The Spot Model for Representation and Processing of Qualitative Data and Semantic Information

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Abstract

This article discusses the basis of the mathematical apparatus of the recently proposed mathematical model of spots [1], which represent vague regions. The main feature of our approach is that both the shape of the spot and its surrounding space can be initially unknown. They can be only identified using qualitative information about the spot's elementary relations with other spots. We regard that crisp figures are a special limiting case of spots that provides a way for the creation of a theory of "qualitative geometry".

The suggested model is considered as a mathematical object that is adequate for the representation of qualitative data or the semantic aspect in data of any nature. Although the concept under consideration is substantially new, it shares common ideas with other theories, related to the vagueness aspect, including Region Connection Calculus, Fuzzy Sets, Rough Sets, Soft Sets, and Granular Computing.

The present work introduces L4 numbers, which are 2x2 logical tables, for the description of elementary relations between spots. Based on them we also define L4 vectors, L4 matrices, and corresponding mathematical operations that can be a base for apparatus of Granular Computing. At least, developed mathematical model is a movement towards creation of Artificial General Intelligence and can be used for knowledge representation and processing, for modeling qualitative reasoning, learning, and natural language processing.

The introduced apparatus was verified by solving problems of the image reconstruction of *unknown* plane crisp figures, utilizing only qualitative data of its elementary relations with many other *known* figures.

Keywords

Mental representation, semantic images, semantic information, vague figures, qualitative data, Artificial Intelligence, Granular Computing

1. Introduction

In this paper, the basis of the mathematical apparatus of the recently proposed new mathematical model of spots for the presentation and processing of *qualitative information* is considered [1]. In particular, this theory offers a universal basis for representing the *semantic information* of human knowledge and thought in the form of "semantic space". The spot is a model of a vague spatial object, which allows us to represent and process qualitative (non-numerical) data starting from an extremely low, elementary, information level. On the other hand, the considered mathematical theory can represent the vagueness of the *categories* and *mental images* that allows us to model human reasoning in Artificial Intelligence (AI).

The notion of a *mental representation* is a basic concept of the Computational Theory of Mind and can be in general interpreted as a mental object with semantic properties [2]. The *concepts* are regarded as the building blocks of thoughts that are crucial to such psychological processes as *categorization,*

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inference, memory, learning, and decision-making. One of the basic models for representations of concepts or ideas is *mental images*.

Kosslyn [3] regards mental representations as mental imagery that themselves have spatial properties – i.e., pictorial representations, or images, which are not literally pictures in the head but represent in a way that is relevantly like the way pictures represent [4]. Mental imagery has often been believed to play a big, even pivotal, role in both memory and motivation. According to a long-dominant philosophical tradition, it plays a crucial role in all thought processes and provides the semantic grounding for language [5]. Authors of [6, 7] investigated a similar concept - *secondary images,* which are *mental images* that formed in previous experience and embodied in concepts extracted from memory, or representations, accumulating various semantics in themselves.

Intuitively, the spots model is based on the qualitative and vague properties of the mind or *semantic images,* which models concepts or ideas and are objects of research in the Representational Theory of Mind and Cognitive Neuropsychology [2−7]. In this paper, we will apply the term *semantic images* which are not direct perception images (visual, sounds, or smells), but represent the meaning, and consider the following concept of them. More generally, we regard *semantic images* as objects for representation the *semantic information* [8, 9] and as building elements for the *semantic space*.

The creation of any *semantic image* is associated with the understanding of some object or phenomenon, while:

- each concept corresponds to a *semantic image,*
- each word corresponds to a *semantic image* (or several, for polysemous words, such as a spring, etc.),
- each phrase corresponds to a *semantic image* or several *images,*
- the text of a book or article forms a system of *semantic images*.

We consider that *semantic images* have elementary spatial properties, corresponding to some vague regions located in a vague environment. The spatial properties of concepts are intuitively reflected in natural language. For example, we are talking about the facets and different sides of a subject matter, about points of view, about the area of knowledge, about the proximity or remoteness of concepts, about the connection of concepts, or about the fact that two concepts in question are not connected or from different areas, etc. The obvious geometric analogy to relations of concepts of *general* and *specific* is the relation between the geometrical figure and its part: a more general concept includes specific concepts. This corresponds to the *taxonomy* in biology. For example, the class of ray-finned fish is part of the animal kingdom but is divided into orders, families, genera, and species of fish. Similarly, the *taxonomy* of concepts is introduced in AI [10, p. 440].

It should be also noted that even complex concepts of the semantic content of colloquial speech or text can have a simple geometric interpretation. For example, the dependence of the meaning of the narrative on the context is similar to a model where identical geometric figures have different positions in space. Then the meaning of a statement is represented not only by the “shape of the figure”, but also by its environment in the *semantic space* (that is, in context).

The proposed theory turned out to be ideologically close to the research directions of mereotopology and qualitative special representation, the idea of which was laid down by Whitehead in 1929 [11–18]. On the other hand, basic ideas of the spots model are also close to the rough set theory [28-33], the formal concept analysis [32], and the fuzzy set theory [34], including fuzzy geometry [35, 36], and also to the soft set model [37, 38]. Moreover, the suggested concept is in good agreement with the ideology of Granular Computing [39-48].

The mereotopology has been developed particularly within the research community of the Qualitative Spatial Reasoning (QSR), which is an approach that excludes the processing of quantitative data. One of the important fields of mereotopology is Region Connection Calculus (RCC) [15, 16], which defines eight qualitative spatial relations between regions (RCC-8), including disconnection, overlapping, part of, external connection, and some others. Although RCC-8 relations represent qualitative information, traditionally, regions are crisp continuous-space models. However, some papers consider vague regions and discrete quantitative data [17, 18].

Bennett [19], and then Jonsson and Drakengren [20] considered a shortened version of these relations - RCC-5, which does not include a connection (touch) of the boundaries of regions. The peculiarity of their approach is the consideration of calculus for spatial regions with indeterminate boundaries when it is impossible to distinguish interior points from boundary points there.

Although most authors considered spatial relations as binary elements, Egenhofer et al. [21, 22] encoded them in form of logical tables. Namely, they introduced the concepts of 4-intersection [21] and 9-intersection [22] matrixes, which are logical matrices that encode topological spatial relations between spatial regions. Notice that these matrixes are similar but differ from the L4 numbers I have proposed in that they also include relations with the boundaries of the regions (see below Par. 2.2). Clementini et al. [23] generalized 9-intersection matrixes, replacing intersections for the crisp boundary with the intersections for broad boundaries. Stell [24] also considered the way of representation for spatial relations using 3x1 logical vectors created on the base of notions *part* and *compliment* only. At least, Butenkov [24] considered a 2x2 logical table for the Cartesian granules that is equivalent to my definition of L4 number for spots and applied them in algorithms for intelligent data analysis.

There are many publications dedicated to the creation of vague mathematical objects, including vague regions [17, 25-48]. In particular, the vague regions are regarded as models for certain indeterminacy of spatial data in the field of spatial reasoning and are applied in several domains such as computer vision, robot navigation, image information systems, and GIS [25].

Rough sets of Z. Pawlak [28-33] are based on the concept of indiscernibility that provides the creation of granules from subsets. Then a rough set is defined as an image of a subset that is plotted on these granules. For the rough set, Pawlak introduced the regions of lower approximation, the upper approximation, and the boundary. From the point of view of the spot model, the rough set is an image of some subset on the granules basis.

The egg-yolk model [15, 27] in the RCC domain is a vague region model, which consists of an inner part, an outer part, and an indeterminate boundary. “The egg is the maximal extent of a vague region and the yolk is its minimal extent, while the white is the area of indeterminacy” [15]. That is, the egg-yolk model of the vague region has a similar structure to the rough set.

A similar model of the vague region was considered in the paper of Clementini et al. [23] as a region with a broad boundary, which is defined as a region as well. Authors defined an “inner boundary and an outer boundary, where the inner boundary is surrounded by the outer boundary”.

The soft set model [37, 38] is, in essence, a collection of mappings of the set under consideration onto subsets of another set, which are considered approximations of the mapped set. This is analogous to the representation of spot in the form of its representation onto a set of base spots.

Lotfi Zadeh in [39, 40] firstly carried out a general formulation and consideration of granules, including the problem of information granulation, which was later called the concept of granular computing. His definition of granules: “the information may be said to be granular in the sense that the data points within a granule have to be dealt with as a whole rather than individually” corresponds to the equivalence classes of the universe. Zadeh regards both crisp and fuzzy granules and “considers granular computing as a basis for computing with words, i.e., computation with the information described in natural language” [41].

Elements of a granule are indiscernible and “depends on available knowledge” [46] that is similar to the spots concept. The importance of the application of granulation and granular computing relates to the fact that such approximation leads to simplification in solving practical problems. Note that the concept of the spots fits very well into all details of the granules concept which will become clear in below Par. 2.

The graph model is convenient for the representation of the structure of the relations between elements of the system under study [49] and nowadays is widely used in AI. For example, graph theory is actively used to define semantic properties of network objects, modeling *semantic networks*, which are used in the *knowledge base* (KB); such KB is called Knowledge graphs [50-52]. Note that, unlike spots, the graph is not a specific spatial object and is only an abstraction for the representation of the relations of the structure between the entities.

Despite the ideological closeness with other considered theories, the proposed theory has a significantly different base. The main difference is that the spots model is not defined on the base of the sets or fuzzy sets and developed for the representation of qualitative information, starting from the elementary level of information. Whereas the graphs are discrete mathematical objects, the spots combine both the properties of discreteness and continuity. Nevertheless, the existence of mentioned close mathematical models permits us to compare, study, and apply some approaches and ideas.

2. Definitions and Apparatus of Spots

2.1. Basic Philosophy of Spots

We want to define spots as mathematical objects that have elementary spatial properties. Therefore, spots can be defined as mathematical objects for which the following attributes exist:

1. *interior* and *environment* that do not intersect;
2. for any two spots connection between their internal parts and environments is defined.

Such a definition permits to regard as spots the following mathematical models:

- crisp and fuzzy subsets;
- geometric figures and regions;
- rough sets;
- granules;
- soft sets.

However, as far as the main aim of the spots theory is to model vague space objects, we introduced peculiar principles for constructing the spots that permit the expansion of the list of existing mathematical models. Among the basic principles of building a spot model are the following.

- The spot is considered as a mathematical object with elementary spatial properties 1.-2 but it does not contain elements like the set. The inner spatial structure of a spot and its environment is determined using their relations with other spots; it is analogous to a projection image. From this point of view, for example, the sets can be regarded as spots represented on the basis of spots-elements. Thus, the set itself and its elements are considered equal mathematical objects (spots).
- The "shape" of the spot is determined by the imaging of this spot on the spots of some basis. This allows you to represent incomplete information about the shape and structure of objects.
- There is a possibility of increasing the crispness of the spot's image, using additional data on its relations with other spots.

In contradiction with traditional geometry, the spot can not possess prior full information about its shape and environment, which is a spot as well. For example, the space dimensions or curvature of the environment can not be initially defined. Instead, the proposed theory allows us to obtain only qualitative information about the spot and its environment in the form of their elementary spatial relations with some other known spots. Note, that a large amount of such qualitative data allows extracting even numerical information. Therefore, we will consider crisp geometric objects or regions as a special, limiting case of spots. To complete the model, we introduce zero spots that occupy zero regions of space. Notice that a zero environment corresponds to the case when the spot occupies a whole space.

Let us introduce the basis of spots as a set of known spots that can be in some mutual relations. The representation of a spot by their elementary relations with the basis spots we call the mapping or image of the spot on this basis. Note that the system of basis spots is similar to the system of basis functions and the orthogonality of basis functions is analogous to the separated basis spots. Therefore, we call separated spots orthogonal spots.

From a certain point of view, the spot can be compared with some "volumetric object", which is defined by its different images or projections on different bases. Hence, the synthesis of the spot images on various bases into the "volumetric" image provides to improve knowledge of its shape.

Indiscernibility is one of the fundamental concepts for the spots model that is similar to that of the rough set theory. Two spots are considered indiscernible on some basis of spots if their elementary relations coincide with each spot of the basis and with its environment. In other words, two spots are indiscernible on a basis if their images on this basis are the same. Note that indiscernibility, in contrast to equality, is a relative concept and can be a consequence of insufficient information about the spot. For example, the optics analogy is the indiscernibility of small objects in an unfocused image. Note that concerning a certain basis, the spot can be indiscernible with an infinite number of crisp figures.

The following simple conclusions can be drawn from the concept of the indiscernibility of spots.

1) Indiscernible spots on one basis may be discernible on another basis.

2) If two spots on some basis are indiscernible, and we introduce additional spots to this basis, these two spots can become discernible on a new basis.

3) If two spots are discernible on some basis, then after removing some spots from the basis, they can become indiscernible on a new basis.

These properties of spots are also in agreement with common sense when the indiscernibility of two objects is usually associated with a lack of information about them, but when additional information is obtained, they can become discernible.

The relative property of indiscernibility permits us to extend this understanding to the relativity of the concept of the correct conclusion which depends on available information. Based on our knowledge, we make some inferences, which can be true or false depending on used inference rules. But if to consider some additional data, even true conclusions can become false. This understanding is fully consistent with the ideology of non-monotone reasoning in AI [10] and is in full agreement with the process of human cognition and analysis, in particular, the work of archaeologists, paleontologists, historians, detectives, and analysts.

The principle that information of a spot is determined using their relations with other spots is analogous to the case that the semantic meaning of a word or concept can be determined using the meanings of other concepts, as is done in an explanatory dictionary. In more detail, the semantic meaning is determined by both the details of the concept and its context, which representation with spots is depicted in Figure 1. All this suggests the possibility of using the proposed theory for modeling semantic images and semantic information in Artificial General Intelligence.

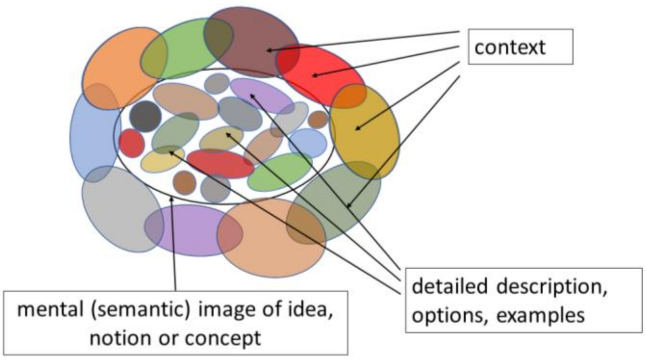


Figure 1: Euler-Venn diagrams [53] for the representation of the semantic meaning using spots model.

2.2. Definition of L4 numbers, L4 vectors, and L4 matrixes

The mathematical model of spots is based on the concept of logical connection between spots that is like that in the mereotopology. However, as far as the model of spots is not based on the set theory, we define connection as the fact of the presence of a common part between spots. The connection of two spots a and b, which we denoted as ab can be equal to logical 1 or logical 0, depending on the connection fact.

Elementary relations of two spots can be defined, based on the notation of connection between spots a, b and their environments that we denoted by $\tilde{a}, \tilde{b}$. Axiomatically, we regard that spots do not connect their environments, that is

$$a\tilde{a} = 0, \qquad b\tilde{b} = 0\,, \tag{1}$$

Let us define the elementary relations between spots a and b, which is denoted by a symbol $\langle a|b\rangle$, using a 2x2 logical table composed of the following logical connections between spots and their environments:

$$\langle a|b\rangle = \begin{bmatrix} ab & a\tilde{b} \\ \tilde{a}b & \tilde{a}\tilde{b} \end{bmatrix} \tag{2}$$

Such logical tables, which we call L4 numbers, make it possible to distinguish 16 different elementary relations between spots. The important point is that we regard L4 numbers as universal logical numbers for the representation of *qualitative information* [14], including *semantic information* [8, 9]. Examples of elementary relations described with L4 numbers are shown in Table 1. We call these spatial relations as elementary relations because they carry low-level quality information about spots. However, a large amount of such data allows you to display quality information of a higher level, and even numerical.

Table 1
Some elementary relations of spots

Elementary Relations	$\langle a\|b\rangle$
Intersection, $a >< b$	$\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$
Separation, $a <> b$	$\begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}$
Inclusion (more), $a > b$	$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$
Inclusion (less), $a < b$	$\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$
Indiscernibility, $a = b$	$\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

Notice that relations of spots presented in Table 1 are equivalent to RCC-5 relations [19, 20]. However introduced L4 numbers permits the representation of up to 16 elementary relations between spots, including relations for zero spots and zero environments.

As it was noticed above, indiscernibility is a relative concept and has the following definition. Two spots a and b are indiscernible on certain spots basis $X = \{x_i\}$ that we denote by $(a = b)_X$ if

$$(a = b)_X \stackrel{\text{def}}{=} \forall x_i \in X\ (\langle a|x_i\rangle = \langle b|x_i\rangle) \tag{3}$$

The proposed theory uses analogies with matrix analysis. For example, elementary relations of spots, analogous to the dot product of numerical vectors where L4 numbers are used instead of reals. Following this idea, we define the L4 vector representation for the spot (4), elements of which are L4 numbers of the spot relations with the basis spots. Note that such elements of the L4 vector correspond to "coordinates" of the spot on the basis and the L4 vector describes a mapping of the spot on this basis. For example, the image of a spot a on the basis $\{x_i\}$ can be represented in the form of a vector $\boldsymbol{a}$ with L4 coordinates:

$$\boldsymbol{a} \equiv [\alpha_1;\ \alpha_2; \dots \alpha_n] \tag{4}$$

where L4 numbers $\alpha_i = \langle a|x_i\rangle$ and symbol "; " denotes the column vector $\boldsymbol{a}$ representation.

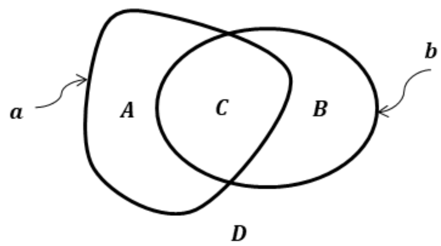


Figure 2: Euler-Venn diagram for the elementary relations between spots.

A system of spots with elementary relations can be represented as a graph where the vertices correspond to the spots and the edges labeled by L4 numbers. Note that this is similar to the weighted graphs with real weights denoting the degree of relatedness between vertices [51]. Another representation of the relations between the spots can be done by the Euler-Venn diagram [53]. Figure 3 illustrates the semantic of the definition of the relations, where logic numbers in the matrix (2) correspond to a binary measure of the intersection parts A, B, C, and D of the spots a, b, and their environments. Hence, we can present (2) in the following form:

$$\langle a|b\rangle = \begin{bmatrix}C & A\\B & D\end{bmatrix} \tag{5}$$

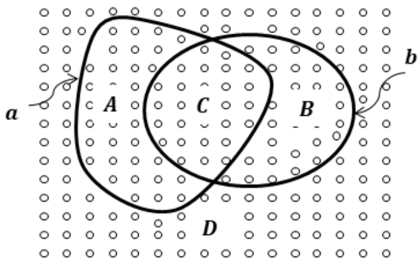


Figure 3: Euler-Venn diagrams for the structural relations of spots, mapped at some basis of spots, which conditionally illustrated by grid of circles.

Consider the simplest case when the basis spots cannot intersect each other and other spots. Such properties permit us to call atomic these spots and basis. Therefore, the atomic spots can be in the mutual relations of separation and indiscernibility and with other spots, in the relations of inclusion (less), separation, and indiscernibility. Note that specified properties make the atomic spots to be like points, pixels (for 2D figures), voxels (for 3D bodies), or elements of sets. Although the atomic basis is an idealized, limited case of the spot's basis, we can consider the intersection parts of spots, which are orthogonal, as some approximation for the atomic basis.

Application of the atomic basis allows us to introduce the numerical characteristics of spots, which simply can be done by counting the number of atomic spots inside spots A, B, C (Figure 3). It should be emphasized that this approach is a generalization of the concept of measure of sets, which demonstrates the ability to extract quantitative information by processing qualitative data. This allows us to introduce the concepts of quasi-measurement and quasi-probability. Also, we can define a quasi-membership function, which is analogous to that of the fuzzy set or rough set theories. It can be formulated the general idea that in the case of an atomic basis the apparatus of the spots model must coincide with those of sets, fuzzy sets, rough sets, and the mereotopology.

Let us define operations union ∨ and the intersection ∧ for the spots, which helps to create new spots. We suggest the following definitions, which are similar but different from those of the set theory:

$$\begin{aligned} c = a \vee b \;&\leftrightarrow\; \forall x \; (cx = ax + bx) \\ c = a \wedge b \;&\leftrightarrow\; \forall x \; \left(\tilde{c}x = \tilde{a}x + \tilde{b}x\right) \end{aligned} \qquad (6)$$

where symbol + denotes the logical disjunction operation. Note that, in contrast to the sets, (6) does not define the image of spots c directly, because it depends on spots basis $\{x_i\}$. Following the equality $c\tilde{c} = 0$, see (1), it is possible to derive the following equation from (6):

$$c = \bigvee x, \qquad x : \left(\tilde{a}x + \tilde{b}x = 0\right) \qquad (7)$$

The definitions (6) permit to derive simple properties for zero spots $\emptyset$:

$$a \vee \emptyset = a, a \wedge \emptyset = \emptyset \qquad (8)$$

and to determine intersection parts A, B, C, and D in Figure 3(a), using the intersection operation also:

$$A = a \wedge \tilde{b},\; B = \tilde{a} \wedge b, C = a \wedge b, D = \tilde{a} \wedge \tilde{b} \qquad (9)$$

It is possible to generalize the concept of elementary relations (2), (5) if to represent the spots A, B, C, and D in the vector forms (4) on some basis. Figure 3 provides a vivid illustration of these extended relations if to regard the atomic basis spots, on which the spots intersection is mapped. We call these new relations structural because they provide information about the inner structure of the spots A, B, C, and D.

Let us introduce an L4 matrix by analogy with a numerical matrix where numerical elements are replaced with L4 numbers. L4 matrix can be used to transform the L4 vector (or spot image) from one basis to another basis. For this purpose, the rows of the L4 matrix must correspond to the vector representation of the spots of the "new" basis mapped on the "current" basis. If the new and current bases are the same, the L4 matrix represents the mutual elementary relations between all spots of the basis. L4 matrix of mutual relations between basis spots is like the graph's weighted adjacency matrix with the real connection weights between vertices [51].

The L4 matrix allows us to simulate such transformations of the spot "shape" that are similar to the movement, rotation, and deformation or to transfer the spot image to another "space" also. It also permits solving other problems of the theory, including spot "shape" reconstruction, based on his images, and inverse problems of the matrix equation for spots.

Note that the L4 matrix can play a role like that of the data transformation between layers in a neural network. For example, each layer of the deep feedforward learning networks includes both the linear unite (or matrix) and the nonlinear unite (activation function) [54]. The application of the L4 matrix permits the elimination of the activation function in the architecture, due to the inherent non-linear transformation property of such a matrix. It should also be emphasized that the developing mathematical apparatus for the L4 matrixes and L4 vectors helps to model human reasoning in AI.

2.3. Operations with L4 objects

Let us introduce two types of vector products for L4 vectors (4) of spots a and b, at first not strict.

a) The scalar product, resulting in an L4 number c, which corresponds to the relation between the spots relatively to basis $X = \{x_i\}$ which is also used for the representation of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$c = \boldsymbol{a} \cdot \boldsymbol{b} = \langle a|b \rangle_X, \tag{10}$$

b) The elementwise (Hadamard) product of L4 vectors, resulting in an L4 vector $\boldsymbol{c}$:

$$\boldsymbol{c} = \boldsymbol{a} \odot \boldsymbol{b} = [\alpha_1 \cdot \beta_1;\ \alpha_2 \cdot \beta_2; \dots; \alpha_n \cdot \beta_n] = [\langle a|b \rangle_{x1};\ \langle a|b \rangle_{x2}; \dots; \langle a|b \rangle_{xn}] \tag{11}$$

Here $\alpha_i = \langle a|x_i \rangle$, $\beta_i = \langle b|x_i \rangle$, $\odot$ is a symbol of Hadamard product, symbol " $\cdot$ " in (10) and (11) denotes the dot product operation for L4 vectors and L4 numbers and it must be defined. Following the previous comment, we call the elementwise product of (11) as the structural product of L4 vectors, since it can be used to determine L4 vectors for parts A, B, C, and D (Figure 3), which defines the structure relation between of the spots a and b.

Based on the rules (10) and (11) for L4 vectors, we can also define two types of multiplications of the L4 matrix and the L4 vector. The first type of product is similar to that for numerical matrices and vectors, resulting in the L4 vector:

$$\boldsymbol{y} = \mathbf{A} \cdot \boldsymbol{x} \tag{12}$$

The second type of multiplication, resulting in a new L4 matrix $\mathbf{M}$, we call the structural product, which is based on the rule (11):

$$\mathbf{M} = \mathbf{A} \odot \boldsymbol{x} \tag{13}$$

Notice that the definition of product operations for L4 vectors is not a trivial task in the general case. Hence, firstly, we define it for the case an *atomic basis* $A = \{u_i\}$ when the scalar product (10) must be equal to the following elementary relation $\langle a|b \rangle_A$ relatively to basis A:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \langle a|b \rangle_A = \sum_{i=1}^{n} \alpha_i \cdot \beta_i = \sum_{i=1}^{n} \langle a|b \rangle_{u_i} \tag{14}$$

where $\langle a|b \rangle_{u_i}$ is the elementary relation relative to the spot u_i.

The next step is to define the relations $\langle a|b \rangle_A$ and $\langle a|b \rangle_{u_i}$ in agreement with (2) and (5), using the following equations:

$$\langle a|b \rangle_A \equiv \begin{bmatrix} (a \wedge b) \bigvee_{i=1}^{n} u_i & (a \wedge \tilde{b}) \bigvee_{i=1}^{n} u_i \\ (\tilde{a} \wedge b) \bigvee_{i=1}^{n} u_i & (\tilde{a} \wedge \tilde{b}) \bigvee_{i=1}^{n} u_i \end{bmatrix} = \sum_{i=1}^{n} \langle a|b \rangle_{u_i} \tag{15}$$

where

$$\langle a|b \rangle_{u_i} \equiv \begin{bmatrix} (a \wedge b) u_i & (a \wedge \tilde{b}) u_i \\ (\tilde{a} \wedge b) u_i & (\tilde{a} \wedge \tilde{b}) u_i \end{bmatrix} \tag{16}$$

If to introduce L4 numbers α_i and β_i for the atomic basis A,

$$\alpha_i = \langle a|u_i\rangle = \begin{bmatrix} au_i & a\tilde{u}_i \\ \tilde{a}u_i & \tilde{a}\tilde{u}_i \end{bmatrix} \tag{17}$$

$$\beta_i = \langle b|u_i\rangle = \begin{bmatrix} bu_i & b\tilde{u}_i \\ \tilde{b}u_i & \tilde{b}\tilde{u}_i \end{bmatrix}$$

then, it is followed from (15) and (16) that

$$\langle a|b\rangle_A = \begin{bmatrix} \sum_{i=1}^{n} au_i \cdot bu_i & \sum_{i=1}^{n} au_i \cdot \tilde{b}u_i \\ \sum_{i=1}^{n} \tilde{a}u_i \cdot bu_i & \sum_{i=1}^{n} \tilde{a}u_i \cdot \tilde{b}u_i \end{bmatrix} = \sum_{i=1}^{n} \alpha_i \cdot \beta_i \tag{18}$$

Hence, we defined the following rule for product operator for L4 numbers α_i, β_i:

$$\alpha_i \cdot \beta_i \equiv \langle a|b\rangle_{u_i} = \begin{bmatrix} au_i \cdot bu_i & au_i \cdot \tilde{b}u_i \\ \tilde{a}u_i \cdot bu_i & \tilde{a}u_i \cdot \tilde{b}u_i \end{bmatrix} \tag{19}$$

where the symbol " $\cdot$ " denotes the logical conjunction.

Now let us regard an orthogonal basis $Y = \{u_i\}$ and we define the product $\boldsymbol{a} \cdot \boldsymbol{b}$ of L4 vectors on it, using the same equations (18) and (19). In contrast with atomic basis, the spots of an orthogonal basis can intersect a testing spot, which makes (15), (16) not equivalent to (18), (19). Nevertheless, we can assume that the spot image represented on this basis has a vague boundary, which is outlined by the "sizes" of the spots u_i. Notice that such vagueness can be made smaller if to consume the relations data with additional basis spots.

Finally, let us regard the general case when the spots of the basis $X = \{x_i\}$ can intersect each other and test spots. Obviously, the direct application of the rules (18), (19) does not provide correct results. However, there is a simple method to create an auxiliary orthogonal basis $Y = \{u_i\}$ which contains all the intersection parts of the basis spots. Application of such a basis Y instead of X improves the imaging "resolution" because each spot u_k is a part of some "parent" spot x_i. Hence, by increasing the number n of the spots belonging to basis X we increase the number of their intersection parts that makes imaging resolution finer. Note that such a method helps to solve the problem of the processing and fusion of qualitative data in the general case.

Let us apply a convenient numbering method of the spot's parts u_k utilizing a binary code that describes the structure of the intersections. Namely, generalizing (9), each u_k can be defined by a formula that includes spots x_i or $\tilde{x}_j$ connected with the intersection operators $\wedge$. For example, the code 101 ... 0 corresponds to binary number $k = 101 \dots 0_2$ that corresponds to the following spot u_k of the orthogonal basis:

$$u_k \equiv u_{101\dots0} = x_1 \wedge \tilde{x}_2 \wedge x_3 \dots \wedge \tilde{x}_n \tag{20}$$

That is, the code number "1" corresponds to elements x_i and the code number "0" correspond to elements $\tilde{x}_i$ in (20). Note that if $x_1\tilde{x}_2 \dots \tilde{x}_n = 0$ then $u_{101\dots0} = \emptyset$ and the maximal amount of the intersection parts u_k is 2^n-1. Hence, the total amount of such parts is $N \le 2^n - 1$ if to ignore zero spots.

We can define the relation $\langle a|u_k\rangle$ basing on (6) and (20), that permits to combine connections of spots a, $\tilde{a}$ with x_i, $\tilde{x}_i$. Then inserting $\langle a|u_k\rangle$ to (18) and (19), we obtain the operation rules for the basis $X = \{x_i\}$ in the general case. Although this method of the definition relation $\langle a|u_k\rangle$ is strictly equivalent only for the atomic basis, such an approach does not contradict our philosophy with non-monotonic logic: conclusions about $\langle a|u_k\rangle$ can be wrong if we have insufficient data. However, the inaccurate inference can be corrected by increasing the number n of elements in X that provides to decrease "sizes" of spots from the orthogonal basis Y of the intersection parts.

3. Results of Reconstruction of Crisp Figures

Despite the aim of the proposed theory to represent vague figures - spots, it is convenient to demonstrate the validity of its apparatus on the base of crisp figures, which are the limiting case of spots, as it was mentioned above. In this case, we can consider the figure under test as a conditionally *unknown spot,* and the figures, which are used for investigation or "sampling" the spot under test as conditionally *known spots* (or basis spots).

Let us consider the two-dimension (2D) problem of the shape reconstruction of a crisp plane figure (*unknown spot*), utilizing the only qualitative information of its elementary relations with a set of *known* 2D figures. That is, the result of each *sampling* can be only the answer to the question: what is an elementary relation between the basis spot and the spot under test? Note that initially, we cannot get to know details about structural relations between them. However, the sequential increasing number of the basis spots and accumulating qualitative data allows us to refine the shape and boundaries of the figure under test continuously. It may seem surprising, but in the limit of an infinitely large amount of such qualitative data, we can undoubtedly reconstruct the object with absolute accuracy.

To verify and illustrate the suggested theory, we wrote MATLAB programs that help to process elementary relations between the *unknown* and *sampling* figures (or spots) based on algorithms for $\langle a|u_k\rangle$. We utilize the spots basis with quite tight distribution in the plane that makes the orthogonal spots u_k to be relatively small. Thus, according to the results of the previous section, this property helps to reconstruct an image of the *unknown* figure with a better spatial resolution even when the *known* figures are relatively large.

The imaging is produced on the base of a grid of small squares (or pixels) that play the role of the orthogonal or atomic basis. Images in these figures were mapped on the pixels with a size of 0.25 units. After calculations $\langle a|u_k\rangle$, the program's algorithm separates all the *intersection parts* u_k into three subsets, which correspond to certain elementary relations with the *unknown* figure: inclusion, intersection, or separation. Uniting the spots in each subset, we obtain the following regions: the inner region of the figure under test, its boundary, and environment (see Figures 4–6).

Figure 4 demonstrates the results of image reconstruction for the circle where the sampling figures are a set of squares with sizes 4×4 (see Figure 4(b)). These squares were periodically distributed with period 0.25 and form the grid of the intersection parts 0.25×0.25 (see Figure 4(a)), which corresponds to the orthogonal basis of the intersection parts u_k. The different regions are marked with three colors: the inner region corresponds to magenta, the boundary corresponds to brown, and the environment corresponds to green-like colors.

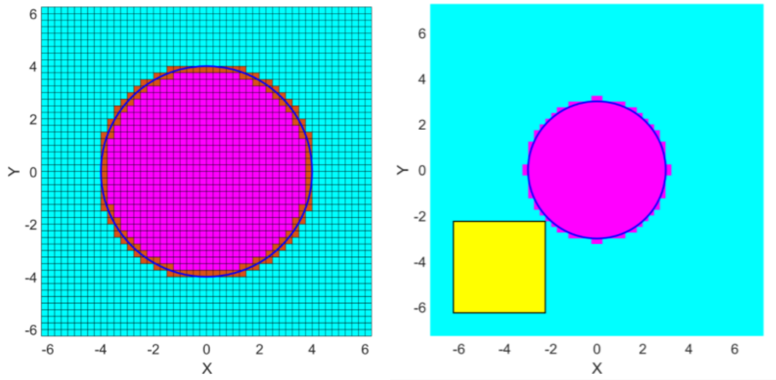
We applied the following additional rules for orthogonal spots u_k belonging to the boundary, which allows us to correct the relations $\langle a|u_k\rangle$ decreasing width of the boundary region and the shape vagueness of the spot a:

$$\begin{gathered} \forall x_i \in X: \{x_i <> a, u_k x_i = 0\} \Rightarrow a > u_k \\ \forall x_i \in X: \{x_i < a, u_k x_i = 0\} \Rightarrow a <> u_k \end{gathered} \tag{21}$$

where symbols $<>$ and $>$ denote relations the separation (SP) and inclusion (more) (IM), correspondingly (see Table 1). Note that the image in Figure 4(a) contains both an inner region and the boundary of the vague image (brown color) where images were created using calculations $\langle a|u_k\rangle$. However, Figure 4(b) demonstrates the result of additional processing (of the qualitative data to reduce the image vagueness in Figure 4(a). Although rules (21) are not the results of rigorous mathematical inference and depend on a specific basis, they undoubtedly help to reduce the uncertainty of the boundary related to a lack of information, as shown in the figure in Figure 4(b).

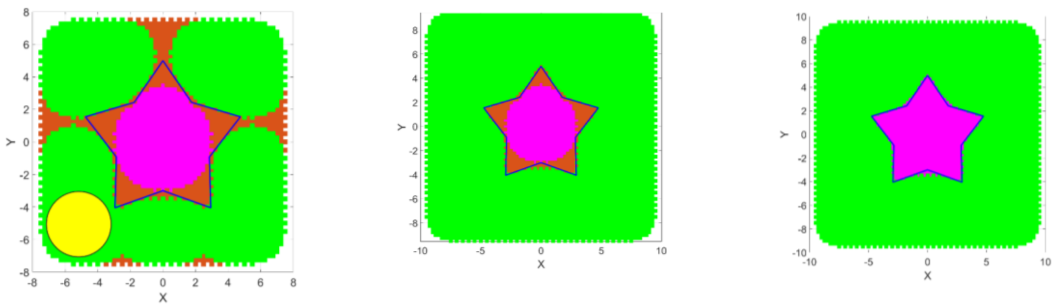
Figure 5 demonstrates the results of the image reconstruction for a five-pointed star when the sampling figures were circles with a diameter of 4 units (an example is a yellow circle in Figure 5(a)). These circles make up the set of periodically arranged spots in the plane and form the basis of their intersection parts u_k. The reconstructed image in Figure 5(a) corresponds to a relatively coarse distribution period of the circles, which is equal to 1 and demonstrates a blurred image caused by the insufficient relations data. In particular, we can see some mistaken boundary regions outside the star.

The image in Figure 5(b) was obtained for the smaller distribution period 0.5 of the circles providing sufficient qualitative data for good image quality. Figure 5(c) demonstrates the result of additional processing of the qualitative data using the rules (21) that permits to reduce the image vagueness in Figure 5(b).



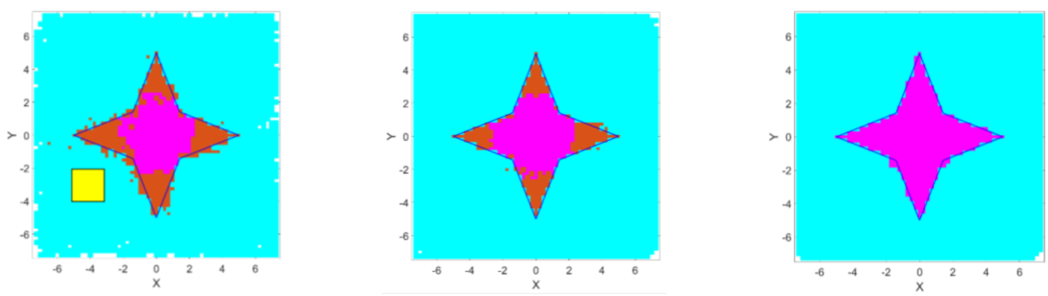
(a) (b)

Figure 4: Results of the circle reconstruction after processing data of its elementary relations with set of squares, periodically distributed with period 0.25. The magenta color corresponds to the inner region, the brown – to the boundary and green-like – to the environment. (a) Result of reconstruction by algorithm for $\langle a|u_k\rangle$. A grid of small squares corresponds to the orthogonal basis of the intersection parts $\mathrm{u_k}$. (b) Result of reconstruction by algorithm for $\langle a|u_k\rangle$ and (21). The yellow square is an example of the testing square.



(a) (b) (c)

Figure 5: Results of reconstruction of a five-pointed star processing data on its elementary relations with set of circles (example is the yellow circle in Figure 5(a)). The magenta color corresponds to the inner region, the brown – to the boundary and green – to the environment. (a) The sampling figures are distributed with period 1, algorithm for $\langle a|u_k\rangle$. (b) The distribution period is 0.5, algorithm for $\langle a|u_k\rangle$. (c) The distribution period is 0.5, algorithm for $\langle a|u_k\rangle$ and (21).



(a) (b) (c)

Figure 6: Results of reconstruction of a four-pointed star using only information about its elementary relations with set of a irregular figures that are randomly distributed incide of 2×2 squares (example of them is yellow square in Figure 6(a)). The magenta color corresponds to the inner region, the brown – to the boundary and green-like – to the environment. (a) Total number of sampling figures is 1000. (b) Total number of sampling figures is 3000, algorithm for $\langle a|u_k\rangle$. (c) Total number of sampling figures is 3000, but algorithm for $\langle a|u_k\rangle$ and (21).

Results of image reconstruction for a four-pointed star are demonstrated in Figure 6 where the sampling figures are irregular random figures, randomly distributed in the plane. These random figures were created as unions of 0.25×0.25 squares randomly distributed inside 2×2 square boundaries (see Figure 6(a)). Figure 6(a) corresponds to the case of a total of 1000 sampling tests and demonstrates a poor image quality that is caused by an insufficient number of tests. Figure 6(b) confirms this conclusion, demonstrating a better image quality corresponding to 3000 sampling tests. Figure 6(c) shows a result of additional image processing, exploiting rules (21).

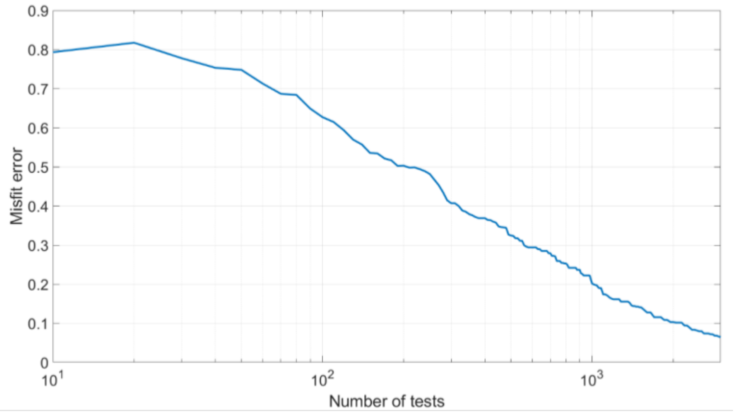


Figure 7: Example of dependence of the misfit error (22) on the number of the sampling tests of the four-pointed star in Figure 6 for random distributed irregular sampling figures.

We also investigated the dependence of the misfit error for the reconstructed image as a function of the number of sampling tests, using the MATLAB program that used both algorithms for $\langle a|u_k\rangle$ and (21) rules. Figure 7 demonstrates an example of such a dependence, where the misfit error was calculated by the following equation:

$$err = \frac{|N_{image} - N_{star}|}{N_{star}} \tag{22}$$

Here N_{star} and N_{image} are numbers of 0.25×0.25 pixels that correspond to the inner region of the star and the inner region of the image, correspondingly.

4. Conclusions

This paper is devoted to the description of the concept and basis of the apparatus of new mathematical objects – spots, which are adequate to represent and process qualitative data, modeling the human mental images and semantic information.

In general, the spot model allows representing semantic information contained in data of various nature. The proposed model utilizes the elementary relations between the spots as qualitative information about them and introduces logical L4 numbers that describe these relations. Based on the L4 numbers, the theory introduces L4 vectors and L4 matrices, utilizing an analogy with the numerical matrix algebra. Although L4 numbers correspond to the elementary level of qualitative data, processing a big number of them permits the extraction of higher-level information, including numerical data.

It is important that the introduced mathematical apparatus of L4 numbers, vectors, and matrices, as well as the operations with them, can be a basis for the knowledge representation and modeling of human reasoning in Artificial General Intelligence. Undoubtedly, the developed mathematical apparatus is very promising for its application in many other areas of AI, including such complex tasks as data mining.

The suggested mathematical apparatus of the spots theory was verified solving the problem of the image reconstruction of the crisp figures, processing only qualitative data about their elementary relations with a set of sampling crisp figures.

Further development of the proposed theory can be carried out in the development of qualitative geometry and as an applied science for use in many areas of AI. Among the urgent problems of the theory, we can select the solving of matrix equations with L4 matrices and L4 vectors, the introduction of quasi-measure, and quasi-probabilities. Further development of the theory expects to define concepts that are qualitative analogies of those in the crisp geometry, including line, surface, dimensionality and curvature of space, etc. The most actual application of the proposed model is the creation of neural networks that utilize algorithms based on the spot's apparatus.

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6. References

[1] N. A. Simonov, Spots concept for problems of artificial intelligence and algorithms of neuromorphic systems, Russian Microelectronics, 49.6 (2020), pp. 431–444. doi: 10.1134/S106373972005008X

[2] D. Pitt, Mental representation, E. N. Zalta (Ed.), The Stanford Encyclopedia of Philosophy, 2020. URL: https://plato.stanford.edu/archives/spr2020/entries/mental-representation.

[3] M. Kosslyn, Mental images and the brain, Cognitive Neuropsychology, 22.3/4 (2005), pp. 333-347.

[4] E. Margolis, S. Laurence, Concepts, E. N. Zalta (Ed.), The Stanford Encyclopedia of Philosophy, 2019. URL: https://plato.stanford.edu/archives/sum2019/entries/concepts.

[5] N. J. T. Thomas, Mental Imagery, E. N. Zalta (Ed.), The Stanford Encyclopedia of Philosophy, 2019. URL: https://plato.stanford.edu/archives/sum2019/entries/mental-imagery.

[6] A. A. Gostev, V. F. Rubakhin, Classification of figurative phenomena in the light of a systems approach, Voprosy psikhologii [Psychology Issues], no. 1, pp. 33–43 (in Russian).

[7] A. A. Gostev, Secondary image psychology (2019). [Electronic resource]/ Gostev A.A. — Electronic text data, Institute of Psychology RAS: Moscow, Russia (in Russian).

[8] M. L. Minsky, Semantic information processing, MIT Press, Cambridge, Mass., London, 1968.

[9] Y. Zhong, A theory of semantic information, China communications, 14.1, 2017, pp.1-17.

[10] S. J. Russell, P. Norvig, Artificial Intelligence: A Modern Approach, 3rd ed, Prentice Hall: Upper Saddle River, US, 2010.

[11] A. N. Whitehead, Process and reality: An essay in cosmology, Cambridge University Press, Cambridge, UK, 1929.

[12] B. L. Clarke, A calculus of individuals based on connection, Notre Dame Journal of Formal Logic, 22 (1981), pp. 204-218.

[13] N. Asher, L. Vieu, Toward a geometry of common sense: A semantics and a complete axiomatization of mereotopology, in: Proceedings of IJCAI'95, pp. 846-852, 1995.

[14] J. G. Stell, Mereotopology and computational representations of the body, Computational Culture, 6 (2017).

[15] A. G. Cohn, B. Bennett, J. Gooday, N. Gotts, Qualitative spatial representation and reasoning with the Region Connection Calculus, GeoInformatica, 1 (1997), pp. 275–316.

[16] A. G. Cohn, J. Renz, Qualitative spatial representation and reasoning, in: F. van Harmelen, V. Lifschitz, B. Porter, (Eds.), Handbook of Knowledge Representation, Elsevier B.V., Amsterdam, Netherlands, 2008, Ch. 13, pp. 551 – 596. doi: 10.1016/S1574-6526(07)03013-1.

[17] A. G. Cohn, N. M. Gotts, A mereological approach to representing spatial vagueness, in: J. D. Aiello, S. Shapiro, (Eds.), Proceedings of the 5th Conference Principles of Knowledge Representation and Reasoning, Morgan Kaufmann, Burlington, US, 1996, pp. 230–241.

[18] A. Galton, Discrete mereotopology, in: C. Calosi, P. Graziani (Eds.), Mereology and the Sciences, volume 371 of Synthese Library (Studies in Epistemology, Logic, Methodology, and Philosophy of Science), Springer, Cham, Switzerland, 2014, pp. 293–321.
[19] B. Bennett, Spatial reasoning with propositional logics, in: J. Doyle, E. Sandewall, P. Torasso (Eds.), Proceedings of the 4th International Conference on Principles on Knowledge Representation and Reasoning (KR-94), Morgan Kaufmann, Bonn, Germany, 1994, pp. 165-176.
[20] P. Jonsson, Th. Drakengren, A complete classification of tractability in RCC-5, Journal of Artificial Intelligence Research, 6 (1997), pp. 211-221.
[21] M. J. Egenhofer, R. Franzosa, Point-set topological spatial relations, International Journal of Geographical Information Systems, 5 (1991), pp. 161-174.
[22] M. J. Egenhofer, J. R. Herring, Categorizing binary topological relations between regions, lines and points in geographic databases, Technical Report, Department of Surveying Engineering, University of Maine, 1991.
[23] E. Clementini, P. Di Felice, 1997. Approximate topological relations. *International journal of approximate reasoning*, *16*(2), pp.173-204.
[24] J. G. Stell, Part and complement: fundamental concepts in spatial relations", Annals of Mathematics and Artificial Intelligence, 41 (2004), pp. 1-17.
[25] S. A. Butenkov, N. S. Krivsha, V. V. Krivsha, Topological space relations for multivariate data granulation models, Bulletin of the Southern Federal University, Technical science, 107.6 (2010), pp. 200-211 (in Russian).
[26] C. Schlieder, Qualitative shape representation, in: Geographic objects with indeterminate boundaries, CRC Press, 2020, pp. 123-140.
[27] A. G. Cohn, N. M. Gotts, The ‘egg-yolk’ representation of regions with indeterminate boundaries, in: *Geographic Objects with Indeterminate Boundaries*, CRC Press, 2020, pp. 171-187.
[28] Z. Pawlak, Rough sets, International Journal of Computer and Information Sciences, 11 (1982), pp. 341-356.
[29] Z. Pawlak, Rough set theory and its applications to data analysis, Cybernetics and Systems, 29.7 (1998), pp. 661-688. DOI: 10.1080/019697298125470.
[30] Z. Suraj, An introduction to rough set theory and its applications: a tutorial, in: Proceedings of the 1st International Computer Engineering Conference (ICENCO’) New Technologies for the Information Society, Cairo, Egypt, 2004, pp. 1-39.
[31] G. Peters, P. Lingras, D. Slezak, Y. Y. Yao (Eds.), Rough sets: selected methods and applications in management and engineering, Springer, London, UK, 2012.
[32] Y. Y. Yao, Rough set approximations: a concept analysis point of view, in: H. Ishibuchi (Ed.) Computational Intelligence, volume I of Encyclopedia of Life Support Systems (EOLSS), 2015, pp. 282-296.
[33] A. Skowron, S. Dutta, Rough sets: past, present, and future, Natural Computing, 17 (2018), pp. 855–876. doi:10.1007/s11047-018-9700-3.
[34] L. A. Zadeh, Fuzzy sets, Inf. Control, 8.3 (1965), pp. 338-35.
[35] J. J. Buckley, E. Eslami, Fuzzy plane geometry I: points and lines, Fuzzy Sets Syst, 86 (1997), pp. 179-187.
[36] J. J. Buckley, E. Eslami, Fuzzy plane geometry II: circles and polygons, Fuzzy Sets Syst, 87 (1997), pp. 79-85.
[37] D. Molodtsov Soft set theory—first results, Computers & Mathematics with Applications, 37.4-5 (1999), pp. 19-31.
[38] D. A. Molodtsov, The theory of soft sets, Editorial URSS Publ., Moscow, 2004. (in Russian)
[39] L. A. Zadeh, Fuzzy Sets and Information Granularity, in: M. Gupta, R. Ragade, R. Yager, (Eds.), Advances in Fuzzy Set Theory and Applications, North-Holland, Amsterdam, 1979, pp. 3-18.
[40] L. A. Zadeh, Towards a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic, Fuzzy Sets Syst, 19 (1997), pp. 111-127.
[41] J. T. Yao, A Ten-year Review of Granular Computing, in: Proceedings of the IEEE Int. Conf. on Granular Computing, Silicon Valley, USA, 2007, pp. 734-739.
[42] T. Y. Lin, Data mining: granular computing approach, in: Lect. Notes Comput. Sci., 1574.1 (1999), pp. 24–33.

[43] Y. Y. Yao, Information granulation and rough set approximation, International Journal of Intelligent Systems, 16.1 (2001), pp. 87-104.
[44] Y. Y. Yao, Granular Computing, Computer Science, Ji Suan, Ji Ke Xue (Eds.), 31 (2004), pp. 1-5, Proceedings of the 4th Chinese National Conference on Rough Sets and Soft Computing.
[45] S. H. Nguyen, A. Skowron, J. Stepaniuk, Granular computing: a rough set approach, Comput. Intell., 17.3 (2001), pp. 514–544.
[46] Y. Y. Yao Granular computing for data mining, in: Proceedings of the SPIE, 6241, Data Mining, Intrusion Detection, Information Assurance, and Data Networks Security, 624105, 2006.
[47] J. T. Yao, A.V. Vasilakos, W. Pedrycz, Granular computing: perspectives and challenges, IEEE Trans. on Cybernetics, 43.6, (2013) pp. 1977- 1989.
[48] A. Skowron, A. Jankowski, S. Dutta, Interactive granular computing, Granul. Comput., 1.2 (2016), pp. 95–113.
[49] R. Diestel, Graph Theory, 3rd ed., Springer-Verlag, Berlin, New York, 2005.
[50] T. Hamaguchi, H. Oiwa, M. Shimbo, Y. Matsumoto, Knowledge transfer for out-of-knowledge-base entities: a graph neural network approach, in: Proceedings of the IJCAI 2017, 2017, pp. 1802–1808.
[51] S. Cao, W. Lu, Q. Xu, Deep neural networks for learning graph representations, in Proceedings of the 30th. AAAI Conference on Artificial Intelligence, AAAI Press, 2016, pp. 1145–1152.
[52] M. Defferrard, X. Bresson, P. Vandergheynst, Convolutional neural networks on graphs with fast localized spectral filtering, in: Advances in Neural Information Processing Systems, Proceedings of the 30th. Conf. on Neural Information Processing Systems (NIPS 2016), Barcelona, Spain, 2016, pp. 3844–3852.
[53] S.-J. Shin, O. Lemon, J. Mumma, Diagrams, E. N. Zalta (Ed.), The Stanford Encyclopedia of Philosophy, 2018. URL: https://plato.stanford.edu/archives/win2018/entries/diagrams.
[54] I. Goodfellow, Y. Bengio, A. Courville, Deep Learning, MIT Press, Cambridge, UK, 2016.