Spline Wavelets and Integration Operators

Elena P. Ushakova[a,b,c] and Sergey E. Zhukovskiy[a]

[a] V.A. Trapeznikov Institute of Control Sciences of RAS, 65 Profsoyuznaya street, Moscow, 117997, Russia
[b] Computing Center of FEB RAS, 65 Kim Yu Chen street, Khabarovsk, 680000, Russia
[c] Steklov Mathematical Institute of RAS, 8 Gubkina street, Moscow, 119991, Russia

Abstract

A class of spline wavelet systems of Battle-Lemarié type is discussed with application to the study of norm related inequalities in weighted Besov spaces involving integration operators.

Keywords 1

B-spline, Battle-Lemarié wavelet system, wavelet basis, Muckenhoupt weight, Besov space, Riemann-Liouville operators, norm inequalities.

1. Introduction

We are interested in a class of orthogonal spline wavelet systems of Battle-Lemarié type [1], [6]. Basing on the ideas taken from [8] and [9], the detailed description of such systems was established in [16] (see also [17] and [18]). Any Battle-Lemarié spline system of natural order n consists of a scaling function and a wavelet function - ϕ_n^{BL} and ψ_n^{BL}- satisfying (2) and (3), respectively. The ϕ_n^{BL} and ψ_n^{BL} are infinite linear combinations of B-splines (basic splines) of order n having supports' lengths equal to n+1. In contrast, the Battle-Lemarié scaling and wavelet functions have unbounded supports. A type of localization property was discovered for ϕ_n^{BL} and ψ_n^{BL}in [16]-[18]. The property is an algorithm resulting in new functions Φ_n^{BL} and Ψ_n^{BL}, which are particular finite linear combinations of integer shifts of ϕ_n^{BL} and ψ_n^{BL}, respectively. The point is that Φ_n^{BL} and Ψ_n^{BL} are compactly supported on $\mathbb{R}$. Moreover, similarly to the initial scaling and wavelet functions $\phi_n^{BL}$and ψ_n^{BL}, they form a Riesz basis in $L^2(\mathbb{R})$. These two facts allowed to establish in [17] and [18] new decomposition theorems for weighted Besov and Triebel-Lizorkin function spaces expressed in terms of Φ_n^{BL} and Ψ_n^{BL}.

This work is devoted to applications of Battle-Lemarié spline wavelet systems, as well as decomposition theorems mentioned above, to the study of norm related inequalities involving images and pre-images of integration operators of natural orders in weighted Besov spaces. Our results are based on the following differentiation (integration) property

$$B'_n(\cdot) = B_{n-1}(\cdot) - B_{n-1}(\cdot - 1),$$

which binds B-splines of natural orders between each other along the smoothness scale. Recall that B-splines B_n of order n are building blocks for the Battle-Lemarié scaling ϕ_n^{BL} and wavelet ψ_n^{BL} functions as well as for their compactly supported counterparts Φ_n^{BL} and Ψ_n^{BL}.

Connections between images and pre-images of integral and differential operators in Besov and Triebel-Lizorkin function spaces have been studied in [13, Theorem 2.3.8], [2], [11, Theorem 2.20], [5, § 4], [16, p. 23], [7]. In particular, a norm related inequality involving a differentiation operator of natural order was established in [11, Theorem 2.20] (see also [5, § 4]). In § 4 we extend this result to arbitrary smoothness parameter s of Besov spaces $B^s_{pq}(\mathbb{R}, w)$ with Muckenhoupt weights w (see

VI International Conference Information Technologies and High-Performance Computing (ITHPC-2021),
September 14–16, 2021, Khabarovsk, Russia
EMAIL: elenau@inbox.ru (A. 1
ORCID: 0000-0002-3497-3762 (A. 1)

Definitions 3.1 and 3.2). Our main result is a type of the reverse implications – inequalities which connect norms of images an pre-images of integration operators of natural orders in $B_{pq}^{s}(\mathbb{R},w)$.

Instruments for obtaining the results of our work are Battle-Lemarié spline wavelet systems. In § 2 we recall their exact formulae and describe their basic properties involving the localization one. Definition of weighted Besov spaces is given in § 3. New results are stated in § 4. For their proofs one can consult [19]. Characteristics of weighted discrete Hardy inequalities were also used for obtaining our statements in § 4. In settings of general measures they can be found in [10, § 1].

We use signs := and =: for determining new quantities. For positive functionals F and G we write $F \ll G$ if $F \le \alpha G$ with some constant $\alpha > 0$ depending, possibly, on irrelevant parameters only. Relations of the type $F \approx G$ mean $F \ll G \ll F$ or $F = \alpha G$.

2. Battle-Lemarié spline wavelet systems

Battle-Lemarié scaling functions are polynomial splines with simple knots at $\mathbb{Z}$ obtained by orthogonalisation process of the B-splines. For $n \in \mathbb{N}$ the n-th order B-spline is defined recursively by

$$B_n(x) = (B_{n-1} * B_0)(x) = \int_0^1 B_{n-1}(x-t)dt, \qquad \text{where} \qquad B_0 = \chi_{[0,1)}.$$

It is known [3] that supp $B_n = [0, n+1]$ and $B_n(x) > 0$ for all $x \in (0, n+1)$ (see Figure 1).

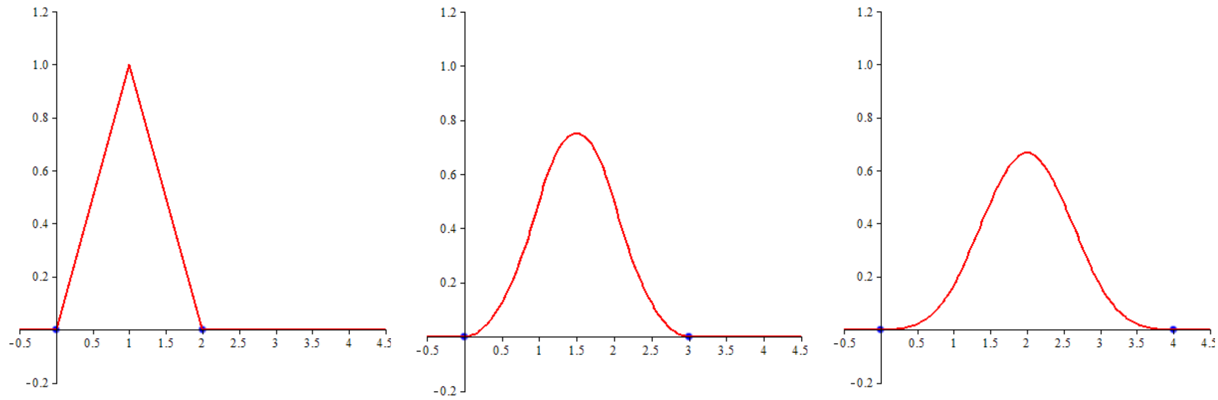

Figure 1: B-splines of orders n=1, n=2 and n=3, respectively.

For a function $g \in L^1(\mathbb{R})$ its Fourier transform has the form

$$\hat{g}(\omega) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-i\omega x} g(x)dx, \qquad x \in \mathbb{R}. \tag{1}$$

The Battle-Lemarié scaling function ϕ_n^{BL} must satisfy the condition:

$$\hat{\phi}_n^{BL}(\omega) = \hat{B}_n(\omega)\left(\sum_{m\in\mathbb{Z}} \left|\hat{B}_n(\omega + 2\pi m)\right|^2\right)^{-\frac{1}{2}}. \tag{2}$$

The n-th order Battle-Lemarié wavelet is a function ψ_n^{BL} whose Fourier transform is

$$\hat{\psi}_n^{BL}(\omega) = -e^{-i\omega/2} \frac{\overline{\hat{\phi}_n^{BL}(\omega + 2\pi)}}{\overline{\hat{\phi}_n^{BL}(\omega/2 + \pi)}} \hat{\phi}_n^{BL}(\omega/2). \tag{3}$$

Integer translations of ϕ_n^{BL} and ψ_n^{BL} form an orthonormal system within the multiresolution analysis of $L^2(\mathbb{R})$ generated by B_n ([3], [20]). To establish explicit forms of elements of Battle-Lemarié spline wavelet systems $\{\phi_n^{BL}, \psi_n^{BL}\}$ we fix $n \in \mathbb{N}$. For each $j = 1, \ldots, n$ we introduce $\alpha_j(n) > 1$ and define $r_j(n) := \left(2\alpha_j(n) - 1\right) - 2\sqrt{\alpha_j(n)\left(\alpha_j(n) - 1\right)} \in (0,1)$. The collection $\left\{\alpha_j(n)\right\}_{j=1}^{n}$ and, respectively, the set of numbers $\left\{r_j(n)\right\}_{j=1}^{n}$ are uniquely defined dependent on n (see [16, § 2, p. 179] so that the sequence $\left\{-r_j(n)\right\}_{j=1}^{n}$ is formed by the roots of Euler's polynomial

$$\sum_{m\in\mathbb{Z}} \left|\hat{B}_n(\omega+2\pi m)\right|^2 = \frac{\left|1+e^{\pm i\omega}r_1(n)\right|^2 \dots \left|1+e^{\pm i\omega}r_n(n)\right|^2}{2^{2n}\alpha_1(n)r_1(n)\dots\,\alpha_n(n)r_n(n)}. \tag{4}$$

Relying on (4), we define the n-th order Battle-Lemarié scaling function $\phi_n^{BL} = \phi_n$ via its Fourier transform as follows (see [16] and [18]):

$$\hat{\phi}_n(\omega) := \frac{2^n\sqrt{\alpha_1(n)r_1(n)\dots\,\alpha_n(n)r_n(n)}\,\hat{B}_n(\omega)}{(1+e^{i\omega}r_1(n))\dots(1+e^{i\omega}r_n(n))}. \tag{5}$$

The Fourier transform of a wavelet $\psi_n^{BL} = \psi_n$ related to ϕ_n has the form (see [16, § 2], [17, § 3.2])

$$\begin{aligned}\hat{\psi}_n(\omega) &:= \frac{\sqrt{\alpha_1(n)r_1(n)\dots\,\alpha_n(n)r_n(n)}}{2e^{i\omega 2}e^{i\pi(n+1)}}\\ &\times \frac{\left[\left|1-e^{i\omega/2}r_1(n)\right|^2\dots\left|1-e^{i\omega/2}r_n(n)\right|^2\right]\left(e^{i\omega/2}-1\right)^{n+1}\hat{B}_n(\omega/2)}{(1+e^{-i\omega}r_1(n))(1+e^{i\omega}r_1^2(n))\dots(1+e^{-i\omega}r_n(n))(1+e^{i\omega}r_n^2(n))}.\end{aligned} \tag{6}$$

The Fourier pre-images of (5) and (6) can be viewed in [16]-[19] (see also Figures 2 and 3 below).

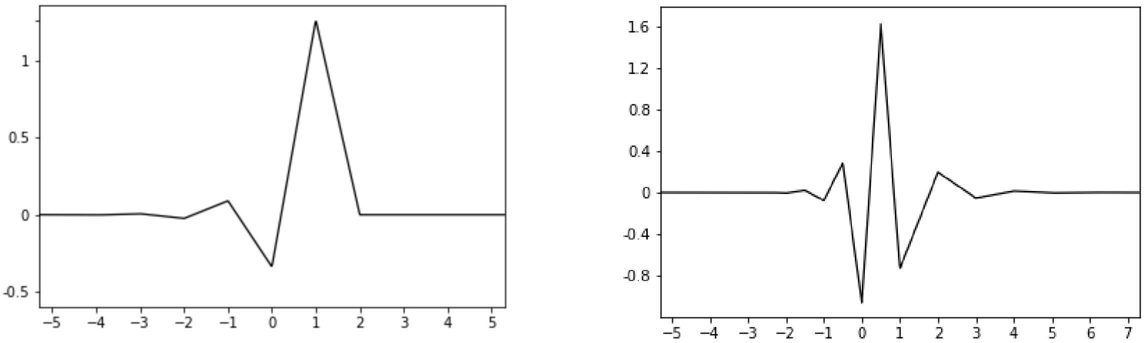

Figure 2: Graphs of $\phi_1$and ψ_1.

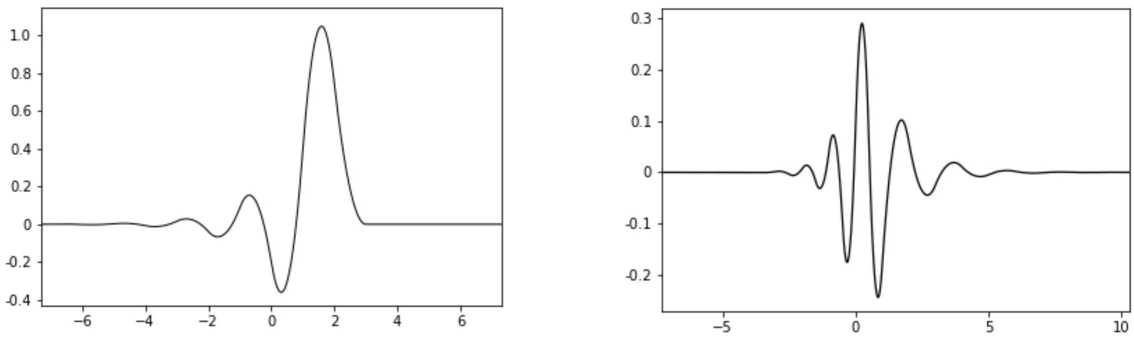

Figure 2: Graphs of $\phi_2$and ψ_2.

As it was already mentioned before, ϕ_n and ψ_n for all $n\in\mathbb{N}$ have unbounded supports on $\mathbb{R}$. Therefore, one can operate, if applicable, with their localized analogs Φ_n and Ψ_n, instead:

$$\begin{aligned}\hat{\Phi}_n(\omega) &:= \hat{\phi}_n(\omega)\left(1+e^{i\omega}r_1(n)\right)\dots\left(1+e^{i\omega}r_n(n)\right)\\ &= 2^n\sqrt{\alpha_1(n)r_1(n)\dots\,\alpha_n(n)r_n(n)}\,\hat{B}_n(\omega),\end{aligned} \tag{7}$$

$$\begin{aligned}\hat{\Psi}_n(\omega) &:= \hat{\psi}_n(\omega)\left(1+e^{-i\omega}r_1(n)\right)\left(1+e^{i\omega}r_1^2(n)\right)\dots\left(1+e^{-i\omega}r_n(n)\right)\left(1+e^{i\omega}r_n^2(n)\right)\\ &= (-1)^{n+1}r_1(n)\dots\,r_n(n)\sqrt{\alpha_1(n)r_1(n)\dots\,\alpha_n(n)r_n(n)}\\ &\times\left|1-e^{-i\omega/2}r_1(n)\right|^2\dots\left|1-e^{-i\omega/2}r_n(n)\right|^2\left(e^{i\omega/2}-1\right)^{n+1}\hat{B}_n(\omega/2).\end{aligned} \tag{8}$$

There exist some alternative versions of (7) and (8) (see [16]-[19]). They were applied in [17], [18] as dictionaries for decomposing elements of weighted Besov and Triebel-Lizorkin function spaces.

3. Weighted Besov function spaces

3.1. Muckenhoupt weights $\mathcal{A}_p, 1 \leq p \leq \infty$

Let w be a locally integrable function, positive almost everywhere (a weight) on $\mathbb{R}$.

Let $L^r(\mathbb{R})$, $0 < r \leq \infty$, denote the Lebesgue space of all measurable functions f on $\mathbb{R}$ quasi-normed by $\|f\|_{L^r(\mathbb{R})} := \left(\int_{-\infty}^{\infty} |f(x)|^r \, dx\right)^{1/r}$ with the usual modification if $r = \infty$. For $p > 1$ we put $p' := p/(p-1)$. Let $B \subset \mathbb{R}$ be a ball in $\mathbb{R}$, and $|B|$ stand for its volume.

Definition 3.1. [12, Chapter V] (i) A weight w belongs to the Muckenhoupt class $\mathcal{A}_{\mathrm{p}}$, $1 < p < \infty$, if

$$\mathcal{A}_p(w) := \sup_{B \subset \mathbb{R}} \left(\frac{1}{|B|} \int \chi_B w\right)^{1/p} \left(\frac{1}{|B|} \int \chi_B w^{1-p'}\right)^{1/p'} < \infty;$$

(ii) $w \in \mathcal{A}_1$ if

$$\mathcal{A}_p(w) := \sup_{B \subset \mathbb{R}} \frac{1}{|B|} \int \chi_B w \, \|1/w\|_{L_\infty(B)} < \infty;$$

(iii) Muckenhoupt class $\mathcal{A}_\infty$ is given by $\mathcal{A}_\infty := \bigcup_{p \geq 1} \mathcal{A}_p$.

For basic properties and examples of weights from $\mathcal{A}_\infty$ we refer to [12] (see also references there).

3.2. Besov spaces with Muckenhoupt weights

For $0 < p < \infty$ and a weight w on $\mathbb{R}$ we denote $L_w^p(\mathbb{R})$ the weighted Lebesgue space quasi-normed by $\|f\|_{L_w^p(\mathbb{R})} := \left\|w^{1/p} f\right\|_{L^p(\mathbb{R})}$ with usual modification if $p = \infty$.

For the definitions of unweighted Besov spaces $B_{pq}^s(\mathbb{R})$ we refer to [13] and [14]. Their weighted counterparts $B_{pq}^s(\mathbb{R}, \mathrm{w})$ with $\mathrm{w} \in \mathcal{A}_\infty$ can be introduced in framework of the Schwartz space $\mathcal{S}'(\mathbb{R})$.

Let $\mathcal{S}(\mathbb{R})$ be Schwartz space of all complex-valued rapidly decreasing, infinitely differentiable functions $C^\infty(\mathbb{R})$. By $\mathcal{S}'(\mathbb{R})$ we denote its topological dual, the space of tempered distributions on $\mathbb{R}$. For $\varphi \in \mathcal{S}(\mathbb{R})$ the inverse Fourier transform $\check{\varphi}$ is given by the right hand side of (1) with i in place of $-i$. Both $\hat{\varphi}$ and $\check{\varphi}$ are extended to $\mathcal{S}'(\mathbb{R})$ in the standard way.

To give a definition of Besov spaces with Muckenhoupt weights we fix $\varphi_0 = \varphi \in \mathcal{S}(\mathbb{R})$ such that

$$\operatorname{supp} \varphi \subset \{y \in \mathbb{R} : |y| < 2\} \quad \text{and} \quad \varphi(y) = 1 \quad \text{if} \quad |y| \leq 1,$$

and let $\varphi_d(y) = \varphi\left(\frac{x}{2^d}\right) - \varphi\left(\frac{x}{2^{d-1}}\right), d \in \mathbb{N}$. Then $\{\varphi_d\}_{d=0}^\infty$ is a smooth dyadic resolution of unity.

Definition 3.2. Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$, $\mathrm{w} \in \mathcal{A}_\infty$ and $\{\varphi_d\}_{d=0}^\infty$ be a smooth dyadic resolution of unity. Weighted Besov space $\mathrm{B}_{\mathrm{pq}}^{\mathrm{s}}(\mathbb{R}, \mathrm{w})$ is the collection of all $\mathrm{f} \in \mathcal{S}'(\mathbb{R})$ such that

$$\|f\|_{B_{pq}^s(\mathbb{R},w)} := \left(\sum_{d=0}^\infty 2^{dsq} \left\|\widetilde{(\varphi_d \hat{f})}\right\|_{L_w^p(\mathbb{R})}^q\right)^{1/q} \tag{9}$$

(with the usual modification if $q = \infty$) is finite.

Definition of the above space is independent of the choice of φ, up to equivalence of quasi-norm (9). The theory and properties of $B_{pq}^s(\mathbb{R}, w)$ can be found in [4].

4. Inequalities for integration operators

For $f \in L_{loc}^1(\mathbb{R})$ we consider the left- and the right-hand side Riemann-Liouville operators

$$I_+^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} f(y)\, dy, \qquad x > 0, \tag{10}$$

and

$$I_-^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_x^0 (y-x)^{\alpha-1} f(y)\, dy, \qquad x < 0, \tag{11}$$

of positive orders α. For natural $\alpha = l$ the I_+^l and I_-^l are integration operators of natural orders.

The main results of this work are inequalities between norms of images and pre-images of operators (10) and (11) in $\mathrm{B}_{\mathrm{pq}}^{\mathrm{s}}(\mathbb{R}, \mathrm{w})$. For simplicity we assume that $\mathrm{f} \equiv 0$ outside of $\operatorname{supp} \mathrm{I}_\pm^\alpha \mathrm{f}$.

Theorem 4.1. [19] *Let* $1 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, *weights* $u, w \in \mathcal{A}_\infty$ *and* $f \in L_{loc}^1(\mathbb{R})$. *For* $\alpha \in \mathbb{N}$ *let* I_+^α *be defined by (10). Suppose that* $f(x) \equiv 0$ *for* $x < 0$.

(i) *Assume that for the both* $v = u$ *and* $v = w$ *it holds*

$$\int_{m/2^{d-1}}^{m/2^d} v \approx v(m/2^d)/2^d \quad for\ all \quad d \in \mathbb{N} \quad and\ \ any\ \ m \in \mathbb{Z}. \tag{12}$$

Then $I_+^\alpha f \in B_{pq}^s(\mathbb{R}, w)$ *if* $f \in B_{pq}^{s+l}(\mathbb{R}, u)$ *provided*

$$\mathcal{N}_+ := sup_{n\ge 0} \left(\sum_{m \ge n} (m-n+1)^{p(2l-1)} w(m) \right)^{\frac{1}{p}} \left(\sum_{0 \le m \le n} [u(m)]^{1-p'} \right)^{\frac{1}{p'}}$$
$$+ sup_{n\ge 0} \left(\sum_{m \ge n} w(m) \right)^{\frac{1}{p}} \left(\sum_{0 \le m \le n} (m-n+1)^{p'(2l-1)} [u(m)]^{1-p'} \right)^{\frac{1}{p'}} < \infty.$$

Moreover,

$$\|I_+^\alpha f\|_{B_{pq}^s(\mathbb{R},w)} \ll C \|f\|_{B_{pq}^{s+l}(\mathbb{R},u)}, \qquad where \qquad C := \mathcal{N}_+.$$

(ii) *If* $I_+^\alpha f \in B_{pq}^s(\mathbb{R}, w)$ *then* $f \in B_{pq}^{s-l}(\mathbb{R}, w)$, *besides,*

$$\|f\|_{B_{pq}^{s-l}(\mathbb{R},w)} \ll \|I_+^\alpha f\|_{B_{pq}^s(\mathbb{R},w)}.$$

Analogous result is valid for the left-hand side Riemann-Liouville operator I_-^α.

Theorem 4.2. [19] *Let* $1 < p < \infty$, $0 < q \le \infty$, $s \in \mathbb{R}$, *weights* $u, w \in \mathcal{A}_\infty$ *and* $f \in L_{loc}^1(\mathbb{R})$. *For* $\alpha \in \mathbb{N}$ *let* I_-^α *be defined by (11). Suppose that* $f(x) \equiv 0$ *for* $x > 0$.

(i) *Assume that (12) holds for* $v = u$ *and* $v = w$. *Then* $I_-^\alpha f \in B_{pq}^s(\mathbb{R}, w)$ *if* $f \in B_{pq}^{s+l}(\mathbb{R}, u)$ *provided*

$$\mathcal{N}_- := sup_{n\le 0} \left(\sum_{m \le n} (n-m+1)^{p(2l-1)} w(m) \right)^{\frac{1}{p}} \left(\sum_{n \le m \le 0} [u(m)]^{1-p'} \right)^{\frac{1}{p'}}$$
$$+ sup_{n\le 0} \left(\sum_{m \le n} w(m) \right)^{\frac{1}{p}} \left(\sum_{n \le m \le 0} (m-n+1)^{p'(2l-1)} [u(m)]^{1-p'} \right)^{\frac{1}{p'}} < \infty.$$

Moreover,

$$\|I_-^\alpha f\|_{B_{pq}^s(\mathbb{R},w)} \ll C \|f\|_{B_{pq}^{s+l}(\mathbb{R},u)}, \qquad where \qquad C := \mathcal{N}_+.$$

(ii) *If* $I_-^\alpha f \in B_{pq}^s(\mathbb{R}, w)$ *then* $f \in B_{pq}^{s-l}(\mathbb{R}, w)$, *besides,*

$$\|f\|_{B_{pq}^{s-l}(\mathbb{R},w)} \ll \|I_-^\alpha f\|_{B_{pq}^s(\mathbb{R},w)}.$$

The results of this section can be generalized to some other cases of integration operators and various types of weighted function spaces (see [19, Theorems 5.2, 5.3 and Remark 5.4]).

5. Acknowledgements

The research work of the first author related to Section 4 was performed at Steklov Mathematical Institute of Russian Academy of Sciences under financial support of the Russian Science Foundation (project 19-11-00087).

The studies were carried out using the resources of the Center for Shared Use of Scientific Equipment "Center for Processing and Storage of Scientific Data of the Far Eastern Branch of the Russian Academy of Sciences", funded by the Russian Federation represented by the Ministry of Science and Higher Education of the Russian Federation under project No. 075-15-2021-663.

6. References

[1] G. Battle. "A block spin construction of ondelettes, Part I: Lemarié functions." Comm. Math. Phys. 110 (1987): 601-615.
[2] M. Z. Berkolaiko, I. Ya. Novikov. "Images of wavelets under the action of convolution operators." Math. Notes 55.5-6 (1994): 446-454.
[3] C.K. Chui, An Introduction to Wavelets, Academic Press, NY, 1992.
[4] D.E. Edmunds and H. Triebel, Function spaces, entropy numbers, differential operators, Cambridge University Press, 1996.
[5] M. Izuki, Y. Sawano. "Atomic decomposition for weighted Besov / Triebel-Lizorkin spaces with local class of weights." Math. Nachr. 285 (2012): 103-126.
[6] P.G. Lemarié-Rieusset. "Ondelettes et poids de Muckenhoupt." Studia Math. 108.2 (1994): 127-147.
[7] M.G. Nasyrova, E.P. Ushakova. "Wavelet basis and entropy numbers of Hardy operator." Anal. Math. 44.4 (2018): 543-576.
[8] I. Ya. Novikov, S. B. Stechkin. "Basic constructions of wavelets." Fundam. Prikl. Mat. 3.4 (1997): 999-1028.
[9] I. Ya. Novikov, S. B. Stechkin, "Basic wavelet theory." Russian Math. Surveys 53.6 (1998): 1159-1231.
[10] D. V. Prokhorov, V .D. Stepanov, E. P. Ushakova. "Hardy-Steklov integral operators: Part I." Proc. Steklov Inst. Math. 300 (2018): 1-112.
[11] V. S. Rychkov. "Littlewood-Paley theory and function spaces with $\mathrm{A}_{\mathrm{p}}^{\mathrm{loc}}$ weights." Math. Nachr. 224 (2001): 145-180.
[12] E.M. Stein, Harmonic analysis, Princeton University Press, Princeton, 1993.
[13] H. Triebel, Theory of Function Spaces, Birkhäuser Verlag, Basel, 1983.
[14] H. Triebel, Theory of function spaces II, Birkhäuser Verlag, Basel, 1992.
[15] H. Triebel, Tempered Homogeneous Function Spaces, European Math. Soc. Publishing House, Zurich, 2015.
[16] E. P. Ushakova, K. E. Ushakova. "Localisation property of Battle–Lemarié wavelets sums." J. Math. Anal. Appl. 461.1 (2018): 176–197.
[17] E. P. Ushakova. "Spline wavelet bases in function spaces with Muckenhoupt weights." Rev. Mat. Complut. 33 (2020): 125-160.
[18] E. P. Ushakova "Spline Wavelet Decomposition in Weighted Function Spaces." Proc. Steklov Inst. Math. 312 (2021): 301–324.
[19] E. Ushakova, Images of integration operators in weighted function spaces, 2020. URL: https://arxiv.org/abs/2011.14981.
[20] P. Wojtaszczyk, A Mathematical Introduction to Wavelets, Cambridge University Press, Cambridge, UK, 1997.