation conditions are finally eliminated from problem formulation. A numerical example is justified the permissibility of replacing the conjugation conditions with bushing coupling.

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