Natural non-group symmetry in modern applications

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Abstract

Intuitively perceived symmetry is formalized for effective application in physics, mathematics, and engineering. In this regard, several scientific research directions are indicated, which are expressed by three generalizations: a) the concept of symmetry using the example of normalized Hadamard matrices; b) cross vector product for the cases of three arguments and seven-dimensional space, c) Lorentz transformations for doubling the spacetime dimension. To generalize and formalize the concept of symmetry, the preservation of the symmetry of matrices under permutations of rows (columns) is studied. It is shown that the set of symmetry-preserving permutations does not constitute a group. For the development of the octonion toolkit and the best generalization of the vector product, based on symmetry considerations, the decomposition of the triple product of octonions into the sum of a triple anticommutator, a triple commutator (generalized vector product) and an associator is deduced. To begin the generalization of Lorentz transformations Lorentz boost is recorded in terms of quaternions so that the treated expressions retain their meaning in the octonionic space. To speed up the assimilation of the research results, the paper proposes some elementary information on the three listed topics, which it is desirable to place in reference books, as well as bring to the attention of students in general education courses at technical universities.

Keywords

Symmetrixes, symmetry, cross product, Lorentz boosts, quaternions

1. Introduction

More than 30 years ago the author has been starting research in mathematical physics with the formalization of *natural* (intuitively perceived) symmetry and published out the results in the USSR patent [1] for a game series. Later, this type of games was reinvented in Japan and was called "Symmetrixes" [2]. It was conceived, starting with games, then to publish the results related to the properties of spacetime symmetry. However, to our surprise, at the moment we found that some early scientific results partially retained their novelty. So, in this paper we reveal the scientific background of the mentioned games and offer the ideas of its further development and utilization.

In Section 2, the additive decomposition of an operator into self-adjoint and skew-symmetric parts is generalized, the notions of permutation and reassignment are refined, as well as the notion of permutational matrix symmetry is introduced. Section 3 deals with the development of the apparatus of hypercomplex numbers by generalizing the cross vector product. Section 4 describes the progress of work on the generalization of the Lorentz transformations. In the Conclusion some corrections and additions to commonly used reference books, as well as classic textbooks on the related topic, are discussed using specific examples.

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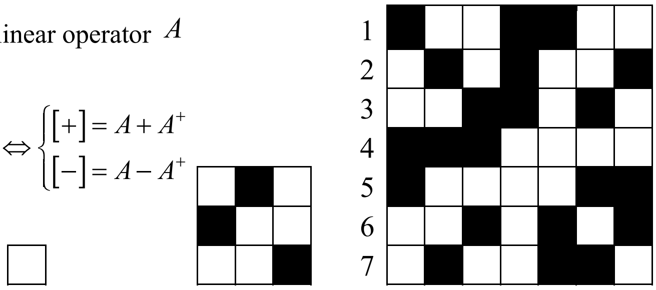
2. Permutable symmetric truncated normalized Hadamard matrices

A *truncated* (down-sized) normalized Hadamard matrix (NHM) is understood as a normalized Hadamard matrix [3] without the row and the column of only $+1$ that are deleted. NHMs arise in linear algebra when generalizing the additive decomposition of a linear operator into *symmetric* (self-adjoint) and skewsymmetric parts (Figure 1, [1,4]).

Expansion of a linear operator A

$$\begin{cases} A = \dfrac{[+]+[-]}{2} \\ A^{+} = \dfrac{[+]-[-]}{2} \end{cases} \Leftrightarrow \begin{cases} [+] = A + A^{+} \\ [-] = A - A^{+} \end{cases}$$



2	7	4	5	5	1	6	3
4	4	5	2	7	7	7	7
7	2	1	6	3	6	1	5
1	3	3	7	6	5	3	1
3	5	6	3	2	4	4	2
6	1	2	1	4	2	2	4
5	6	7	4	1	3	5	6

Figure 1: Permutable symmetric truncated NHMs

Black and white cells in Figure 1 denote $+1$ and, -1 respectively. At the top left in Figure 1, the well-known additive decomposition of a linear operator is written out. It is described by the truncated 1×1 NHM, that contains the single -1, and shown at the bottom left. The next, really black-and-white truncated 3×3 NHM describes the decomposition of a linear operator into four symmetric-skewsymmetric parts, which either change or retain their sign under the action of each of the two operations, for example, Hermitian conjugation " $^{+}$ " and the operator inversion " $^{-1}$ ". This 3×3 matrix preserves symmetry for any row permutation and is therefore always symmetric. The next black-and-white symmetric truncated 7×7 NHM with numbered rows describes the additive operator expansion into an octet of symmetric-skewsymmetric parts for three commuting operations of Hermitian conjugation type, which form an abelian group of the self-inverse operations. The extreme right table lists eight digital columns obtained by permutations of the rows of the truncated 7×7 NHM that preserve its symmetry. Cyclic repetition of any of eight specified permutations preserves the matrix symmetry. Each column from the columns, isolated by bold lines, gives, respectively, two, three, and six additional symmetric matrices. A total of 28 symmetric matrices are obtained.

It's remarkable, that the symmetry-preserving row permutations of 7×7 matrix in Figure 1 do not form a group, but are only a union of cyclic subgroups of some general group of 168 permutations that preserve the so-called "hidden symmetry" [1, 4]. The example of a matrix possessing hidden symmetry is shown in Figure 2

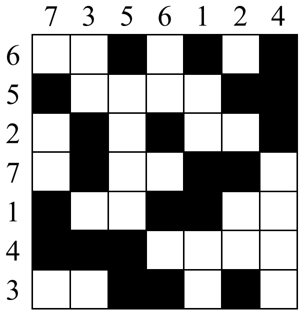


Figure 2: Asymmetrical matrix with hidden symmetry

The asymmetric matrix in Figure 2 is obtained by composition of a pair of permutations from different cyclic subgroups Figure 1.

Regarding the expected group properties of the permutable Hadamard matrices, note that the set of their rows, together with the unit row containing only the $+1$s, form an Abelian group of the self-inverse elements with respect to termwise row multiplication.

It is verified that among the symmetric black-and-white 7×7 matrices from all the different rows, the symmetric truncated 7×7 NHM are maximally permutable [1].

As it is easy to establish, the permutability of symmetric truncated 7×7 NHM determines the set of rows and the matrices themselves, accurate to the reversal of the roles of black and white cells. Moreover, to restore the septet of rows and the matrices, it is enough to use just three digital columns, selected from 28 ones according to the specific algorithm [5].

For the convenience of checking the statements under consideration, it is useful to distinguish between the concepts of "substitution", "permutation" and "reassignment".

As usual, *substitution* here refers to the reversible mapping of a set of some elements onto themselves, for example, for digital elements: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix} \equiv \begin{pmatrix} 2 & 4 & 1 & 3 \\ 4 & 2 & 3 & 1 \end{pmatrix}$. Let's order the top row in substitutions, as in the left one in the given example. Omitting the natural series of digits in the top row of substitutions, we represent the product of substitutions as the product of rows. For example $\begin{pmatrix} 3 & 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 4 & 2 \end{pmatrix}$. Similarly, the product of substitutions is represented as the product of columns:

$$\begin{matrix} 3 & 3 \\ 4 & 1 \\ 1 & 2 \\ 2 & 4 \end{matrix} = \begin{matrix} 1 \\ 3 \\ 4 \\ 2 \end{matrix}. \qquad (1)$$

Let's treat the product of a pair of substitutions on the left side (1) either as a *permutation* of the left column of the pair, or as a *reassignment* of its right column, by which the column on the right side (1) is obtained.

The meaning of the formulated permutation and reassignment definition manifests itself when multiplying by a given column on the right or/and on the left of some table of several columns and lies in the fact that the columns of the table can be transformed term by term, or the entire table can be transformed with the same result. Moreover, due to the associativity of a substitution composition, when performing multiplication on the right and on the left, the result does not depend on the order in which the right and left multiplication is performed.

For a pair α, β of any symmetry-preserving permutations from the set S, it is true that their composition $\alpha\beta\alpha$ also belongs to S:

$$\alpha \in S, \beta \in S \quad \Rightarrow \quad \alpha\beta\alpha \in S \qquad (2)$$

Property (2) means that the complete permutability table of 28 columns for Figure 1 is preserved up to the permutation of the columns when multiplying from the left and also from the right by any of its columns.

Property (2) implies that cyclic permutation groups are subsets of the set S:

$$\alpha \in S \quad \Rightarrow \quad \alpha^k \in S, \quad k = 1, 2, ... \qquad (3)$$

A specific feature of just the 7×7 black-and-white matrix in Figure 1 is that it is antisymmetric with respect to the secondary diagonal, wherein antisymmetry notion is defined with the necessary reservations regarding the elements occupying the secondary diagonal [1]. According to the [1], non-trivial permutations of the columns of the truncated 7×7 NHM in Figure 1, provide to get eight such symmetric-antisymmetric matrices of the 7th order. Are there such symmetric-antisymmetric matrices of the 15th order, that is the current important question.

3. Twofold generalization of a cross vector product

The above-mentioned decomposition of the operator into symmetric-skewsymmetric parts turned out to be useful in a twofold generalization of the vector product to the case of three arguments, as well as to the seven-dimensional subspace of the eight-dimensional octonionic space [6,7].

In [7], a linear operator is considered, which is produced by conjugation of a vector argument with subsequent multiplication by fixed vectors on the left and on the right. Then, the decomposition of the product of three octonions into four symmetric-skewsymmetric parts is obtained using a pair of suitable operations of the Hermitian conjugation type. One of four part turned out to be zero. The rest of the parts are: triple anticommutator, triple commutator and associator. Triple commutator $[u_1,u_2,u_3]$ is defined in two ways by the formulae:

$$[u_1,u_2,u_3]=\frac{(u_1\bar{u}_2)u_3-u_3(\bar{u}_2u_1)}{2}\equiv\frac{u_1(\bar{u}_2u_3)-(u_3\bar{u}_2)u_1}{2}, \tag{4}$$

where u_1, u_2, u_3 are arbitrary octonions, $\bar{u}_2=2(u_2,i_0)i_0-u_2$ is conjugated octonion u_2, i_0 is the multiplicative identity, i.e. a unit vector along the real axis.

The triple commutator $[u_1,u_2,u_3]$ possesses the property of antipermutability of its arguments, is orthogonal to each argument, and turns into an ordinary two-argument cross vector product when the central argument is replaced by the multiplicative identity i_0. It also has other properties that are characteristic for an ordinary two-argument cross vector product that are detailed in [7]. So, the triple commutator $[u_1,u_2,u_3]$ is exactly what it is a doubly generalized cross vector product, coinciding with the conventional two-argument cross vector product $[u_1,u_3]$ written in the space of quaternions or octonions [7]. And the expressions for $[u_1,u_2,u_3]$ taken from [7] complete the prolonged search [8–11] for most perspective generalization of cross vector product.

4. The problem to generalize Lorentz transformations

It was William Rowan Hamilton who first discovered quaternions as the spacetime [12]. Nowadays the generalization of quaternions (octonions) and the double generalization of the cross vector product have been invented, which provides convenient work with non-associative octonions. So, a generalization of the Lorentz transformations suggests itself in order to clarify and develop the motion laws. On the way to this goal, a quaternionic record of Lorentz boost was found in [13,14]. It turns out that the Lorentz boost is decomposed into a linear combination of rotation and orthogonal multiplicative transformation, expressing in twofold ways by the formulae:

$$Lu=\bar{a}ua-\mathrm{sh}\,\theta\cdot n\bar{u}=L^{+}u=au\bar{a}-\mathrm{sh}\,\theta\cdot\bar{u}n, \tag{5}$$

where the cross denotes the Hermitian conjugation, $a=i_0\cdot\mathrm{ch}\frac{\theta}{2}+n\cdot\mathrm{sh}\frac{\theta}{2}$, n is the unit vector along the speed, θ is the rapidity: $\mathrm{th}\,\theta=\frac{\mathrm{v}}{\mathrm{c}}$, v is the speed magnitude, c is the speed of light.

It should be noted that in [15] the Lorentz boost is expressed by the half-expression (5), but this is done by increasing the dimension of the vector space.

It is noteworthy that quaternionic expressions of Lorentz boost, as well as the rotation expression $V\{u\}=\bar{b}ub,\ (b,b)=1$ do not depend on the multiplication order and retain their meaning in octonions, that exemplify the possible generalization of Lorentz transformations. At first glance, this is quite sufficient for the eight-dimensional generalization of the Lorentz transformations VL as a superposition of rotation V and Lorentz boost L. This may be true, but other options should also be considered for comparison.

In four-dimensional space, both the rotation V and the Lorentz boost L are the elementary transformations that modify some two-dimensional plane and do not change the orthogonal vectors in another two-dimensional plane.

In both quaternions and octonions, Lorentz boost (5) describes the stretching by a certain number of times of one vector and the contraction by the same number of times of another vector that belong to the same complex plane while preserving purely spatial vectors that are orthogonal to this complex plane. The transformation of the rotation $V\{u\}=\bar{b}ub, (b,b)=1$ in octonions is more complicated than the Lorentz boost, because it modifies the rest six-dimensional subspace of pure spatial vectors, while maintaining the complex plane defined by the rotational axis.

Lorentz boost L differs significantly from the rotation V in that it has a full quartet of basis eigenvectors, while the rotation V through a nontrivial angle preserves only the directions of the rotational and the time axes. For this reason, the general Lorentz transformations in the form of a composition VL of rotation V and boost L are subdivided into boost-like transformations with a quartet of basis vectors, and the rest ones, referring to as rotation-like. The most interesting is that the composition L_1L_2 of Lorentz boosts L_1 and L_2 is always a boost-like transformation [14].

The eigenvectors for the composition L_1L_2 of the Lorentz boosts together with the corresponding eigenvalues are listed in Table 1.

Table 1

Eigenvectors for the composition of Lorentz boosts L_1L_2

Notation	Eigenvector	Eigenvalue
c_0	$i_0 - d\big\|_{\xi=\exp(\chi)}$	$\exp(\chi)$
c_1	$i_0 - d\big\|_{\xi=\exp(-\chi)}$	$\exp(-\chi)$
c_2	$i_0 - n_1\dfrac{\coth\frac{\theta_1}{2}+(n_1,n_2)\coth\frac{\theta_2}{2}}{1-(n_1,n_2)^2}+n_2\dfrac{\coth\frac{\theta_2}{2}+(n_1,n_2)\coth\frac{\theta_1}{2}}{1-(n_1,n_2)^2}$	1
c_3	$[n_1,n_2]$	1

In Table 1 n_1 and n_2 are the unit spatial vectors along the considered intersecting velocities, such that $(n_1,n_1)=(n_2,n_2)=1$ and $(n_1,i_0)=(n_2,i_0)=0$. The cross vector product $[n_1,n_2]$ is directed along the Wigner rotational axis [16], so that $[n_1,n_2]=v\sqrt{1-(n_1,n_2)^2}$. The spatial part of the eigenvectors c_0 and c_1 depends on the eigenvalue ξ and, up to the sign, coincides with the unit vector d_ξ that is defined as a function of eigenvalue ξ in the form [14]:

$$d_\xi=\frac{n_1\sqrt{\xi}\sinh\frac{\theta_1}{2}+n_2\sinh\frac{\theta_2}{2}}{\sqrt{\xi}\cosh\frac{\theta_1}{2}-\cosh\frac{\theta_2}{2}}. \tag{6}$$

The spatial parts $-d\big|_{\xi=\exp(\chi)}$ and $-d\big|_{\xi=\exp(-\chi)}$ of the eigenvectors c_0 and c_1 are obtained by substituting in (6) the values ξ by $\exp(\chi)$ and $\exp(-\chi)$, respectively. The scalar parameter χ is defined in accordance with well-known cosine rule:

$$\cosh\frac{\chi}{2}=\cosh\frac{\theta_1}{2}\cosh\frac{\theta_2}{2}+(n_1,n_2)\sinh\frac{\theta_1}{2}\sinh\frac{\theta_2}{2}. \tag{7}$$

and the scalar parameters θ_1 and θ_2 are the rapidities, such that the velocities v_1, v_2 divided by scalar speed of light c are expressed as $\mathrm{v}_1/\mathrm{c}=n_1\tanh\theta_1$, $\mathrm{v}_2/\mathrm{c}=n_2\tanh\theta_2$. Note that (7) refers to the half hyperbolic angles $\chi/2$, $\theta_1/2$ and $\theta_2/2$, while the well-known velocity addition is expressed via holistic hyperbolic angles θ, θ_1 and θ_2 [17,18].

It should be noted that the expressions for the eigenvectors of Table 1 turned out to be laconic. However, they are obtained by rather cumbersome intermediate calculations in terms of quaternions [14]. Meanwhile, it is precisely the laconism of calculations that is usually the main advantage of quaternions. So, the use of quaternions alone is not enough for a transparent derivation of the formulae Table 1. Apparently, for a transparent representation of obtaining of the eigenvectors Table 1 it will be useful to decompose L_1L_2 into symmetric-skewsymmetric parts according to Figure 1 and accompanying symmetry considerations. To generalize Lorentz transformations, it seems worth trying to generalize the laconic formulae (5) for the Lorentz boost to the case of a composition L_1L_2 of boosts L_1 and L_2, say, expressing them in terms of eigenvectors and eigenvalues from Table 1.

The eigenvectors from Table 1 form the basis of the considered space R^4. Their pairwise pseudoscalar products $(c_i,\bar{c}_k),\quad i,k=0,1,2,3$, accounting for commutativity, are given by the formulae (8).

$$\begin{gathered}
(c_0,\bar{c}_0)=(c_0,\bar{c}_2)=(c_0,\bar{c}_3)=(c_1,\bar{c}_1)=(c_1,\bar{c}_2)=(c_1,\bar{c}_3)=(c_2,\bar{c}_3)=0,\\
(c_0,\bar{c}_1)=-2\frac{\sinh^2\frac{\chi}{2}}{\cosh^2\frac{\theta_1}{2}+\cosh^2\frac{\theta_2}{2}-2\cosh\frac{\chi}{2}\cosh\frac{\theta_1}{2}\cosh\frac{\theta_2}{2}},\\
(c_2,\bar{c}_2)=1-\frac{\coth^2\frac{\theta_1}{2}+\coth^2\frac{\theta_2}{2}+2(n_1,n_2)\coth\frac{\theta_1}{2}\coth\frac{\theta_2}{2}}{1-(n_1,n_2)^2},\\
(c_3,\bar{c}_3)=(n_1,n_2)^2-1.
\end{gathered} \tag{8}$$

Using formulae (8), it is easy to expand an arbitrary vector in the basis of the eigenvectors of the composition L_1L_2 of boosts L_1 and L_2, and then obtain an expression for the composition L_1L_2 in terms of its eigenvectors $c_0\ c_1\ c_2\ c_3$:

$$L_1L_2u=\frac{c_0\exp(\chi)(u,\bar{c}_1)+c_1\exp(-\chi)(u,\bar{c}_0)}{(c_0,\bar{c}_1)}+c_2\frac{(u,\bar{c}_2)}{(c_2,\bar{c}_2)}+c_3\frac{(u,\bar{c}_3)}{(c_3,\bar{c}_3)}. \tag{9}$$

Expression (9) determines the composition of the Lorentz boosts L_1L_2 in terms of its own eigenvectors. Obviously, this formula can be rewritten as a linear combination of orthogonal transformations, which are elegantly expressed in terms of quaternionic multiplication. In this case, we obtain a generalization of the quaternionic record of the single Lorentz boost (5) to the case of a composition of a pair of Lorentz boosts, which is useful for further generalization to the case of an eight-dimensional octonionic space.

5. Conclusions

The current tasks formulated in the final paragraphs of the second and fourth sections are quite capable of solving by interested senior students.

And it is didactic, that in [19], in the last paragraph of the section "Quaternions in Vector Symbolics", Erwin Madelung refers the rotation $bub,\ (b,b)=1$ of the complex plane in the quaternionic space to the class of Lorentz transformations. It would be appropriate to supplement this paragraph with the formulae (5) for Lorentz boost, so that the beginners do not mix it with the orthogonal transformations. When assimilating hypercomplex numbers (quaternions and octonions) according to the well-known book [6], one should pay attention to the fact that formula (8) is written out twice in the book, but when used for the first time the unfortunate mistake was made in this formula (the comma is omitted), which radically changes the meaning. The definition and examples of utilization of the doubly generalized vector product (4) can be presented as a useful application to [6]. Formal clarification (1) of difference between the concepts of permutation and reassignment (redesignation) compensate for the rather confused interpretation of these concepts in most reference books on mathematics and physics.

The latter examples show the need for some modernization of classical reference books and guidelines, aids for lecturers teaching students to solve the problems of current interest. The method of additive expansion of a linear operator and concomitant symmetry considerations Figure 1 deserve increased attention, as well as an introduction to the vector product in terms of quaternions. This is so because the conventional cross vector product appeals to intuition in the "left-hand rule," which prevents its generalization. Compared to the conventional one, the quaternionic cross vector product was invented earlier and is more promising for the effective development of scientific and, all the more so, engineering research. In particular, it is promising for generalizing of classical transformations of coordinates to better understand the laws of motion. Let's take this into account in the future.

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