On Counting L-Convex Polyominoes (short paper)*

Valentina Dorigatti[1] and Paolo Massazza[1]

Department of Theoretical and Applied Sciences
University of Insubria, Varese, Italy
{vdorigatti,paolo.massazza}@uninsubria.it

Abstract. A convex polyomino P is L-convex if any two cells of P can be joined by a monotone path inside P with at most one change of direction. In this paper we show that the problem of computing the number of L-convex polyominoes of area n can be solved in polynomial time using $O(n^4)$ space. We designed a C++ program to significantly extend the counting sequence of L-convex polyominoes and to improve the estimate of the associated growth constant.

Keywords: convex polyominoes · counting problem · integer sequences

1 Introduction

A polyomino is a geometrical figure consisting of a finite set of connected unitary squares (called cells) in the plane $\mathbb{Z} \times \mathbb{Z}$, considered up to translations. Polyominoes gained popularity after the paper of S. Golomb [8]. They are widely studied by physicists, mathematicians, computer scientists and biologists.

The problem of counting the number of polyominoes with n cells (*i.e.* of area n) is probably one of the fundamental open problems in combinatorial geometry, see problem 37 in [1]. The problem has been solved up to $n \leq 56$ [9] and no closed-form expression is known for the general case. Due to the difficulty of the problem, suitable classes of polyominoes have been introduced and widely studied. In particular, the class of hv-convex polyominoes (polyominoes where the intersection with an infinite horizontal or vertical stripe is a finite segment) and some of its subclasses have been deeply investigated [2,3,11,6].

In this paper we deal with the problem of computing the coefficient c_n of the generating function (with respect to the area) of the class LConv of L-convex polyominoes. This class has been introduced in [5] as the first level in the hierarchy of hv-convex polyominoes, and studied in the context of discrete tomography in [7], where it has been shown that an L-convex polyomino is uniquely determined by two integer vectors representing its vertical and horizontal projections. Lastly, in [4] a non-closed form expression for the generating function (with respect to the area) of LConv has been provided.

We show a simple decomposition of L-convex polyominoes that is used to obtain a set of recurrence equations for computing the number of L-convex polyominoes of area n in polynomial time and $O(n^4)$ space. Indeed, we designed a C++ program that uses dynamic programming to extend sequence A126764 in OEIS from $n < 19$ up to $n \leq 120$, so obtaining a better estimate of the growth constant of LConv, $\lambda = 1.2207$.

2 Notation and preliminaries

Let P be a polyomino with a $r \times c$ minimal bounding rectangle. The r rows (resp., c columns) of P are numbered from bottom to top (resp., from left to right). The *area* of P is the number of its cells, denoted by $\mathrm{A}(P)$. A cell is identified by a pair of integers (i, j), where i (resp., j) is the row (resp., column) index. Two cells $a = (i, j)$ and $a' = (i', j')$ are *adjacent* if $|i - i'| + |j - j'| = 1$. Given two cells a and b of P, a *path* from a to b is a sequence $q_1, \ldots, q_k$ of cells of P, with $q_1 = a$ and $q_k = b$, such that q_i and q_{i+1} are adjacent for all i, with $1 \leq i < k$. A *step* is a sequence of two adjacent cells $(i, j), (i', j')$, and it's called *North step* (resp. *South step*) if $i' = i + 1$ (resp., $i' = i - 1$) and $j' = j$. Similarly, for a *West step* (resp. *East step*) one has $j' = j - 1$ (resp., $j' = i + 1$) and $i' = i$.

A path of lenght r in P is uniquely identified by indicating a starting cell and a string $\beta \in \{\mathsf{N}, \mathsf{W}, \mathsf{S}, \mathsf{E}\}^r$. The number of *changes of direction* in $\beta = \beta_1\beta_2 \ldots \beta_r$ is defined as the number of indices i such that $\beta_i \neq \beta_{i+1}$, with $1 \leq i < r$. A path is *monotone* if $\beta \in \{\mathsf{N}, \mathsf{W}\}^+ \cup \{\mathsf{N}, \mathsf{E}\}^+ \cup \{\mathsf{S}, \mathsf{E}\}^+ \cup \{\mathsf{S}, \mathsf{W}\}^+$ (the symbol '+' denotes the positive closure).

A polyomino P belongs to the class Conv of *hv-convex* (convex, for short) polyominoes if any column and any row of P is a segment (a sequence of adjacent cells). It has been proved [5, Proposition 1] that P is convex if and only if any two cells of P are joined by a monotone path in P. Monotone paths allow to define the *zigzag-distance* between two cells a, b of P, denoted by $D(a, b)$, as the least integer k such that there exists a monotone path from a to b with k changes of direction (we use the term *distance* in an informal way, since P equipped with D is not a metric space).

From here on we consider only convex polyominoes. The *degree of convexity* of P, denoted by $D_c(P)$, is the least integer k such that any two cells of P can be joined by a monotone path with at most k changes of direction, that is, $D_c(P) = \max\{D(a, b) : a, b \in P\}$. Furthermore, P is called *k-convex* if its degree of convexity is at most k. When $k = 1$ we have the class LConv of L-convex polyominoes introduced in [5]. We denote by $\mathsf{LConv}(n)$ the subset of LConv containing polyominoes of area n.

A polyomino P is a *stack* (resp., *Ferrers diagram*) if it shares exactly two (resp, three) adjacent vertices with its smallest bounding rectangle B. The *height* of a stack P, denoted by $\textsc{height}(P)$, is the area of its largest column. A stack P is a *left* (resp., *right*) stack if the area of its last (resp., first) column is $\textsc{height}(P)$. We denote by L (resp., R) the set of left (resp., right) stacks, whereas F stands for the class of Ferrers diagrams.

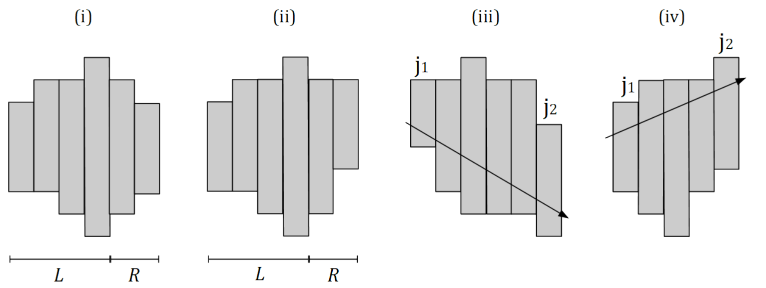

Fig. 1. From left to right: a polyomino in LR_1 (no disjoint or overlapping columns), a polyomino in LR_2 (no disjont columns, at least one column includes any other column), a descending polyomino and an ascending polyomino.

By $\textsc{low}(j)$ (resp., $\textsc{high}(j)$) we indicate the row index of the bottom cell (resp., top cell) of column j of P. Furthermore, by $\textsc{first}(P)$ (resp., $\textsc{last}(P)$) we indicate the first (resp., last) column of P.

Given two columns i, j, we say that: i *includes* j, denoted by $j \subseteq i$, if and only if $\textsc{low}(i) \leq \textsc{low}(j)$ and $\textsc{high}(i) \geq \textsc{high}(j)$; i and j are *overlapping* if and only if $\textsc{low}(j) < \textsc{low}(i) \leq \textsc{high}(j) < \textsc{high}(i)$ or $\textsc{low}(i) < \textsc{low}(j) \leq \textsc{high}(i) < \textsc{high}(j)$; i and j are *disjoint* if and only if $\textsc{low}(i) > \textsc{high}(j)$ or $\textsc{low}(j) > \textsc{high}(i)$. If neither $i \subseteq j$ nor $j \subseteq i$ then i and j are disjoint or overlapping.

A convex polyomino P is called *descending* (resp., *ascending*) if it contains two overlapping columns j_1 and j_2 such that $j_1 < j_2$ and $\textsc{low}(j_2) < \textsc{low}(j_1)$ (resp., $\textsc{low}(j_2) > \textsc{low}(j_1)$), and there is not a column $\bar{\jmath}$ of P such that $j \subseteq \bar{\jmath}$ for all columns j of P, see Fig. 1 (iii) and (iv). If P is neither descending nor ascending then P is uniquely decomposed as concatenation of two polyominoes, $P = P_1 \cdot P_2$, where the last column of P_1 includes any column of P, with $\textsc{first}(P_2) \subsetneq \textsc{last}(P_1)$, see Fig. 1 (i) and (ii). Notice that P_1 (resp., P_2) is a rectagle or belongs to $\in \mathsf{L} \cup \mathsf{F}$ (resp., $P_2 \in \mathsf{R} \cup \mathsf{F}$). We denote by LR the class of these polyominoes. Notice that LR includes both R and L and F.

The degree of convexity of $P \in \mathsf{LR}$ is at most 2 as P contains a column $\bar{\jmath}$ such that $j \subseteq \bar{\jmath}$ for all columns j of P. Analogously, if P is descending or ascending then $D(P) \geq 2$ since P contains two overlapping columns j_1, j_2 and $D(a, b) = 2$, where a (resp., b) is the top (resp., bottom) cell of j_1 (resp., j_2).

As a matter of fact, LConv is a subset of LR. So, we consider the partition $\mathsf{LR} = \mathsf{LR}_1 \cup \mathsf{LR}_2$, where $\mathsf{LR}_i = \{P \in \mathsf{LR} : D_c(P) = i\}$. In other words, one has $\mathsf{LConv} = \mathsf{LR}_1$. The following property is the basis of the decomposition used to obtain a formula for $|\mathsf{LConv}(n)| = |\mathsf{LR}_1(n)|$.

Property 1. A convex polyomino P is in LR_1 if and only if for any two columns i and j of P one has $i \subseteq j$ or $j \subseteq i$.

Proof. If P is in LR and contains two columns that are disjoint or overlapping then $D_c(P) = 2$, that is $P \in \mathsf{LR}_2$.

3 Polyominoes decomposition and counting

In this section we provide a decomposition for $P \in \mathsf{LR}_1(n)$ that is useful to obtain a set of recurrence equations for computing $|\mathsf{LR}_1(n)|$. We consider the partition

$$\mathsf{LR}_1(n) = \bigcup_{p,q,i} \mathsf{LR}_1(n,p,q,i),$$

where $\mathsf{LR}_1(n,p,q,i)$ is the set of all $P \in \mathsf{LR}_1(n)$ such that $p = \mathrm{A}(\textsc{first}(P))$, $q = \mathrm{A}(\textsc{last}(P))$ and $i = |\textsc{low}(\textsc{first}(P)) - \textsc{low}(\textsc{last}(P))|$. Without loss of generality, in the following we suppose that $p \geq q$. Indeed, by symmetry one has $|\mathsf{LR}_1(n,p,q,i)| = |\mathsf{LR}_1(n,q,p,i)|$.

Consider a polyomino $P \in \mathsf{LR}_1(n,p,q,i)$. If P does not contain a column of area larger than p then P belongs to the set $\mathsf{R}(n,p,q,i)$ comprising all polyominoes of area n that are rectangles or belong to $\mathsf{R} \cup \mathsf{F}$, and such that $p = \textsc{height}(P)$, $q = \mathrm{A}(\textsc{last}(P))$ and $i = \textsc{low}(\textsc{last}(P)) - \textsc{low}(\textsc{first}(P))$. Otherwise, consider the rightmost column in P with area $r > p$ and write P as concatenation of two polyominoes, $P = P' \cdot R'$, with $P' \in \mathsf{LR}_1(n-z,p,r,j)$ and $R' \in \mathsf{R}(z,e,q,k)$, where $e,z,k,j \in \mathbb{N}$ are uniquely identified ($q \leq e \leq p$, $q \leq z \leq n-p-r$), see Fig. 2.

By considering the decomposition $P = P' \cdot R'$, we get the following partition:

$$\mathsf{LR}_1(n,p,q,i) = \mathsf{R}(n,p,q,i) \cup \bigcup_{r,j,z,e,k} \mathsf{LR}_1(n-z,p,r,j) \cdot \mathsf{R}(z,e,q,k), \qquad (1)$$

where the union is taken over all values r,j,z,e,k such that:

- $p < r \leq n-p-q$ (remark: $\mathrm{A}(\textsc{first}(P')) = p$ and $\mathrm{A}(\textsc{last}(R')) = q$);
- $0 \leq j \leq r-p$ ($j = r-p$ if $\textsc{high}(\textsc{first}(P')) = \textsc{high}(\textsc{last}(P'))$, whereas $j = 0$ if $\textsc{low}(\textsc{first}(P')) = \textsc{low}(\textsc{last}(P'))$);
- $q \leq z \leq n-p-r$ (remark: $\mathrm{A}(P') \geq p+r$, R' contains at least one column of area q);
- $0 \leq k \leq i$ (due to convexity);
- $q+k \leq e \leq k+p-i$ (remark: $\textsc{first}(R') \subseteq \textsc{first}(P')$).

We refer to Fig. 2 for a better understanding of the conditions on r,j,z,e,k. It is immediate that in (1) all unions (resp. products) are disjoint (resp. unambiguous). In particular, given $P' \in \mathsf{LR}_1(n-z,p,r,j)$ and $R' \in \mathsf{R}(z,e,q,k)$ there is only one way of obtaining a polyomino in $\mathsf{LR}_1(n,p,q,i)$: concatenate R' to P' so that $\textsc{low}(\textsc{first}(R')) - \textsc{low}(\textsc{last}(P')) = i-k+j$.

Since the set $\mathsf{R}(z,e,q,k)$ appears in (1) we consider also the decomposition

$$\mathsf{R}(n,p,q,i) = \bigcup_{\substack{q+j \leq r \leq p-i+j \\ 0 \leq j \leq i}} \mathsf{R}(p,p,p,0) \cdot \mathsf{R}(n-p,r,q,j) \qquad (2)$$

Indeed, $R \in \mathsf{R}(n,p,q,i)$ is the concatenation of a one-column rectangle $R' \in \mathsf{R}(p,p,p,0)$ and a polyomino $R'' \in \mathsf{R}(n-p,r,q,j)$ for suitable r and j. Notice

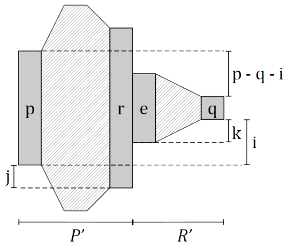

Fig. 2. Decomposition of a polyomino $P \in \mathsf{LR}_1(n)$. R' has area z, while P' has area $n - z$.

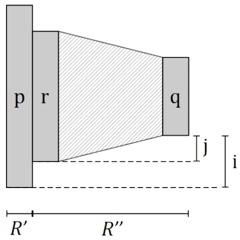

Fig. 3. Decomposition of a polyomino in R. Considering n as the total area, the stack R' has area p (since it includes only column p), while the stack R'' has area $n - p$.

that the position of R'' with respect to R' is uniquely determined by i and j, since $i - j = |\text{LOW}(\text{FIRST}(R'')) - \text{LOW}(\text{FIRST}(R'))\,|$, see Fig. 3.

The recurrence equations used to compute $|\mathsf{LConv}(n)|$ follow directly from the above decompositions.

4 Conclusions

By applying dynamic programming it is straightforward to develop a program that uses tables of size $O(n^4)$ to store the cardinalities of the sets $\mathsf{R}(n, p, q, i)$ and $\mathsf{LR}(n, p, q, i)$. Since each entry in a table is computed in polynomial time, we conclude that there exists a polynomial algorithm to compute $|\mathsf{LConv}(n)|$. We developed a C++ program that computes $|\mathsf{LConv}(n)|$ for large n. The source code is available at:

`https://sites.google.com/view/polyominoesgeneration/home`

Table 1 shows the first 121 values of the sequence $\{|\mathsf{LConv}(n)|\}_{n\geq 0}$, (previously computed [4] only for $n < 19$ - sequence A126764 in OEIS). The sequence has been computed in 20 minutes on an entry-level laptop.

The knowledge of more values of the counting sequence for LConv is of great importance since it allow to obtain a better approximation of the growth constant λ of the class LConv. Indeed, we recall that for any class of polyominoes with counting sequence $\{c_n\}$ it holds $\lim_{n\mapsto\infty} c_n^{1/n} = \lambda$ [10]. In particular, if a class of polyominoes satisfies Axioms Ca1–Ca5 in [12] one has the stronger result

Table 1. $|\mathsf{LConv}(n)|$ for $0 \leq n \leq 120$.

1, 1, 2, 6, 15, 35, 76, 156, 310, 590, 1098, 1984, 3515, 6094, 10398, 17434, 28837, 47038, 75820, 120794,
190479, 297365, 460056, 705576, 1073473, 1620680, 2429352, 3616580, 5349359, 7863564, 11491946, 16700534,
24140606, 34716813, 49682700, 70766326, 100343410, 141665826, 199172140, 278897192, 389023478, 540606678,
748543820, 1032844616, 1420315158, 1946761600, 2659891076, 3623095094, 4920401713, 6662912574, 8997185064,
12116088688, 16272878335, 21799355950, 29129313088, 38828680480, 51634288623, 68503548132, 90678083896,
119765026500, 157840769584, 207583098066, 272439268276, 356839386632, 466466992155, 608601537859,
792551372020, 1030200202402, 1336695959993, 1731317801706, 2238566012418, 2889530083444, 3723603926469,
4790633313200, 6153601295172, 7891981954408, 10105923985926, 12921462874542, 16497007238584,
21031401126748, 26773934116473, 34036755475474, 43210253221164, 54782085062392, 69360703770136,
87704407338787, 110757174873692, 139692827677312, 175969395713746, 221395981085892, 278214911971700,
349202586196148, 437793140235291, 548229968120604, 685751192735588, 856816491322044, 1069384249807199,
1333249914358312, 1660458696133736, 2065808536877134, 2567462560505766, 3187694221776132,
3953793160381426, 4899165525940288, 6064669456588960, 7500234685505662, 9266825196490576,
11438815758244096, 14106867431540154, 17381404200712580, 21396813283137772, 26316516037970698,
32339085501320858, 39705621296067648, 48708634082324850, 59702741064184742, 73117532860023593,
89473042013023344, 109398326670950676, 133653781930341812, 163157908951796052

$\lim_{n\mapsto\infty}\frac{c_{n+1}}{c_n} = \lambda$. In our case, this means a significant improvement of the estimate of λ, from $\frac{c_{18}}{c_{17}} = 1.6118\ldots$ to $\frac{c_{120}}{c_{119}} = 1.2207$.

We plan to extend the approach here presented to obtain suitable recurrences for computing the number of k-convex polyominoes of given area in polynomial time, for any $k > 1$.

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