On fuzzy truth-values and quasi-standard completeness

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Abstract
Propositional many-valued logics constitute formalisation of fuzzy logics, as the intended set of truth-values is the real unit interval $[0, 1]$, or meaningful subsets of it. In this paper we propose to frame some intuitive notion about fuzzy truth-values in formal logic and algebraic definitions, inducing some reflections about the usual notion of standard completeness.

Keywords
Fuzzy logic, truth values, standard completeness, MTL

1. Introduction

The main feature of fuzzy logic is to have fuzzy truth-values, that is, the classical notion of false/true (or $0/1$) membership of an element to a set is generalised to a wide spectrum of values in $[0, 1]$. From a mathematical point of view, in the last decades several many-valued logics have been introduced, whose natural semantics is evaluated in the real unit interval equipped with reasonable generalisations of classical two-valued connectives. In particular, the hierarchy of schematic extensions of Esteva and Godo's *Monoidal t-norm based logic MTL* is widely considered as a mature framework for studying truth-functional, $[0, 1]$-valued fuzzy logic from a purely formal, algebraic-logical approach. As a matter of fact MTL is sound and complete with respect to *standard structures*, that is, algebraic systems whose universe is the real unit interval $[0, 1]$, equipped with a left-continuous t-norm as conjunction, and its residuum as implication (and the constants 0 and 1 with the obvious meaning of, resp., crisp falsity and crisp truth). This property, motivating the introduction of MTL, is known as *standard completeness* of the logic MTL, and it can be applied to its schematic extensions, too.

Definition 1.1. A schematic extension L of MTL is *standard complete* iff there is a class $Std(L)$ of standard structures such that for any formula φ, it holds that φ is a theorem of L iff $\varphi = 1$ is a valid identity in every algebra in $Std(L)$. We refer to $Std(L)$ as the *standard models* of L.

Theorem 1.2. *MTL is standard complete. $Std(MTL)$ can be chosen as the class of all standard structures.*

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Definition 1.1 seems justly to capture adequately the notion of a logic whose truth-values are fuzzy, in the sense that the set of truth-values coincides with the real unit interval $[0, 1]$, totally ordered in the natural way. But this is not the end of the story for what concerns logics and their truth-value sets. Some observations are in order.

First, not all schematic extensions of MTL are standard complete. Foremost examples are finite-valued logics, among which, classical two-valued logic, which is then nicely considered as a particular case of fuzzy logic. Actually, finitely valued logics are generally considered authentic fuzzy logics, both for their extensive use in applications, and for the general tameness of their treatment, both in applicative contexts and in more theoretical ones. But there are other examples, where the schematic extension of MTL considered cannot be given a full $[0, 1]$ semantics, while being actually infinitely-valued. These logics are usually considered only by those theoreticians which explore the structure of the whole lattices of subvarieties of MTL, but they are seldom considered as actual fuzzy-valued logics.

In this work we propose some observations on the notion of standard completeness showing that it can be reasonably strengthened and also weakened, providing us with a sort of classification of schematic extension of MTL for what regards their fitness with respect to $[0, 1]$-valued semantics. In particular we shall argument that there is a very strong notion of being $[0, 1]$-valued, which is satisfied exactly by one schematic extension of MTL, namely Łukasiewicz logic. On the other hand we shall propose sound and complete semantics for some standard complete extensions which are very far from having the whole interval $[0, 1]$ as intended truth-value set. Further, we shall consider a weakening of the notion of standard completeness to show that some non-standard complete extensions of MTL, which are for this reason usually not considered as actual fuzzy logics, are indeed very close, in a precise technical sense, to have full $[0, 1]$-valued semantics.

2. Preliminaries

A *t-norm* is a binary operation from $[0, 1]^2$ into $[0, 1]$ that is associative, commutative, non-decreasing in both arguments, and has 0 as absorbing element and 1 as unit. Given a left-continuous t-norm $\odot$, its associated *residuum* is the binary operation $x \to y = \max\{z \mid z \odot x \leq y\}$. The algebra $[0, 1]_\odot = ([0, 1], \odot, \to, \wedge, 0)$, where $x \wedge y = \min(x, y)$, is called a *standard algebra* and it is completely determined by the left-continuous t-norm $\odot$.

A t-norm $\odot$ is *Archimedean* if it has the Archimedean property, that is, if for each $x, y \in (0, 1)$ there is a natural number n such that $x^n \leq y$, where by x^n we mean $x \odot \cdots \odot x$, n times. A t-norm $\odot$ is *nilpotent* if for each $x \in [0, 1)$ there is a natural number n such that $x^n = 0$. Clearly, each nilpotent t-norm is Archimedean[1].

Two t-norms $\odot_1$ and $\odot_2$ are isomorphic if there is a strictly increasing bijective map $f\colon [0, 1] \to [0, 1]$ such that $f(x \odot_1 y) = f(x) \odot_2 f(y)$ for every $x, y \in [0, 1]$. Two standard algebras are isomorphic if their t-norms are isomorphic.

Among the examples of t-norms and corresponding residua (hence of standard algebras), we mention the following:

[1] In the t-norm literature (see [1]), the definitions of Archimedean and nilpotent t-norms are applied only to continuous ones. Here, we generalise these definitions to all t-norms.

- Gödel t-norm $a \odot_G b = \min\{a, b\}$ with residuum $a \to_G b = 1$ if $a \le b$ and $a \to_G b = b$ otherwise. The algebra $[0,1]_G = ([0,1], \odot_G, \to_G, \wedge, 0)$ is the *standard Gödel algebra.*
- Product t-norm $a \odot_P b = a \cdot b$ (that is the usual product), that is a strictly monotone continuous t-norm having residuum $a \to_P b = 1$ if $a \le b$ and $a \to_P b = b/a$ otherwise. The algebra $[0,1]_P = ([0,1], \odot_P, \to_P, \wedge, 0)$ is the *standard Product algebra.*
- Łukasiewicz t-norm $a \odot_L b = \max\{0, a + b - 1\}$, that is a nilpotent continuous t-norm having residuum $a \to_L b = \min\{0, 1 - a + b\}$. The algebra $[0,1] = ([0,1], \odot, \to, \wedge, 0)$ is the *standard Łukasiewicz algebra*, also called the standard MV-algebra.
- Nilpotent minimum, that is a non-continuous but left-continuous t-norm that, despite its name, is not a nilpotent t-norm: $a \odot_{NM} b = \min(a, b)$ if $a + b > 1$ and $a \odot_{NM} b = 0$ otherwise, with residuum $a \to_{NM} b = 1$ if $a \le b$ and $a \to_{NM} b = \max\{1 - a, b\}$, otherwise. The algebra $[0,1]_{NM} = ([0,1], \odot_{NM}, \to_{NM}, \wedge, 0)$ is the *standard NM-algebra.*
- Drastic product t-norm, that is a non-continuous but right-continuous and as such it does not have a residuum: $a \odot_{DP} b = b$ if $a = 1$, $a \odot_{DP} b = a$ if $b = 1$ and $a \odot_{DP} b = 0$ otherwise.

Proposition 2.1. *Any left-continuous nilpotent t-norm is isomorphic with Łukasiewicz t-norm.*

Proof. In [2] it is proved that any left-continuous Archimedean t-norm is continuous. Since nilpotent t-norms are archimedean, any left-continuous nilpotent t-norm is continuous. In [1], Prop. 5.10 it is proved that any continuous nilpotent t-norm is isomorphic with Łukasiewicz t-norm. ■

Monoidal t-norm based logic (MTL, for short), axiomatized in [3], was proved in [4] to be complete with respect to the set of all standard algebras (this is stated as Theorem 1.2 in the introduction). The algebraic counterpart of MTL, via the usual Lindenbaum construction, is the variety $\mathbb{V}(MTL)$ of MTL-algebras. An *MTL-algebra* $(A, *, \to, \wedge, 0)$ is a prelinear commutative bounded integral residuated lattice. Any standard algebra $([0,1], \odot, \to, \wedge, 0)$ is an MTL-algebra and by Theorem 1.2 $\mathbb{V}(MTL)$ is generated by the set of standard algebras. In any MTL-algebra we set $1 := 0 \to 0$.

A *filter* F of an MTL-algebra $\mathbf{A} = (A, \odot, \to, \wedge, 0)$ is a subset of A containing 1 and such that if $a \le b$ and $a \in F$ then also $b \in F$ and if $a, b \in F$ also $a \odot b \in F$. A proper filter $\mathfrak{p}$ of $\mathbf{A}$ is *prime* iff for each pair of elements $x, y \in A$ either $x \to y \in \mathfrak{p}$ or $y \to x \in \mathfrak{p}$. The set of prime filters of $\mathbf{A}$ is called its *prime spectrum* $Spec(\mathbf{A})$ and can be topologised by setting as a base of closed sets all subsets of the form $\{\mathfrak{p} \in Spec(\mathbf{A}) \mid a \in \mathfrak{p}\}$, for $a \in A$. We denote by $Max(\mathbf{A})$ the set of filters of A that are maximal with respect to set inclusion, endowed with the topology inherited by restriction from $Spec(\mathbf{A})$. An MTL-algebra is *simple* if its only proper filter is $\{1\}$. Each axiomatic extension L of MTL determines a subvariety $\mathbb{V}(L)$ of $\mathbb{V}(MTL)$. We shall denote the free n-generated algebra in a variety $\mathbb{V}(L)$ by $\mathbf{F}_n(L)$.

Hájek's *Basic logic* (BL for short, [5]) is the axiomatic extension of MTL by means of the *divisibility* axiom $(\varphi \wedge \psi) \to (\varphi \odot (\varphi \to \psi))$. The algebraic counterpart of BL is the variety $\mathbb{V}(BL)$ of BL-algebras. BL is the logic of all continuous t-norms and their residua, in the sense that $\mathbb{V}(BL)$ is generated by all standard algebras $[0,1]_\odot$ for $\odot$ any continuous t-norm [6].

Gödel logic (G for short) is the axiomatic extension of BL given by adding the *idempotency* axiom $\varphi \to (\varphi \odot \varphi)$. The variety $\mathbb{V}(G)$ of Gödel algebras is formed by the BL-algebras satisfying the equation $x \odot x = x$. Gödel logic is standard complete and further, the standard Gödel algebra generates $\mathbb{V}(G)$.

Nilpotent Minimum logic (NM for short) is the axiomatic extension of MTL obtained by adding the *involutiveness* axiom $\neg\neg\varphi \to \varphi$ and the so-called *weak nilpotent minimum* axiom $\neg(\varphi \odot \psi) \vee ((\varphi \wedge \psi) \to (\varphi \odot \psi))$. In [3] it is proved that NM is standard complete since the standard algebra $[0,1]_{NM}$ generates $\mathbb{V}(NM)$. NM^{-} is the extension of NM by the axiom $(\neg(\neg\varphi \odot \neg\varphi)) \odot (\neg(\neg\varphi \odot \neg\varphi)) \to \neg(\neg(\varphi \odot \varphi) \odot \neg(\varphi \odot \varphi))$.

While Drastic product t-norm is not residuated, there are MTL-chains obtained by restricting this t-norm to suitable subsets of $[0,1]$. These chains generate the variety $\mathbb{V}(DP)$, associated with the logic DP, axiomatised by $\varphi \vee \neg(\varphi \odot \varphi)$.

Łukasiewicz logic (Ł for short) is the axiomatic extension of BL given by adding the axiom $\neg\neg\varphi \to \varphi$. The variety $\mathbb{V}$(Ł) of MV-algebras is formed by the BL-algebras satisfying the equation $\neg\neg x = x$. We refer the reader to [7, 8] for all background on MV-algebras. *Łukasiewicz logic* is standard complete and further, by Chang's algebraic completeness, the standard MV-algebra generates $\mathbb{V}$(Ł). Every MV-algebra is the interval of some lattice-ordered group. Indeed, the functor Γ implements the equivalence between the category of MV-algebras and the category of lattice-ordered abelian groups (abelian ℓ-groups) with strong unit. For every abelian ℓ-group $(G, +, 0, \leq)$ with strong unit u the functor Γ equips the unit interval $[0,u] = \{0 \leq x \leq u \mid x \in G\}$ with the operations $x \odot y = \max(0, x + y - u)$ and $x \to y = \min(u - x + y, u)$. It is easy to see that the resulting structure $\Gamma(G,u) = ([0,u], \odot, \to, \wedge, u)$ is an MV-algebra.

We are particularly interested in the simple MV-algebra $S_n = \Gamma(\mathbb{Z}, n-1)$ and in the non-simple MV-algebras $S_n^{\omega} = \Gamma(\mathbb{Z} \vec{\times} \mathbb{Z}, (n-1, 0))$ and $S_n^{\mathfrak{c}} = \Gamma(\mathbb{Z} \vec{\times} \mathbb{R}, (n-1, 0))$ for $n \geq 2$, where $\vec{\times}$ stands for the lexicographic product (i.e., the direct product with the order relation defined lexicographically: $(n,m) \leq (n',m')$ if and only if $n < n'$ or $n = n'$ and $m \leq m'$). We denote the operations of S_n^{ω} and $S_n^{\mathfrak{c}}$ respectively by $\odot_n^{\omega}$, $\to_n^{\omega}$ and $\odot_n^{\mathfrak{c}}$, $\to_n^{\mathfrak{c}}$.

Komori fully classified all subvarieties of MV-algebras. In particular, a proper variety of MV-algebras is generated by a set of chains $I \cup J$ where I is a finite set of chains of the form S_k and J a finite set of chains of the form S_k^{ω}. Notice that $S_k^{\mathfrak{c}}$ generates the same variety as S_k^{ω}.

3. Single standard completeness, and truly $[0,1]$-valued logics

We start strengthening the notion of standard completeness, as follows.

Definition 3.1. A schematic extension L of MTL is singly standard complete iff there exists a single standard structure $\mathbf{S}(L)$ such that the set of standard models of L can be chosen as $Std(L) = \{\mathbf{S}(L)\}$.

Proposition 3.2. *A schematic extension L of MTL is singly standard complete iff, for all integers $n \geq 0$, the free n-generated algebra in $\mathbb{V}(L)$, $\mathbf{F}_n(\mathbb{V}(L))$, is isomorphic with the subalgebra of the algebra of all functions $[0,1]^n \to [0,1]$, generated by the projections $x_i \colon (t_1, \ldots, t_n) \mapsto t_i$, using the operations of a standard algebra.*

When concrete representation matters, we shall identify $\mathbf{F}_n(\mathbb{V}(L))$ with the algebra of functions given in Proposition 3.2.

MTL itself is standard complete while is not known if it is singly standard complete, and most likely it is not. On the other hand BL is singly standard complete, and one can choose $\mathbf{S}(BL)$ in several distinct, not mutually isomorphic ways. A rather canonical choice is the ordinal sum of ω copies of the standard MV-algebra, which is used in [9], by applying Proposition 3.2, to characterise the free BL-algebras. Many other extensions of MTL are standard complete but not singly so. Some examples could be found in the paper [10], as subvarieties of $DNMG$. On the other hand there are singly standard complete extensions of MTL which are formally $[0,1]$-valued, but we shall argument here that these values are not fully satisfactory *truth values*.

We begin this discussion recalling what happens in classical propositional logic, where truth-values are in bijection with maximal theories, and in turn with maximal filters of the Lindenbaum algebra, that is, the free Boolean algebra over a denumerable infinite set of free generators.

Theorem 3.3. *Let* $v\colon Var \to \{0,1\}$ *be a truth-value assignment in classical propositional logic and let* Θ_v *be* $\{x_i \mid v(x_i) = 1\} \cup \{\neg x_i \mid v(x_i) = 0\}$*. Then there is a unique maximal theory extending* Θ_v *and moreover, the correspondence* $v \mapsto \Theta_v$ *is a bijection between the set of all truth-value assignments and maximal theories.*

Clearly, truth-value assignments can be identified with points in $\{0,1\}^\omega$, and maximal theories are in bijection with quotients of the Lindenbaum algebra over maximal filters. Further, these correspondences still hold on all the fragments with a finite number of variables: let Var_n, $Form_n$ and $\mathbf{F}_n(B)$ denote respectively the set of the first n variables, the set of all formulas over these variables, and the free Boolean algebra over n free generators.

Theorem 3.4. *For each* $n \in \omega \cup \{\omega\}$*, each pair of the following sets are in bijective correspondence.*

- *Points* $\mathbf{p} \in \{0,1\}^n$.
- *Truth-value assignments* $v\colon Var_n \to \{0,1\}$.
- *Maximal theories* $\Theta \subset Form_n$.
- *Maximal filters* $\mathfrak{p} \in Max(\mathbf{F}_n(B))$.

Notice in particular that for each truth-value assignment $v\colon Var_n \to \{0,1\}$, the quotient of $\mathbf{F}_n(B)$ over the filter $\mathfrak{p}_v = \{f \in \mathbf{F}_n(B) \mid f(v) = 1\}$ is isomorphic with the set of restrictions of elements in $\mathbf{F}_n(B)$, thought as functions $f\colon \{0,1\}^n \to \{0,1\}$, to the singleton $\{(v(x_1),\ldots,v(x_n))\}$. So, truth-values are the same as points in the domain of the functions forming the free Boolean algebra, and the evaluation of a formula under a given truth-value assignment is the same as restricting the function corresponding to that formula to the singleton formed by the point corresponding to that truth-value assignment. Given distinct assignments, there are formulas distinguishing them. These observations can be applied to the $[0,1]$-valued setting, by requiring that each point in $[0,1]^n$ behaves as a unique truth-value assignment, and *versa vice*, each assignment corresponds uniquely to a point. Further, the evaluation of formulas should undergo the same above-mentioned process, and, in particular, distinct points should be discerned by formulas.

Definition 3.5. A standard complete schematic extension L of MTL is *truly* $[0,1]$*-valued* iff, for every integer $n > 0$ and for every point $v \in [0,1]^n$ the correspondence $v \mapsto \{f \in \mathbf{F}_n(\mathbb{V}(L)) \mid f(v) = 1\}$, is a homeomorphism between $[0,1]^n$ and $Max(\mathbf{F}_n(\mathbb{V}(L)))$.

Theorem 3.6. *The only logic truly* $[0,1]$*-valued is Łukasiewicz infinite-valued logic.*

Proof. It is well known that the map $v \mapsto \{f \in \mathbf{F}_n(MV) \mid f(v) = 1\}$ is a homeomorphism between $[0,1]^n$ and $Max(\mathbf{F}_n(MV))$, whence Łukasiewicz logic is truly $[0,1]$-valued.

Now, assume L is truly $[0,1]$-valued. Since L is singly standard complete, $\mathbb{V}(L)$ is generated by a single standard structure $\mathbf{S}(L)$, and by Proposition 3.2, $\mathbf{F}_n(\mathbb{V}(L))$ is identifiable with the subalgebra of the algebra of all the functions $[0,1]^n \to [0,1]$ generated by the projections. Then in particular, maximal filters of $\mathbf{F}_n(\mathbb{V}(L))$ are in bijection with points of $[0,1]^n$ via the map $v \mapsto \{f \in \mathbf{F}_n(\mathbb{V}(L)) \mid f(v) = 1\}$. This implies that, for each $v \in [0,1]^n$, each algebra $\mathbf{A}(v)$ of the form $\{f \in \mathbf{F}_n(\mathbb{V}(L)) \mid f \restriction \{v\}\}$, being isomorphic to $\mathbf{F}_n(\mathbb{V}(L))$ modulo the maximal filter determined by the point v, is a simple algebra. But $\mathbf{A}(v)$ is obtained by substituting in $\mathbf{F}_n(\mathbb{V}(L))$ each generator x_i with the element $v_i \in [0,1]$. Whence $\mathbf{A}(v)$ is the chain generated by $\{v_1, \ldots, v_n\} \subseteq [0,1]$. Since $\mathbf{A}(v)$ is simple, for each $v_i \neq 1$ there is an integer k_i such that $v_i^{k_i} = 0$, otherwise v_i would generate a proper filter. But this property must hold for any n and any set $\{v_1, \ldots, v_n\} \subseteq [0,1]$, that is, for each $v \in [0,1)$ there is k such that $v^k = 0$. Then the conjunction of $\mathbf{S}(L)$ is a nilpotent t-norm, which in turn implies, by Proposition 2.1 that L is Łukasiewicz logic. ■

4. Quasi-standard complete logics

While in the previous section we have dealt with a reasonable strenghtening of the notion of standard completeness, in this section we propose and apply a weakening of the same notion in order to stress that some schematic extensions of MTL, which are usually considered only from the purely technical algebraic point of view, are almost as *fuzzy*-valued logic as major mathematical fuzzy logics such as Product and Gödel logics. We start by emphasising that singly standard complete logics may be given a sound and complete semantics which is very far from being $[0,1]$-valued. We recall that, when concrete representation matters, we identify free algebras $\mathbf{F}_n(L)$, for L a singly standard complete logic, with the algebra of functions $f : [0,1]^n \to [0,1]$ generated by the projections. Free Gödel algebras $\mathbf{F}_n(G)$ have been described in [11], and free product algebras $\mathbf{F}_n(P)$ in [12].

Fix any real number $\epsilon \in (0,1)$. Let $\iota_\epsilon(0) = \{0\}$ and $\iota_\epsilon(1) = [1-\epsilon, 1]$ be subsets of $[0,1]$. For each integer $n \geq 0$, with each point $\mathbf{b} \in \{0,1\}^n$ we associate the subset $\iota_\epsilon(\mathbf{b}) = \{\mathbf{p} \in [0,1]^n \mid p_i \in \iota_\epsilon(i)\}$.

Example 4.1. Let $\mathbf{G}_n$ be the Gödel algebra of restrictions of functions in $\mathbf{F}_n(G)$ to the set $\bigcup_{\mathbf{b}\in\{0,1\}^n} \iota_\epsilon(\mathbf{b})$. Then $\mathbf{G}_n \cong \mathbf{F}_n(G)$.

Let $\mathbf{P}_n$ be the Product algebra of restrictions of functions in $\mathbf{F}_n(P)$ to the set $\bigcup_{\mathbf{b}\in\{0,1\}^n} \iota_\epsilon(\mathbf{b})$. Then $\mathbf{P}_n \cong \mathbf{F}_n(P)$.

Examples in 4.1 show that a formula is a theorem of Gödel or of Product logic iff it evaluates identically to 1 for all assignments $v\colon Var_n \to \bigcup_{\mathbf{b}\in\{0,1\}^n} \iota_\epsilon(\mathbf{b})$, for arbitrary small values for ϵ. Actually, if we consider infinitesimal elements, living in the non-standard real interval $[0,1]$, in the sense of non-standard analysis, we obtain sound and complete semantics for both Gödel and Product logics, where the interval $\iota_\epsilon(1)$ is replaced by an infinitesimal left neighbourhood of 1. This kind of semantics is actually not a novelty, since it is considered the usual semantics of, for instance, the logic associated with Chang's MV-algebra S_2^ω (with the only difference that $\iota_\epsilon(0)$ is replaced by an infinitesimal right neighbourhood of 0). But this semantics is seldom, if ever, considered for Gödel and Product logics. The close relationship of $\mathbb{V}(S_2^\omega)$ with product algebras amount to a categorial equivalence between the two varieties, as shown in [13], [14]. Notice that the representation of Gödel and Product logics with infinitesimal truth-values around 1 suggests to consider these logics as variants of Boolean logic, as the non-infinitesimal values are exactly the Boolean truth-values. Approaching this observation from a topos-theoretic approach, in [15] is recalled that the subobject classifier in a category dually equivalent to finite Gödel algebras is very close in structure to the subobject classifier in the category of sets, that is the familiar notion of characteristic function of a set. Reversing the traditional interpretation, we shall now propose a semantics for S_2^ω which is very close to be a full standard semantics, and we shall generalise this to a family of other extensions of Łukasiewicz logics.

Definition 4.2. A schematic extension L of MTL is *quasi-standard complete* iff $\mathbb{V}(L)$ is generated by a class of algebras $Q(L)$ such that each $\mathbf{A} \in Q(L)$ has a universe which is a dense subset of $[0,1]$. L is *singly quasi-standard complete* iff $Q(L)$ can be chosen as a singleton.

Theorem 4.3. *For each integer $k > 1$, the logic associated with S_k^ω is singly quasi-standard complete.*

Proof. We start recalling that $S_k^{\mathfrak{c}} = \Gamma(\mathbb{Z} \vec{\times} \mathbb{R}, (k-1, 0))$ generates the same variety as S_k^ω. Now we fix an arbitrarily chosen monotonically non-decreasing bijection $f_k\colon \mathbb{R} \to (\frac{-1}{2k-2}, \frac{1}{2k-2})$ such that $f_k(0) = 0$: for sake of concreteness let $f_k(x) = \frac{\arctan(x)}{\pi(k-1)}$. Let now $h\colon S_k^{\mathfrak{c}} \to [0,1]$ be the function defined by

$$(m, x) \mapsto \frac{m}{k-1} + f_k(x)\,.$$

It is easy to check that h is non-decreasing and injective. The range $h[S_k^{\mathfrak{c}}]$ of h is $[0,1] \setminus \{\frac{2i+1}{2k-2} \mid i = 0, \ldots, k-2\}$, which is a dense subset of $[0,1]$. It remains to equip the range of h with the structure of an MTL-chain isomorphic with $S_k^{\mathfrak{c}}$. To achieve this it suffices to define the conjunction $*_k$ by going back and forth, as $x *_k y = h(h^{-1}(x) \odot_k^{\mathfrak{c}} h^{-1}(y))$, and, analogously, its residuum $\Rightarrow_k$ as $x \Rightarrow_k y = h(h^{-1}(x) \to_k^{\mathfrak{c}} h^{-1}(y))$. We conclude that $\mathbb{V}(S_k^\omega)$ is generated by $(h[S_k^{\mathfrak{c}}], *_k, \Rightarrow_k, 0)$ which is an MTL-chain whose universe is dense in $[0,1]$. ∎

No logic associated with a variety generated by S_k^ω, for all integers $k > 1$, is standard complete, as each subset of the form $\{(m, x) \mid x \in \mathbb{R}\}$, for a fixed m, is unbounded. The paper [16] introduces a family of t-norms defined by combining together chains of the form $h[S_k^{\mathfrak{c}}]$ and S_{k+1}.

Corollary 4.4. *The logic of each variety generated by a set of MV-chains of the form S_k^ω, for any integer $k > 1$, is quasi-standard complete. The only standard complete among them is Łukasiewicz infinite-valued logic.*

Theorem 4.5. *Let V be any variety of MV-algebras. Then exactly one of the following holds.*

1. *V is finitely valued, that is V is generated by a finite set of finite MV-chains.*
2. *V is quasi-standard complete.*
3. *V is the join of one finitely valued variety with one quasi-standard complete variety.*

Proof. By Komori's complete classification of varieties of MV-algebras. ■

Proposition 4.6. *DP is not quasi-standard complete. NM is singly standard complete and NM^- is not standard complete but it is singly quasi-standard complete.*

Proof. Any DP chain has a coatom. Each such a chain singly generates DP. Whence, any non-trivial DP chain whose universe is a subset of $[0, 1]$ must omit an interval of the form $(c, 1)$, where $c \in [0, 1)$ is the coatom. Therefore, every set of chains generating DP omits such an interval, that is, their universe is not dense in $[0, 1]$, and then DP is not quasi-standard complete. Recall that NM is generated by the standard nilpotent minimum algebra, while NM^- is generated by the subalgebra of the standard nilpotent minimum algebra obtained by removing from the universe $[0, 1]$ the point $\frac{1}{2}$ (see [14, 18, 19, 20]). Then the logic NM is singly standard complete, and the logic NM^- is singly quasi-standard complete. ■

Canonical DP chains are defined in [17] as having universe of the form $[0, c] \cup \{1\}$.

5. Conclusion and future work

The full $[0, 1]$ semantics provided by standard completeness lends itself to several applications. Just to mention one such application on the theoretical side, a logic L that is singly standard complete could be endowed with a notion of finitely additive measure over the space of truth-value assignments, allowing the development of a probability theory of non-classical events (or, *states*), where the events are modeled as formulas living in the non-classical, fuzzy, logic L (see [21, 22, 23, 24, 25]). Actually, full $[0, 1]$ semantics can be too strong a requirement for such a development: in future works we shall show how to develop states over logics which are only quasi-standard complete.

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