On Variational Problem with Nonstandard Growth Conditions for the Restoration of Clouds Corrupted Satellite Images

Pavel Khanenko[1,2], Peter I. Kogut[3,4] and Mykola Uvarov[1,5]

[1]Max Planck Institute for Chemical Physics of Solids, 01187 Dresden, Germany

[2]EOS Data Analytics Ukraine, Desyatynny lane, 5, 01001 Kyiv, Ukraine

[3]Department of Differential Equations, Oles Honchar Dnipro National University, Gagarin av., 72, 49010 Dnipro, Ukraine

[4]EOS Data Analytics Ukraine, Gagarin av., 103a, Dnipro, Ukraine

[5]Department of Computational Physics, G. V. Kurdyumov Institute for Metal Physics, Kyiv, Ukraine

Abstract

Sensitivity to weather conditions, and specially to clouds, is a severe limiting factor to the use of optical remote sensing for Earth monitoring applications. Typically, the optical satellite images are often corrupted because of poor weather conditions. As a rule, the measure of degradation of optical images is such that one can not rely even on the brightness inside of the damaged regions. As a result, some subdomains of such images become absolutely invisible. So, there is a risk of information loss in optical remote sensing data. In view of this, we propose a new variational approach for exact restoration of multispectral satellite optical images. We discuss the consistency of the proposed variational model, give the scheme for its regularization, derive the corresponding optimality system, and discuss the algorithm for the practical implementation of the reconstruction procedure. Experimental results are very promising and they show a significant gain over baseline methods using the reconstruction through linear interpolation between data available at temporally-close time instants.

Keywords

Risk of cloud distortion of satellite images, Risk of information loss, Image restoration, Variational approach

1. Introduction

A very serious obstacle to utilization of optical remote sensing satellite images is a risk of cloud and cloud shadow distortion issue (referred to as cloud contamination, hereafter). It has been reported in [1] that over 50% of all the Moderate Resolution Imaging Spectroradiometer (MODIS) instrument aboard the Terra and Aqua satellites are covered by clouds or cloud-contaminated globally. Moreover, it is a typical situation when the measure of degradation of optical images

2^{nd} *International Workshop on Computational & Information Technologies for Risk-Informed Systems, CITRisk–2021, September 16-17, 2021, Kherson, Ukraine*

pavel.khanenko@eosda.com (P. Khanenko); peter.kogut@eosda.com (P. I. Kogut); nikolay.uvarov@eosda.com (M. Uvarov)
https://eos.com (P. Khanenko); https://kogut.uaua.us (P. I. Kogut); https://eos.com (M. Uvarov)
0000-0003-1593-0510 (P. I. Kogut)

is such that we can not rely even on the brightness inside of the damaged regions. As a result, there is a risk of information loss, some subdomains of such images become absolutely invisible. So, there is a great deal of missing information in optical remote sensing data, and a huge gap still exists between the satellite data we acquire and the data we require. Therefore, the reconstruction of missing information in remote sensing data becomes an active research field.

Many solutions have been developed to remove the clouds from multispectral images. (for the technical review, we refer to [2]). Formally, the present traditional algorithms can be primarily classified into four categories: 1) spatial-based methods, without any other auxiliary information source; 2) spectral-based methods, which extract the complementary information from other spectra; 3) multitemporal-based methods, which extract the complementary information from other data acquired at the same position and at different time periods; and 4) hybrid methods, which extract the complementary information by a combination of the three previous approaches. In parallel to the traditional approach, data-driven machine learning algorithms are actively developing since 2014 [3]. However, the necessity of large datasets and volatility to errors in input data limits its performance.

Our main effort in this research is to develop a new variational model for the exact restoration of the damaged multi-band optical satellite images that will meet demands from the agro application, that is, it must be applicable for large areas in different climate zones, and preserves the crop fields borders within damaged regions. In some sense, this model combines the spacial-based method with the multitemporal one. Therefore, in contrast to the standard variational-based methods that are often optimased for a specific region or significantly blur textures, (see, for instance, [4] and the references therein), we focus on the global texture reconstruction inside damage regions. With that in mind we assume that the texture of a corrupted image can be predicted through a number of past cloud-free images of the same region from the time series. In order to describe the texture of background surface in the damage region, we follow the paper [5], where the authors experimentally checked the hypothesis that the essential geometric contents of a color image is contained in the level lines of the corresponding total spectral energy of such image.

We also pay much attention to the faithfulness of the reconstruction problem in the framework of the proposed model and supply this approach by the rigorous mathematical substantiation. The experiments undertaken in this study confirmed the efficacy of the proposed method and revealed that it can acquire plausible visual performance and satisfactory quantitative accuracy for agro scenes with rather complicated texture of background surface.

2. Preliminaries

Let $\Omega \subset \mathbb{R}^2$ be a bounded image domain with a Lipschitz boundary $\partial\Omega$. With each particular image $\vec{u} = [u_1(x), u_2(x), \dots, u_M(x)]^t : \Omega \to \mathbb{R}^M$, where each coordinate represents the intensity of the corresponding spectral channel, we associate the panchromatic image u (the so-called total spectral energy of $\vec{u}$)

$$u(x) = \alpha_1 u_1(x) + \dots \alpha_M u_M(x). \tag{1}$$

Here, $\alpha_1, \dots, \alpha_M$ are some weight coefficients.

Let $D \subset \Omega$ be a Borel set with non empty interior and sufficiently regular boundary and such that $|\Omega \setminus D| > 0$. We call D the damage region. Let $\vec{u}_0 \in L^2(\Omega \setminus D; \mathbb{R}^M)$ be an image of interest which is assumed to be corrupted by clouds, and D is the zone of missing information.

As it was mentioned before, we deal with the case where we have no information about the original image $\vec{u}_0$ inside D. Instead of this, we assume that the texture of background surface in the damage region D can be predicted with some accuracy by a number of past cloud-free images of the same region from the time series of satellite images. Unlike the well-known 'chronochrome method' [6] which essentially assumes that the background in D is stationary in wide sense, we admit that the image time series follows smooth variations over land (background), the time-series data are strictly chronological, and display regular fluctuations.

Let $\{\vec{u}_{t-1}, \dots, \vec{u}_{t-n}\}$ be a given collection of past cloud-free images of the same region, where we set $\vec{u}_t = \vec{u}_0$. We suppose that each cloud-free image of this time series should be well co-registered with $\vec{u}_0 \in L^2(\Omega \setminus D; \mathbb{R}^M)$ in $\Omega \setminus D$ [7]. With each particular image $\vec{u}_{t-k}$ in this series, we associate its total spectral energy u_{t-k} using the standard rule (1). So, each element of the time series $\{u_t, u_{t-1}, \dots, u_{t-n}\}$ is well-defined in Ω.

Let u^* be a predicted total spectral energy of $\vec{u}_0$ in the damage region D. This prediction can be done following the regularized regression model and the available information in the time series $\{u_t, u_{t-1}, \dots, u_{t-n}\}$ (for the details we refer to [8]).

$$\begin{aligned}\mathcal{L}(w) = \frac{\beta_1}{|D|} \int_D \left[u_{t-1} - \sum_{k=2}^{n} w_{k-1} u_{t-k}\right]^2 dx \\ + \frac{\beta_1}{|\Omega \setminus D|} \int_{\Omega \setminus D} \left[u_t - \sum_{k=1}^{n-1} w_k u_{t-k}\right]^2 dx + \lambda \|w\|^2_{\mathbb{R}^{n-1}} \to \inf, \quad (2)\end{aligned}$$

where $\lambda > 0$ is the regularization parameter, $\beta_1 > 0$ and $\beta_2 > 0$ are the parameters that control the importance of the prediction and estimation terms, respectively. Seeing that the prediction and estimation errors in (2) are normalized by the volume of samples contributing to each term, we can constrain the values of $\beta_1 = \beta \in [0, 1]$ and $\beta_2 = 1 - \beta$ to control their relevance with a single parameter β.

As a result, setting $w^0 = \operatorname{Argmin} \mathcal{L}(w)$, the total spectral energy u^* in the damage region D can be estimated as follows $u^* = \widehat{u}$ in D, where

$$\widehat{u}(x) = \sum_{k=1}^{n-1} w_k^0 u_{t-k}(x), \quad \forall\, x \in \Omega.$$

In order to reconstruct the texture (or geometry) of $\vec{u}_0$ in the damage region D, we assume that predicted total energy u^* is a function of bounded variation, i.e. $u^* \in BV(D)$, and all spectral channels of the damaged image should share the geometry of the panchromatic image $u^* \in L^2(D)$ in D. Hence, at most all points of almost all level sets of $u^* \in BV(D)$ we can define a normal vector $\theta(x)$, i.e., it formally satisfies $(\theta, u^*) = |\nabla u^*|$ and $|\theta| \leq 1$ a.e. in D.

3. Problem Statement

In view of the risk of cloud distortion, the problem is to reconstruct the original multi-band image $\vec{u}_0$ in the damage region D using the knowledge of its texture (geometry) on the subset D together with the exact information about this image in $\Omega \setminus D$ (the undamaged region). We say that a function $\vec{u} = [u_1, u_2, \dots, u_M]^t : \Omega \to \mathbb{R}^M$ is the result of restoration of a cloud corrupted image $\vec{u}_0 : D \to \mathbb{R}^M$ if for given regularization parameters $\mu>0$, $\alpha>1$, and $\lambda_j>0$, $j = 1, 2$, each spectral component u_i is the solution of the following constrained minimization problem with the nonstandard growth energy functional

$$
\begin{aligned}
(\mathcal{P}_i) \quad J_i(v,p) := & \int_\Omega \frac{1}{p(x)} |\nabla v(x)|^{p(x)}\, dx + \frac{\mu}{\alpha} \int_{\Omega\setminus D} |v(x) - u_{0,i}(x)|^\alpha \, dx \\
& + \lambda_1 \int_{\Omega\setminus D} \left|\nabla G_\sigma * (v - u_{0,i})\right|^2 dx + \lambda_2 \int_D \left| \left(\theta^\perp, \nabla v\right) \right|^\alpha dx \longrightarrow \inf_{(v,p)\in\Xi}, \quad (3)
\end{aligned}
$$

where

- Ξ stands for the set of feasible solutions to the problem (3) which we define as follows

$$
\Xi = \left\{ (v,p) \;\middle|\; \begin{array}{l} v \in W^{1,p(\cdot)}(\Omega),\ p \in C(\overline{\Omega}), \\ 1 \le \gamma_0 \le v(x) \le \gamma_1 \text{ a.a. in } \Omega, \\ p(x) = \mathfrak{F}(v(x)) \ \text{ in } \Omega. \end{array} \right\}
$$

  Here, $W^{1,p(\cdot)}(\Omega)$ is the Sobolev-Orlicz space,

$$
\mathfrak{F}(v(x)) = 1 + g\left(\left|\left(\nabla G_\sigma * v\right)(x)\right|\right),
$$

  and g:$[0, \infty) \to (0, \infty)$ is the edge-stopping function which we take it in the form of the Cauchy law $g(t) = \frac{1}{1+(t/a)^2}$ with an appropriate $a > 0$;
- $\theta \in L^\infty(D, \mathbb{R}^2)$ is a given vector field such that

$$
|\theta(x)| \le 1 \ \text{ and } \ (\theta(x), \nabla u^*(x))_{\mathbb{R}^2} = |\nabla u^*(x)| \quad \text{a.e. in } D;
$$

- $(G_\sigma * v)(x)$ determines the convolution of function v with the two-dimensional Gaussian filter kernel G_σ, where the parameter $\sigma>0$ determines the spatial size of the image details which are removed by this 2D filter;

By default we assume that the functions $u_{0,i}$ and u^* are extended by zero outside of domains $\Omega \setminus D$ and D, respectively.

The proposed model (3) provides a completely new approach to restoration of non-smooth multi-spectral images $\vec{u}_0$ with the gap in damage region. The main characteristic feature of this model is that we involve into consideration the energy functional with the nonstandard growth condition. The variable character of the exponent $p(x)$ provides more flexibility in terms of regularity for the recovered images. Since the first term in (3) is the regularization and the second one is the so-called data fidelity, it is worth to emphasize the role of the rest terms in

(3). Taking into account the fact that the magnitude $\mathfrak{F}(v(x))$ is close to one at those points, where the spectral energy v is slowly varying, and this value is close to zero at the edges of v, it follows that the edge information in the non-damage zone for the reconstruction is derived from the given image $\vec{u}_0$. So, in view of the estimate

$$\begin{aligned}\int_{\Omega\setminus D} |p(x) - \mathfrak{F}(u_{0,i})|\, dx &= a^2 \int_{\Omega\setminus D} \left| \frac{|(\nabla G_\sigma * v)|^2 - |(\nabla G_\sigma * u_{0,i})|^2}{\left(a^2 + |(\nabla G_\sigma * v)|^2\right)\left(a^2 + |(\nabla G_\sigma * u_{0,i})|^2\right)} \right| dx \\ &\le \frac{1}{a^2} \int_{\Omega\setminus D} \Big(|(\nabla G_\sigma * v)| + |(\nabla G_\sigma * u_{0,i})| \Big) \Big| |(\nabla G_\sigma * v)| - |(\nabla G_\sigma * u_{0,i})| \Big| \, dx \\ &\le \frac{2\|G_\sigma\|_{C^1(\overline{\Omega - \Omega})} \gamma_1 |\Omega|^{\frac{3}{2}}}{a^2} \left(\int_{\Omega\setminus D} \left| \nabla G_\sigma * (v - u_{0,i}) \right|^2 dx \right)^{\frac{1}{2}},\end{aligned}$$

the third term is also fidelity term which forces the texture (or topological map) of minimizer u in domain $\Omega \setminus D$ to stay close to the texture of a given spectral energy $\vec{u}_{0,i}$.

As for the last term in (3), we notice that since $\theta \in L^\infty(D, \mathbb{R}^2)$ is a vector field with indicated properties, it follows that $\theta(x)$ has the direction of the normal to the level lines of u^*. Therefore, the counterclockwise rotation of angle $\pi/2$, denoted by $\theta^\perp$, represents the tangent vector to the level lines of u^*. In this case, if the spectral channels u_i share the geometry of the panchromatic image u^*, we have

$$\left(\theta^\perp, \nabla u_i\right)_{\mathbb{R}^2} = 0, \quad i = 1, \dots, M \text{ in } D.$$

Therefore, we impose them in the energy functional J_i in the form of the last term.

In practice, at the discrete level, θ can be defined by the relation $\theta(x_1, x_2) = \frac{\nabla \widehat{u}(x_1,x_2)}{|\nabla \widehat{u}(x_1,x_2)|}$ when $\nabla \widehat{u}(x_1, x_2) \neq 0$, and $\theta(x_1, x_2) = 0$ when $\nabla \widehat{u}(x_1, x_2) = 0$. However, a better choice would be to compute $\theta(x_1, x_2)$ by first regularizing $\widehat{u}$ by the equation

$$\frac{\partial v}{\partial t} = \operatorname{div}\left(\frac{\nabla v}{|\nabla v|}\right) \quad \text{in } (0, \infty) \times \Omega,$$

coupled with the initial and Neumann boundary conditions

$$v(0, x_1, x_2) = \widehat{u}(x_1, x_2), \quad \text{for a.a. } (x_1, x_2) \in \Omega, \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

Then, for any $t \ge 0$, there is a vector field $\xi(t) \in L^\infty(\Omega)$ with $\|\xi(t)\|_{L^\infty(\Omega)} \le 1$ such that (see [9, 10] for the details)

$$\begin{gathered}(\xi(t), \nabla v(t)) = |\nabla v(t)| \text{ in } \Omega, \quad (\xi(t), \nu) = 0 \text{ on } \partial\Omega, \\ \text{and } \frac{\partial v}{\partial t} = \operatorname{div}(\xi(t)) \text{ in the sense of distributions in } (0, \infty) \times \Omega.\end{gathered}$$

As a result, in order to characterize the texture of the cloud contaminated image $\vec{u}_0$ in the damage region D, we may take $\theta = \xi(t)$ for some small value of $t > 0$. As was mentioned in [9], following this way, we do not not distort the geometry of $\widehat{u}$ in an essential way.

So, the novelty of the model that we propose, is that the edge information for the multispectral restoration in Ω is accumulated in the variable exponent $p(x)$ which we derive from the time series and initial data.

4. Existence Result

In this section we show that constrained minimization problem (3) is consistent and admits at least one solution $(u_i^{rec}, p_i^{rec}) \in \Xi$, where $p_i^{rec}(x) = \mathfrak{F}(u_i^{rec}(x))$ in Ω. We note that because of the specific form of the energy functional $J_i(v,p)$ in (3), the standard approaches are no longer applicable in its study, especially with respect to the existence of minimizers and their basic properties. It makes the minimization problem (3) rather challenging.

We begin with some auxiliary results which will play a crucial role in the sequel.

Lemma 4.1. *Let $\{v_k\}_{k\in\mathbb{N}}$ be a sequence of measurable non-negative functions $v_k : \Omega \to [\gamma_0, \infty)$ such that $\{v_k\}_{k\in\mathbb{N}}$ are uniformly bounded in $L^1(\Omega)$ and $v_k(x) \to v(x)$ almost everywhere in Ω for some $v \in L^1(\Omega)$. Let $\{p_k = 1 + g\left(|(\nabla G_\sigma * v_k)|\right)\}_{k\in\mathbb{N}}$ be the corresponding sequence of variable exponents. Then*

$$\begin{aligned} p_k(\cdot) \to p(\cdot) = 1 + g\left(|(\nabla G_\sigma * v)(\cdot)|\right) \quad &\textit{uniformly in } \overline{\Omega} \textit{ as } k \to \infty, \\ \alpha := 1 + \delta \le p_k(x) \le \beta := 2, \quad &\forall\, x \in \Omega,\ \forall\, k \in \mathbb{N}, \end{aligned} \tag{4}$$

where

$$\delta = \frac{a^2}{a^2 + \|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})} \sup_{k\in\mathbb{N}} \|v_k\|^2_{L^1(\Omega)}}.$$

Proof. Since $\{v_k\}_{k\in\mathbb{N}}$ is the bounded sequence in $L^1(\Omega)$, by smoothness of the Gaussian filter kernel G_σ, it follows that

$$\begin{aligned} |(\nabla G_\sigma * v_k)(x)| &\le \int_\Omega |\nabla G_\sigma(x-y)|\, v_k(y)\, dy \le \|G_\sigma\|_{C^1(\overline{\Omega-\Omega})} \|v_k\|_{L^1(\Omega)}, \\ p_k(x) &= 1 + \frac{a^2}{a^2 + \left(|(\nabla G_\sigma * v_k)(x)|\right)^2} \\ &\ge 1 + \frac{a^2}{a^2 + \|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})} \|v_k\|^2_{L^1(\Omega)}}, \quad \forall\, x \in \Omega. \end{aligned}$$

Then L^1-boundedness of $\{v_k\}_{k\in\mathbb{N}}$ guarantees the existence of a positive value $\delta \in (0,1)$ such that estimate (4) holds true for all $k \in \mathbb{N}$.

Moreover, as follows from the estimate

$$\begin{aligned} |p_k(x) - p_k(y)| &\le a^2 \left| \frac{|(\nabla G_\sigma * v_k)(x)|^2 - |(\nabla G_\sigma * v_k)(y)|^2}{\left(a^2 + |(\nabla G_\sigma * v_k)(x)|^2\right)\left(a^2 + |(\nabla G_\sigma * v_k)(y)|^2\right)} \right| \\ &\le \frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})} \|v_k\|_{L^1(\Omega)}}{a^2} \left| |(\nabla G_\sigma * v_k)(x)| - |(\nabla G_\sigma * v_k)(y)| \right| \\ &\le \frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})} \gamma_1^2 |\Omega|}{a^2} \int_\Omega |\nabla G_\sigma(x-z) - \nabla G_\sigma(y-z)|\, dz,\ \forall\, x,y \in \Omega \end{aligned}$$

and smoothness of the function $\nabla G_\sigma(\cdot)$, there exists a positive constant $C_G > 0$ independent of k such that

$$|p_k(x) - p_k(y)| \le \frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})} \gamma_1^2 |\Omega| C_G}{a^2} |x - y|,\ \forall\, x,y \in \Omega.$$

Setting

$$C := \frac{2\|G_\sigma\|_{C^1(\overline{\Omega-\Omega})}\gamma_1^2|\Omega|C_G}{a^2}, \tag{5}$$

we see that

$$\{p_k(\cdot)\} \subset \mathfrak{S} = \left\{ h \in C^{0,1}(\Omega) \;\middle|\; \begin{array}{c} |h(x)-h(y)| \leq C|x-y|, \ \forall\, x, y, \in \Omega, \\ 1 < \alpha \leq h(\cdot) \leq \beta \text{ in } \overline{\Omega}. \end{array} \right\}$$

Since $\max_{x\in\overline{\Omega}} |p_k(x)| \leq \beta$ and each element of the sequence $\{p_k\}_{k\in\mathbb{N}}$ has the same modulus of continuity, it follows that this sequence is uniformly bounded and equi-continuous. Hence, by Arzelà–Ascoli Theorem the sequence $\{p_k\}_{k\in\mathbb{N}}$ is relatively compact with respect to the strong topology of $C(\overline{\Omega})$. Taking into account that the set $\mathfrak{S}$ is closed with respect to the uniform convergence and $v_k(x) \to v(x)$ almost everywhere in Ω, we deduce: $p_k(\cdot) \to p(\cdot)$ uniformly in $\overline{\Omega}$ as $k \to \infty$, where $p(x) = 1 + g\left(\left|\left(\nabla G_\sigma * v\right)(x)\right|\right)$ in Ω. The proof is complete.

Following in many aspects the resent studies [11, 12], we give the following existence result.

Theorem 4.2. *For each $i = 1, \dots, M$ and given $\mu{>}0$, $\lambda_1{>}0$, $\lambda_2{>}0$, $\theta \in L^\infty(D, \mathbb{R}^2)$, and $u_{0,i} \in L^2(\Omega \setminus D)$, the minimization problem (3) admits at least one solution $(u_i^{rec}, p_i^{rec}) \in \Xi$.*

Proof. Since $\Xi \neq \emptyset$ and $0 \leq J_i(v,p) < +\infty$ for all $(v,p) \in \Xi$, it follows that there exists a non-negative value $\zeta \geq 0$ such that $\zeta = \inf\limits_{(v,p)\in\Xi} J_i(v,p)$. Let $\{(v_k, p_k)\}_{k\in\mathbb{N}} \subset \Xi$ be a minimizing sequence to the problem (3), i.e.

$$(v_k, p_k) \in \Xi,\ p_k(x) = 1 + g\left(\left|\left(\nabla G_\sigma * v_k\right)(x)\right|\right) \ \text{ in } \Omega\, \forall\, k \in \mathbb{N}, \ \text{ and } \lim_{k\to\infty} J_i\left(v_k, p_k\right) = \zeta.$$

So, without lost of generality, we can suppose that $J_i\left(v_k, p_k\right) \leq \zeta + 1$ for all $k \in \mathbb{N}$. From this and the initial assumptions, we deduce

$$\begin{gathered} \int_\Omega |v_k(x)|^\alpha\, dx \leq \int_\Omega \gamma_1^\alpha\, dx \leq \gamma_1^\alpha |\Omega|, \quad \forall\, k \in \mathbb{N}, \\ \int_\Omega |\nabla v_k(x)|^{p_k(x)}\, dx \leq 2\int_\Omega \frac{1}{p_k(x)} |\nabla v_k(x)|^{p_k(x)}\, dx < 2(\zeta+1), \quad \forall\, k \in \mathbb{N}, \end{gathered} \tag{6}$$

where

$$\sup_{k\in\mathbb{N}} \left[\sup_{x\in\Omega} p_k(x)\right] \leq 2.$$

Utilizing the fact that $v_k(x) \leq \gamma_1$ for almost all $x \in \Omega$, we infer the following estimate

$$\|v_k\|_{L^1(\Omega)} \leq \gamma_1|\Omega|, \quad \forall\, k \in \mathbb{N}.$$

Then arguing as in Lemma 4.1 it can be shown that $p_k \in C^{0,1}(\overline{\Omega})$ and

$$\alpha := 1 + \delta \leq p_k(x) \leq \beta := 2, \quad \forall\, x \in \Omega, \quad \forall\, k \in \mathbb{N}, \tag{7}$$

$$\text{with} \quad \delta = \frac{a^2}{a^2 + \|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})}\gamma_1^2|\Omega|^2}. \tag{8}$$

Taking this fact into account, we deduce from (6), (7), and (??) that

$$\begin{aligned}
\|v_k\|_{W^{1,\alpha}(\Omega)} &= \left(\int_\Omega \left[|v_k(x)|^\alpha + |\nabla v_k(x)|^\alpha\right] dx\right)^{1/\alpha} \\
&\le (1+|\Omega|)^{1/\alpha}\left(\int_\Omega \left[|v_k(x)|^{p_k(x)} + |\nabla v_k(x)|^{p_k(x)}\right] dx + 2\right)^{1/\alpha} \\
&\le \left[(1+|\Omega|)\left(\gamma_1^2|\Omega| + 2\zeta + 4\right)\right]^{1/\alpha}
\end{aligned}$$

uniformly with respect to $k \in \mathbb{N}$. Therefore, there exists a subsequence of $\{v_k\}_{k\in\mathbb{N}}$, still denoted by the same index, and a function $u_i^{rec} \in W^{1,\alpha}(\Omega)$ such that

$$\begin{gathered}
v_k \to u_i^{rec} \text{ strongly in } L^q(\Omega) \text{ for all } q \in [1, \alpha^*), \\
v_k \rightharpoonup u_i^{rec} \text{ weakly in } W^{1,\alpha}(\Omega) \text{ as } k \to \infty,
\end{gathered}$$

where, by Sobolev embedding theorem, $\alpha^* = \frac{2\alpha}{2-\alpha} = \frac{2+2\delta}{1-\delta} > 2$.

Moreover, passing to a subsequence if necessary, we have (see Proposition A.3 and Lemma 4.1):

$$\begin{gathered}
v_k(x) \to u_i^{rec}(x) \text{ a.e. in } \Omega. \\
v_k \rightharpoonup u_i^{rec} \text{ weakly in } L^{p_k(\cdot)}(\Omega), \\
\nabla v_k \rightharpoonup \nabla u_i^{rec} \text{ weakly in } L^{p_k(\cdot)}(\Omega; \mathbb{R}^2), \\
p_k(\cdot) \to p_i^{rec}(\cdot) = 1 + g\left(\left|\left(\nabla G_\sigma * u_i^{rec}\right)(\cdot)\right|\right) \quad \text{uniformly in } \overline{\Omega} \text{ as } k \to \infty,
\end{gathered} \tag{9}$$

where $u_i^{rec} \in W^{1,p^{rec}(\cdot)}(\Omega)$ with $p^{rec}(x) = 1 + g\left(\left|\left(\nabla G_\sigma * u_i^{rec}\right)(x)\right|\right)$ in Ω.

Since $\gamma_0 \le v_k(x) \le \gamma_1$ a.a. in Ω for all $k \in \mathbb{N}$, it follows from (9) that the limit function u_i^{rec} is also subjected the same restriction. Thus, u_i^{rec} is a feasible solution to minimization problem (3).

Let us show that (u_i^{rec}, p_i^{rec}) is a minimizer of this problem. With that in mind we note that in view of the obvious inequality

$$|v_k(x) - u_{0,i}(x)|^\alpha \le 2^{\alpha-1}\left(|v_k(x)|^\alpha + |u_{0,i}(x)|^\alpha\right)$$

and the fact that $u_{0,i} \in L^2(\Omega \setminus D)$, we have: the sequence $\{v_k(x) - u_{0,i}(x)\}_{k\in\mathbb{N}}$ is bounded in $L^\alpha(\Omega \setminus D)$ and converges weakly in $L^\alpha(\Omega \setminus D)$ to $u_i^{rec} - u_{0,i}$. Hence, by Proposition A.3 (see (32)), $u_i^{rec} - u_{0,i} \in L^{p^{rec}(\cdot)}(\Omega \setminus D)$ and

$$\liminf_{k\to\infty} \int_{\Omega\setminus D} |v_k(x) - u_{0,i}(x)|^\alpha\, dx \ge \int_{\Omega\setminus D} |u_i^{rec}(x) - u_{0,i}(x)|^\alpha\, dx. \tag{10}$$

As for the rest terms in (3), in view of the strong convergence $v_k \to u_i^{rec}$ in $L^q(\Omega)$ with $q > 2$, we have

$$\int_\Omega \left|\nabla G_\sigma * (v_k - u_i^{rec})\right|^2 dx \le \int_\Omega \left(\int_\Omega |\nabla G_\sigma(x-y)|\, |v_k(y) - u_i^{rec}(y)|\, dy\right)^2 dx$$

$$\leq \int_\Omega \left(\int_\Omega |\nabla G_\sigma(x-y)|^{\frac{q}{q-1}}\, dy \right)^{2-\frac{2}{q}} dx\, \|v_k - u_i^{rec}\|^2_{L^q(\Omega)}$$
$$\leq \|G_\sigma\|^2_{C^1(\overline{\Omega-\Omega})} |\Omega|^{3-\frac{2}{q}} \|v_k - u_i^{rec}\|_{L^q(\Omega)} \to 0 \text{ as } k \to \infty.$$

Hence,

$$\liminf_{k\to\infty} \int_{\Omega\setminus D} \left|\nabla G_\sigma * (v_k - u_{0,i})\right|^2 dx = \int_{\Omega\setminus D} \left|\nabla G_\sigma * (u_i^{rec} - u_{0,i})\right|^2 dx, \tag{11}$$
$$\liminf_{k\to\infty} \int_D \left| \left(\theta^\perp, \nabla v_k\right) \right|^\alpha dx \geq \int_D \left| \left(\theta^\perp, \nabla u_i^{rec}\right) \right|^\alpha dx\, dx, \tag{12}$$

As a result, utilizing relations (10), (11), (12), and the lower semicontinuity property (32), we finally obtain

$$\zeta = \inf_{(v,p)\in\Xi} J_i(v,p) = \lim_{k\to\infty} J_i\left(v_k, p_k\right) = \liminf_{k\to\infty} J_i(v_k, p_k) \geq J_i(u_i^{rec}, p_i^{rec}).$$

Thus, (u_i^{rec}, p_i^{rec}) is a minimizer to the problem (3), whereas its uniqueness remains as an open question.

5. On Relaxation of the Restoration Problem

It is clear that because of the nonstandard energy functional and its non-convexity, constrained minimization problem (3) is not trivial in its practical implementation. The main difficulty in its study comes from the state constraints

$$1 \leq \gamma_0 \leq v(x) \leq \gamma_1 \text{ a.a. in } \Omega, \quad p(x) = 1 + g\left(\left|\left(\nabla G_\sigma * u\right)(x)\right|\right)$$

that we impose on the set of feasible solutions Ξ. This motivates us to pass to some relaxation. In view of this, we propose the following iteration procedure which is based on the concept of relaxation of extremal problems and their variational convergence [13, 14, 15, 16]. At the first step we set up

$$p_0(x) = \left\{ \begin{array}{l} 1 + g\left(\left|\left(\nabla G_\sigma * u_{0,i}\right)(x)\right|\right), \text{ if } x\in\Omega\setminus D, \\ 1 + g\left(\left|\left(\nabla G_\sigma * u^*\right)(x)\right|\right), \text{ if } x\in D, \end{array} \right\}$$
$$u^0 = \operatorname*{Argmin}_{v\in\mathcal{B}_{p_0(\cdot)}} J_i(v, p_0(\cdot)).$$

Then, for each $k \geq 1$, we set

$$p_k(x) = 1 + g\left(\left|\left(\nabla G_\sigma * u^{k-1}\right)(x)\right|\right), \quad \forall\, x \in \Omega, \quad u^k = \operatorname*{Argmin}_{v\in\mathcal{B}_{p_k(\cdot)}} J_i(v, p_k(\cdot)). \tag{13}$$

Here, $\mathcal{B}_{p(\cdot)} = \left\{v \in W^{1,p(\cdot)}(\Omega) : 1\leq\gamma_0 \leq v(x) \leq \gamma_1 \text{ a.a. in } \Omega\right\}$.

Before proceeding further, we set

$$\mathfrak{S} = \left\{ h \in C(\overline{\Omega}) \;\middle|\; \begin{array}{c} |h(x) - h(y)| \leq C|x-y|,\ \forall\, x, y \in \Omega, \\ \alpha := 1 + \delta \leq h(x) \leq \beta := 2, \quad \forall\, x \in \Omega, \end{array} \right\}$$

where $C > 0$ and $\delta > 0$ are defined by (5) and (8), respectively.

Arguing as in the proof of Theorem 1 and using the convexity arguments, it can be shown that, for each $p(\cdot) \in \mathfrak{S}$, there exists a unique element $u_i^{0,p(\cdot)} \in \mathcal{B}_{p(\cdot)}$ such that $u_i^{0,p(\cdot)} = \operatorname{Argmin}_{v\in\mathcal{B}_{p(\cdot)}} J_i(v, p(\cdot))$. Moreover, it can be shown that, for given $i = 1, \dots, M$, $\mu > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, $u^* \in W^{1,\alpha}(D)$, and $\vec{u}_0 \in L^2(\Omega \setminus D, \mathbb{R}^M)$, the sequence $\left\{u^k \in W^{1,p_k(\cdot)}(\Omega)\right\}_{k\in\mathbb{N}}$ is compact with respect to the weak topology of $W^{1,\alpha}(\Omega)$, whereas the exponents $\{p_k\}_{k\in\mathbb{N}}$ are compact with respect to the strong topology of $C(\overline{\Omega})$.

We say that a pair $(\widehat{u}_i, \widehat{p})$ is a weak solution to the original problem (3) if

$$\widehat{u}_i = \operatorname*{Argmin}_{v\in\mathcal{B}_{\widehat{p}(\cdot)}} J_i(v, \widehat{p}(\cdot)),\ \widehat{u}_i \in \mathcal{B}_{\widehat{p}(\cdot)}, \quad \widehat{p}(x) = 1 + g\left(\left|\left(\nabla G_\sigma * \widehat{u}_i\right)(x)\right|\right),\ \forall\, x \in \Omega.$$

Our main result can be stated as follows:

Theorem 5.2. *Let $\mu > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, $u^* \in BV(D)$, and $\vec{u}_0 \in L^2(\Omega\setminus D, \mathbb{R}^M)$ be given data. Then, for each $i \in \{1, \dots, M\}$, the sequence of approximated solutions $\left\{(u^k, p_k)\right\}_{k\in\mathbb{N}}$ possesses the asymptotic properties:*

$$\begin{gathered} u^k(x) \to \widetilde{u}(x) \text{ a.e. in } \Omega, \\ u^k \rightharpoonup \widetilde{u} \text{ in } L^\alpha(\Omega), \text{ and } \nabla u^k \rightharpoonup \nabla\widetilde{u} \text{ in } L^\alpha(\Omega; \mathbb{R}^2), \\ p_k \to \widetilde{p} = \mathfrak{F}(\widetilde{u}(x)) \text{ strongly in } C(\overline{\Omega}) \text{ as } k \to \infty, \end{gathered}$$

where $(\widetilde{u}, \widetilde{p})$ is a weak solution to the original problem (3), that is,

$$\widetilde{u} \in \mathcal{B}_{\widetilde{p}(\cdot)}, \quad \widetilde{u} = \operatorname*{Argmin}_{v\in\mathcal{B}_{\widetilde{p}(\cdot)}} J_i(v, \widetilde{p}(\cdot)),$$

and, in addition, the following variational property holds true

$$\lim_{k\to\infty} J_i(u^k, p_k(\cdot)) = \lim_{k\to\infty}\left[\inf_{v\in\mathcal{B}_{p_k(\cdot)}} J_i(v, p_k(\cdot))\right] = \inf_{v\in\mathcal{B}_{\widetilde{p}(\cdot)}} J_i(v, \widetilde{p}(\cdot)) = J_i(\widetilde{u}, \widetilde{p}(\cdot)). \tag{14}$$

Proof. Let's assume the converse — namely, there is a function $u^\bullet \in \mathcal{B}_{\widetilde{p}(\cdot)}$ such that

$$J_i(u^\bullet, \widetilde{p}(\cdot)) = \inf_{v\in\mathcal{B}_{\widetilde{p}(\cdot)}} J_i(v, \widetilde{p}(\cdot)) < J_i(\widetilde{u}, \widetilde{p}(\cdot)). \tag{15}$$

Using the procedure of the direct smoothing, we set

$$u_\varepsilon(x) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} K\left(\frac{x-z}{\zeta(\varepsilon)}\right)\widetilde{u^\bullet}(z)\,dz,$$

where $\varepsilon > 0$ is a small parameter, K is a positive compactly supported smooth function with properties

$$K \in C_0^\infty(\mathbb{R}^2),\ \int_{\mathbb{R}^2} K(x)\,dx = 1, \text{ and } K(x) = K(-x),$$

and $\widetilde{u^\bullet}$ is zero extension of $u^\bullet$ outside of Ω.

Since $u^\bullet \in W^{1,\widetilde{p}(\cdot)}(\Omega)$ and $\widetilde{p}(x) \geq \alpha = 1+\delta$ in Ω), it follows that $u^\bullet \in W^{1,\alpha}(\Omega)$. Then

$$\begin{gathered} u_\varepsilon \in C_0^\infty(\mathbb{R}^2) \text{ for each } \varepsilon > 0, \\ u_\varepsilon \to u^\bullet \ \text{ in } \ L^\alpha(\Omega), \quad \nabla u_\varepsilon \to \nabla u^\bullet \ \text{ in } \ L^\alpha(\Omega;\mathbb{R}^2) \end{gathered} \tag{16}$$

by the classical properties of smoothing operators (see [17]). From this we deduce that

$$u_\varepsilon(x) \to u^\bullet(x) \ \text{ a.e. in } \Omega. \tag{17}$$

Moreover, taking into account the estimates

$$\begin{aligned} u_\varepsilon(x) &= \int_{\mathbb{R}^2} K(y)\, \widetilde{u^\bullet}(x-\zeta(\varepsilon)y)\, dy \leq \gamma_1 \int_{\mathbb{R}^2} K(y)\ dy = \gamma_1, \\ u_\varepsilon(x) &\geq \int_{y\in\zeta(\varepsilon)^{-1}(x-\Omega)} K(y)\, \widetilde{u^\bullet}(x-\zeta(\varepsilon)y)\, dy \geq \gamma_0 \int_{y\in\zeta(\varepsilon)^{-1}(x-\Omega)} K(y)\ dy \geq \gamma_0, \end{aligned}$$

we see that each element u_ε is subjected to the pointwise constraints

$$\gamma_0 \leq u_\varepsilon(x) \leq \gamma_1 \text{ a.a. in } \Omega, \quad \forall\, \varepsilon > 0.$$

Since, for each $\varepsilon > 0$, $u_\varepsilon \in W^{1,p_k(\cdot)}(\Omega)$ for all $k \in \mathbb{N}$, it follows that $u_\varepsilon \in \mathcal{B}_{p_k(\cdot)}$, i.e., each element of the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ is a feasible solution to all approximating problems $\left\langle \inf_{v\in\mathcal{B}_{p_k(\cdot)}} J_i(v,p_k(\cdot)) \right\rangle$. Hence,

$$J_i(u^k,p_k(\cdot)) \leq J_i(u_\varepsilon,p_k(\cdot)), \quad \forall\, \varepsilon > 0,\ \forall\, k = 0,1,\dots \tag{18}$$

Further we notice that

$$\liminf_{k\to\infty} J_i(u^k,p_k(\cdot)) \geq J_i(\widetilde{u},\widetilde{p}(\cdot)) \tag{19}$$

by Proposition A.3 and Fatou's lemma, and

$$\begin{aligned} \lim_{k\to\infty} J_i(u_\varepsilon,p_k(\cdot)) = \lim_{k\to\infty} \int_\Omega \frac{1}{p_k(x)} |\nabla u_\varepsilon(x)|^{p_k(x)}\, dx + \frac{\mu}{\alpha}\int_{\Omega\setminus D} |u_\varepsilon(x) - u_{0,i}(x)|^\alpha\ dx \\ + \lambda_1 \int_{\Omega\setminus D} \left| \nabla G_\sigma * (u_\varepsilon - u_{0,i}) \right|^2 dx + \lambda_2 \int_D \left| \left(\theta^\perp, \nabla u_\varepsilon\right) \right|^\alpha dx. \end{aligned} \tag{20}$$

Since

$$\frac{1}{p_k(x)} |\nabla u_\varepsilon(x)|^{p_k(x)} \to \frac{1}{\widetilde{p}(x)} |\nabla u_\varepsilon(x)|^{\widetilde{p}(x)} \quad \text{uniformly in } \Omega \text{ as } k\to\infty,$$

it follows from the Lebesgue dominated convergence theorem and (20) that

$$\lim_{k\to\infty} J_i(u_\varepsilon,p_k(\cdot)) = J_i(u_\varepsilon,\widetilde{p}(\cdot)), \quad \forall \varepsilon > 0. \tag{21}$$

As a result, passing to the limit in (18) and utilizing properties (19)–(21), we obtain

$$J_i(\widetilde{u},\widetilde{p}(\cdot)) \leq J_i(u_\varepsilon,\widetilde{p}(\cdot)) = \int_\Omega \frac{1}{\widetilde{p}(x)} |\nabla u_\varepsilon(x)|^{\widetilde{p}(x)}\, dx + \frac{\mu}{\alpha}\int_{\Omega\setminus D} |u_\varepsilon(x) - u_{0,i}(x)|^\alpha\ dx$$

$$+ \lambda_1 \int_{\Omega \setminus D} \left| \nabla G_\sigma * (u_\varepsilon - u_{0,i}) \right|^2 dx + \lambda_2 \int_D \left| \left(\theta^\perp, \nabla u_\varepsilon \right) \right|^\alpha dx, \tag{22}$$

for all $\varepsilon > 0$. Taking into account the pointwise convergence (see (17) and property (16))

$$\begin{gathered} |\nabla u_\varepsilon(x)|^{\widetilde{p}(x)} \to |\nabla u^\bullet(x)|^{\widetilde{p}(x)}, \\ |u_\varepsilon(x) - u_{0,i}(x)|^\alpha \to |u^\bullet(x) - u_{0,i}(x)|^\alpha, \\ \left| \nabla G_\sigma * (u_\varepsilon - u_{0,i}) \right|^2 \to \left| \nabla G_\sigma * (u^\bullet - u_{0,i}) \right|^2, \\ \left| \left(\theta^\perp, \nabla u_\varepsilon \right) \right|^\alpha \to \left| \left(\theta^\perp, \nabla u^\bullet \right) \right|^\alpha \end{gathered}$$

as $\varepsilon \to 0$, and the fact that, for ε small enough,

$$\begin{gathered} |\nabla u_\varepsilon(x)|^{\widetilde{p}(x)} \le (1 + |\nabla u^\bullet(\cdot)|)^{\widetilde{p}(\cdot)} \in L^1(\Omega) \quad \text{a.e. in } \Omega, \\ |u_\varepsilon(\cdot) - u_{0,i}(\cdot)|^\alpha \le \left[2\,(1 + |u^\bullet(\cdot)|)^\alpha + 2\,(1 + |u_{0,i}(\cdot)|)^2 \right] \in L^1(\Omega) \quad \text{a.e. in } \Omega \setminus D, \\ \left| \nabla G_\sigma * (u_\varepsilon - u_{0,i})(x) \right|^2 \le \|G_\sigma\|^2_{C^1(\overline{\Omega - \Omega})} |\Omega|^2 \gamma_0^2 = \text{const}, \quad \forall\, x \in \Omega, \\ \left| \left(\theta^\perp, \nabla u_\varepsilon \right) \right|^\alpha \le \|\theta\|_{L^\infty(D,\mathbb{R}^2)} (1 + |\nabla u^\bullet(\cdot)|)^{\widetilde{p}(\cdot)} \in L^1(\Omega) \quad \text{a.e. in } \Omega, \end{gathered}$$

we can pass to the limit in (22) as $\varepsilon \to 0$ by the Lebesgue dominated convergence theorem. This yields

$$J_i(\widetilde{u}, \widetilde{p}(\cdot)) \le \lim_{\varepsilon \to 0} J_i(u_\varepsilon, \widetilde{p}(\cdot)) = J_i(u^\bullet, \widetilde{p}(\cdot)).$$

Combining this inequality with (22) and (15), we finally get

$$J_i(u^\bullet, \widetilde{p}(\cdot)) = \inf_{v \in \mathcal{B}_{\widetilde{p}(\cdot),i}} J_i(v, \widetilde{p}(\cdot)) < J_i(u^*, \widetilde{p}(\cdot)) \le J_i(u^\bullet, \widetilde{p}(\cdot)),$$

that leads us into conflict with the initial assumption. Thus,

$$J_i(\widetilde{u}, \widetilde{p}(\cdot)) = \inf_{v \in \mathcal{B}_{\widetilde{p}(\cdot)}} J_i(v, \widetilde{p}(\cdot)) \tag{23}$$

and, therefore, $(\widetilde{u}, \widetilde{p})$ is a weak solution to the original problem (3). As for the variational property (14), it is a direct consequence of (23) and (21).

6. Optimality Conditions

To characterize the solution $u^{0,p(\cdot)} \in \mathcal{B}_{p(\cdot)}$ of the approximating optimization problem $\left\langle \inf_{v \in \mathcal{B}_{p(\cdot)}} J_i(v, p(\cdot)) \right\rangle$, we check that the functional $F_{p(\cdot)}$ is Gâteaux differentiable, that is,

$$\lim_{t \to 0} \frac{J_i(u^{0,p(\cdot)} + tv, p(\cdot)) - J_i(u^{0,p(\cdot)}, p(\cdot))}{t} = \int_\Omega \left(|\nabla u^{0,p(\cdot)}(x)|^{p(x)-2} \nabla u^{0,p(\cdot)}(x), \nabla v(x) \right) dx$$
$$+ \mu \int_{\Omega \setminus D} \left| u^{0,p(\cdot)}(x) - u_{0,i}(x) \right|^{\alpha - 2} u^{0,p(\cdot)}(x) v(x)\, dx$$

$$+2\lambda_1 \int_\Omega \Lambda(x) v(x)\, dx + \alpha\lambda_2 \int_\Omega \left| \left(\theta^\perp, \nabla u^{0,p(\cdot)}\right) \right|^{\alpha-1} \left(\theta^\perp, \nabla v\right)\, dx, \tag{24}$$

for all $v \in W^{1,p(\cdot)}(\Omega)$, where

$$\Lambda(x) = \int_\Omega \int_\Omega (\nabla G_\sigma(y-z), \nabla G_\sigma(y-x)) \left(u^{0,p(\cdot)}(z) - u_{0,i}(z)\right) \chi_{\Omega\setminus D}(y)\, dz\, dy.$$

To this end, we note that

$$\frac{|\nabla u^{0,p(\cdot)}(x) + t\nabla v(x)|^{p(x)} - |\nabla u^{0,p(\cdot)}(x)|^{p(x)}}{p(x)t} \\ \to \left(|\nabla u^{0,p(\cdot)}(x)|^{p(x)-2}\nabla u^{0,p(\cdot)}(x), \nabla v(x)\right) \text{ as } t \to 0$$

almost everywhere in Ω. Since, by convexity,

$$|\xi|^p - |\eta|^p \le 2p \left(|\xi|^{p-1} + |\eta|^{p-1}\right) |\xi - \eta|,$$

it follows that

$$\left|\frac{|\nabla u^{0,p(\cdot)}(x) + t\nabla v(x)|^{p(x)} - |\nabla u^{0,p(\cdot)}(x)|^{p(x)}}{p(x)t}\right| \\ \le 2\left(|\nabla u^{0,p(\cdot)}(x) + t\nabla v(x)|^{p(x)-1} + |\nabla u^{0,p(\cdot)}(x)|^{p(x)-1}\right)|\nabla v(x)| \\ \le \text{const}\left(|\nabla u^{0,p(\cdot)}(x)|^{p(x)-1} + |\nabla v(x)|^{p(x)-1}\right)|\nabla v(x)|. \tag{25}$$

Taking into account that

$$\||u^{0,p(\cdot)}(x)|^{p(x)-1}\|_{L^{p'(\cdot)}(\Omega)} \overset{\text{by (33)}}{\le} \left(\int_\Omega \||u^{0,p(\cdot)}(x)|^{p(x)}\, dx + 1\right)^{\frac{1}{\beta'}} \\ \overset{\text{by (??)}}{\le} \left(\|u^{0,p(\cdot)}|^2_{L^{p(\cdot)}(\Omega)} + 2\right)^{\frac{1}{\beta'}},$$

$$\int_\Omega |\nabla u^{0,p(\cdot)}(x)|^{p(x)-1}|\nabla v(x)|\, dx \overset{\text{by (33)}}{\le} 2\||u^{0,p(\cdot)}(x)|^{p(x)-1}\|_{L^{p'(\cdot)}(\Omega)}\|v(x)|\|_{L^{p(\cdot)}(\Omega)},$$

and $\int_\Omega |v(x)|^{p(x)}\, dx \overset{\text{by (??)}}{\le} \|v\|^2_{L^{p(\cdot)}(\Omega)} + 1$, we see that the right hand side of inequality (25) is an $L^1(\Omega)$ function. Therefore,

$$\int_\Omega \frac{|\nabla u^{0,p(\cdot)}(x) + t\nabla v(x)|^{p(x)} - |\nabla u^{0,p(\cdot)}(x)|^{p(x)}}{p(x)t}\, dx \\ \to \int_\Omega \left(|\nabla u^{0,p(\cdot)}(x)|^{p(x)-2}\nabla u^{0,p(\cdot)}(x), \nabla v(x)\right) dx \text{ as } t \to 0$$

by the Lebesgue dominated convergence theorem.

Utilizing the similar arguments to the rest terms in (3), we deduce that the representation (24) for the Gâteaux differential of $J_i(\cdot, p(\cdot))$ at the point $u^{0,p(\cdot)} \in \mathcal{B}_{p(\cdot)}$ is valid.

Thus, in order to derive some optimality conditions for the minimizing element $u^{0,p(\cdot)} \in \mathcal{B}_{p(\cdot)}$ to the problem $\inf\limits_{v \in \mathcal{B}_{p(\cdot)}} J_i(v, p(\cdot))$, we note that $\mathcal{B}_{p(\cdot)}$ is a nonempty convex subset of $W^{1,p(\cdot)}(\Omega)$ and the objective functional $J_i(\cdot, p(\cdot)) : \mathcal{B}_{p(\cdot)} \to \mathbb{R}$ is strictly convex. Hence, the well known classical result (see [18, Theorem 1.1.3]) and representation (24) lead us to the following conclusion.

Theorem 6.1. *Let $p_k(\cdot) \in \mathfrak{S}$ be an exponent given by the iterative rule (13). Then the unique minimizer $u^k \in \mathcal{B}_{p_k(\cdot)}$ to the approximating problem $\inf_{v \in \mathcal{B}_{p_k(\cdot)}} J_i(v, p_k(\cdot))$ is characterized by*

$$\begin{aligned}
\int_\Omega \left(\left| \nabla u^k(x) \right|^{p_k(x)-2} \nabla u^k(x), \nabla v(x) - \nabla u^k(x) \right) dx + 2\lambda_1 \int_\Omega \Lambda(x) \left(v(x) - u^k(x) \right) dx \\
+ \mu \int_{\Omega \setminus D} \left| u^k(x) - u_{0,i}(x) \right|^{\alpha-2} u^k(x) \left(v(x) - u^k(x) \right) dx \\
+ \alpha\lambda_2 \int_\Omega \left| \left(\theta^\perp, \nabla u^{0,p(\cdot)} \right) \right|^{\alpha-1} \left(\theta^\perp, \nabla v - \nabla u^k \right) dx \geq 0, \quad \forall\, v \in \mathcal{B}_{p_k(\cdot)}.
\end{aligned}$$

7. Numerical Experiments

In order to illustrate the proposed algorithm for the restoration of satellite multi-spectral images we have provided some numerical experiments. As input data we have used a series of Sentinel-2 L2A images over the Dnipro Airport area, Ukraine (see Fig. 1, 2). This region represents a typical agricultural area with medium sides fields of various shapes.

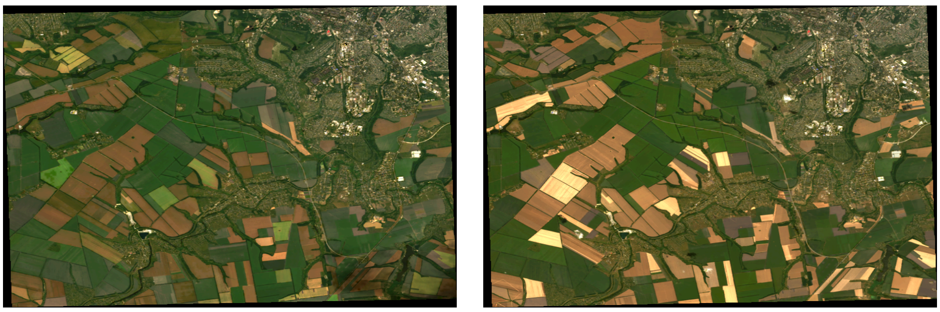

Figure 1: Given collection of past cloud-free images. Date of generation: (left) - 2019/06/15, (right) - 2019/07/01

As a final result, we obtain in Fig. 3. Comparing the restored image and the contaminated one we could see that the texture of original image is well preserved. However, overall colors of different fields are shifted due to colorization part of algorithm. This problem has to be addressed in the following research.

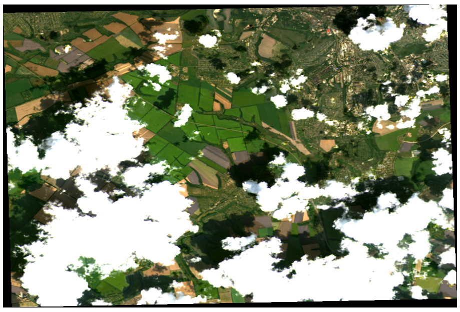

Figure 2: The could contaminated image with date of generation 2019/07/17

8. Conclusion

We propose a novel model for the restoration of satellite multi-spectral images. This model is based on the solutions of special variational problems with nonstandard growth objective functional. Because of the risk of information loss in optical images (see [19] for the details), we do not impose any information about such images inside the damage region, but instead we assume that the texture of these images can be predicted through a number of past cloud-free images of the same region from the time series. So, the characteristic feature of variational problems, which we formulate for each spectral channel separately, is the structure of their objective functionals. On the one hand, we involve into consideration the energy functionals with the nonstandard growth $p(x)$, where the variable exponent $p(x)$ is unknown a priori and it directly depends on the texture of an image that we are going to restore. On the other hand, the texture of an image $\vec{u}$, we are going to restore, can have rather rich structure in the damage region D. In order to identify it, we push forward the following hypothesis: the geometry of each spectral channels of a cloud corrupted image in the damage region is topologically close to the geometry of the total spectral energy that can be predicted with some accuracy by a number of past cloud-free images of the same region. As a result, we impose this requirement in each objective functional in the form of a special fidelity term. In order to study the consistency of the proposed collection of non-convex minimization problems, we develop a special technique and supply this approach by the rigorous mathematical substantiation.

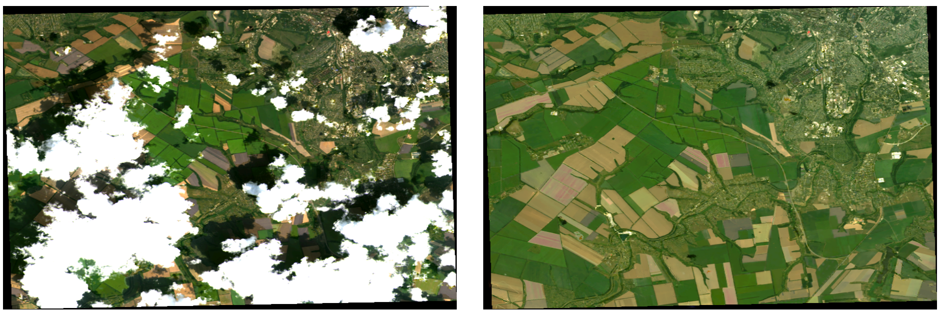

Figure 3: Result of the restoration of image in Fig 2 by the proposed method.

Appendix A. On Orlicz Spaces

Let $p(\cdot)$ be a measurable exponent function on Ω such that $1 \le \alpha \le p(x) \le \beta < \infty$ a.e. in Ω, where α and β are given constants. Let $p'(\cdot) = \frac{p(\cdot)}{p(\cdot)-1}$ be the corresponding conjugate exponent. It is clear that

$$1 \le \underbrace{\frac{\beta}{\beta-1}}_{\beta'} \le p'(x) \le \underbrace{\frac{\alpha}{\alpha-1}}_{\alpha'} \text{ a.e. in } \Omega,$$

where β' and α' stand for the conjugates of constant exponents. Denote by $L^{p(\cdot)}(\Omega)$ the set of all measurable functions $f(x)$ on Ω such that $\int_\Omega |f(x)|^{p(x)}\,dx < \infty$. Then $L^{p(\cdot)}(\Omega)$ is a reflexive separable Banach space with respect to the Luxemburg norm (see [20] for the details)

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 \;:\; \rho_p(\lambda^{-1} f) \le 1\right\}, \tag{26}$$

where $\rho_p(f) := \int_\Omega |f(x)|^{p(x)}\,dx$.

It is well-known that $L^{p(\cdot)}(\Omega)$ is reflexive provided $\alpha > 1$, and its dual is $L^{p'(\cdot)}(\Omega)$, that is, any continuous functional $F = F(f)$ on $L^{p(\cdot)}(\Omega)$ has the form (see [21, Lemma 13.2])

$$F(f) = \int_\Omega f g\,dx, \quad \text{with } g \in L^{p'(\cdot)}(\Omega).$$

As for the infimum in (26), we have the following result.

Proposition A.1. *The infimum in* (26) *is attained if* $\rho_p(f) > 0$. *Moreover*

$$\text{if } \lambda_* := \|f\|_{L^{p(\cdot)}(\Omega)} > 0, \text{ then } \rho_p(\lambda_*^{-1} f) = 1.$$

Taking this result and condition $1 \le \alpha \le p(x) \le \beta$ into account, we see that

$$\frac{1}{\lambda_*^\beta}\int_\Omega |f(x)|^{p(x)}\,dx \le \int_\Omega \left|\frac{f(x)}{\lambda_*}\right|^{p(x)} dx \le \frac{1}{\lambda_*^\alpha}\int_\Omega |f(x)|^{p(x)}\,dx,$$
$$\frac{1}{\lambda_*^\beta}\int_\Omega |f(x)|^{p(x)}\,dx \le 1 \le \frac{1}{\lambda_*^\alpha}\int_\Omega |f(x)|^{p(x)}\,dx.$$

Hence, (see [20] for the details)

$$\|f\|^\alpha_{L^{p(\cdot)}(\Omega)} \le \int_\Omega |f(x)|^{p(x)}\,dx \le \|f\|^\beta_{L^{p(\cdot)}(\Omega)}, \text{ if } \|f\|_{L^{p(\cdot)}(\Omega)} > 1,$$
$$\|f\|^\beta_{L^{p(\cdot)}(\Omega)} \le \int_\Omega |f(x)|^{p(x)}\,dx \le \|f\|^\alpha_{L^{p(\cdot)}(\Omega)}, \text{ if } \|f\|_{L^{p(\cdot)}(\Omega)} < 1,$$

and, therefore,

$$\|f\|^\alpha_{L^{p(\cdot)}(\Omega)} - 1 \le \int_\Omega |f(x)|^{p(x)}\,dx \le \|f\|^\beta_{L^{p(\cdot)}(\Omega)} + 1, \quad \forall f \in L^{p(\cdot)}(\Omega),$$
$$\|f\|_{L^{p(\cdot)}(\Omega)} = \int_\Omega |f(x)|^{p(x)}\,dx, \text{ if } \|f\|_{L^{p(\cdot)}(\Omega)} = 1.$$

The following estimates are well-known: if $f \in L^{p(\cdot)}(\Omega)$ then

$$\|f\|_{L^\alpha(\Omega)} \le (1+|\Omega|)^{1/\alpha}\,\|f\|_{L^{p(\cdot)}(\Omega)},$$
$$\|f\|_{L^{p(\cdot)}(\Omega)} \le (1+|\Omega|)^{1/\beta'}\,\|f\|_{L^\beta(\Omega)}, \quad \beta' = \frac{\beta}{\beta-1}, \quad \forall\, f \in L^\beta(\Omega).$$

Let $\{p_k\}_{k\in\mathbb{N}} \subset C^{0,\delta}(\overline{\Omega})$, with some $\delta \in (0,1]$, be a given sequence of exponents. Hereinafter in this subsection we assume that

$$1 \le \alpha \le p_k(x) \le \beta < \infty \text{ a.e. in } \Omega \text{ for } k = 1,2,\dots, \text{ and } \; p_k(\cdot) \to p(\cdot) \; \text{ in } C(\overline{\Omega}) \text{ as } k \to \infty. \tag{27}$$

We associate with this sequence the following collection $\left\{f_k \in L^{p_k(\cdot)}(\Omega)\right\}_{k\in\mathbb{N}}$. The characteristic feature of this set of functions is that each element f_k lives in the corresponding Orlicz space $L^{p_k(\cdot)}(\Omega)$. We say that the sequence $\left\{f_k \in L^{p_k(\cdot)}(\Omega)\right\}_{k\in\mathbb{N}}$ is bounded if (see [22, Section 6.2])

$$\limsup_{k\to\infty} \int_\Omega |f_k(x)|^{p_k(x)}\,dx < +\infty. \tag{28}$$

Definition A.2. *A bounded sequence $\left\{f_k \in L^{p_k(\cdot)}(\Omega)\right\}_{k\in\mathbb{N}}$ is weakly convergent in the variable Orlicz space $L^{p_k(\cdot)}(\Omega)$ to a function $f \in L^{p(\cdot)}(\Omega)$, where $p \in C^{0,\delta}(\overline{\Omega})$ is the limit of $\{p_k\}_{k\in\mathbb{N}} \subset C^{0,\delta}(\overline{\Omega})$ in the uniform topology of $C(\overline{\Omega})$, if*

$$\lim_{k\to\infty} \int_\Omega f_k \varphi\,dx = \int_\Omega f\varphi\,dx, \quad \forall\,\varphi \in C_0^\infty(\mathbb{R}^2).$$

For our further analysis, we need some auxiliary results (we refer to [21, Lemma 13.3] for comparison). We begin with the lower semicontinuity property of the variable $L^{p_k(\cdot)}$-norm with respect to the weak convergence in $L^{p_k(\cdot)}(\Omega)$.

Proposition A.3. *If a bounded sequence $\left\{f_k \in L^{p_k(\cdot)}(\Omega)\right\}_{k\in\mathbb{N}}$ converges weakly in $L^\alpha(\Omega)$ to f for some $\alpha > 1$, then $f \in L^{p(\cdot)}(\Omega)$, $f_k \rightharpoonup f$ in variable $L^{p_k(\cdot)}(\Omega)$, and*

$$\liminf_{k\to\infty} \int_\Omega |f_k(x)|^{p_k(x)}\,dx \ge \int_\Omega |f(x)|^{p(x)}\,dx. \tag{29}$$

Proof. In view of the fact that

$$\begin{aligned}
&\left| \int_\Omega |f_k(x)|^{p_k(x)}\,dx - \int_\Omega \frac{p(x)}{p_k(x)} |f_k(x)|^{p_k(x)}\,dx \right| \\
&\qquad\qquad \le \|p_k - p\|_{C(\overline{\Omega})} \int_\Omega \frac{1}{p_k(x)} |f_k(x)|^{p_k(x)}\,dx \\
&\qquad\qquad\qquad \le \frac{\|p_k - p\|_{C(\overline{\Omega})}}{\alpha} \int_\Omega |f_k(x)|^{p_k(x)}\,dx \overset{\text{by (28)}}{\to} 0 \text{ as } \; k \to \infty,
\end{aligned}$$

we see that

$$\liminf_{k\to\infty} \int_\Omega |f_k(x)|^{p_k(x)}\,dx = \liminf_{k\to\infty} \int_\Omega \frac{p(x)}{p_k(x)} |f_k(x)|^{p_k(x)}\,dx.$$

Using the Young inequality $ab \le |a|^p/p + |b|^{p'}/p'$, we have

$$\int_\Omega \frac{p(x)}{p_k(x)} |f_k(x)|^{p_k(x)}\,dx \ge \int_\Omega p(x) f_k(x)\varphi(x)\,dx - \int_\Omega \frac{p(x)}{p_k'(x)}|\varphi(x)|^{p_k'(x)}\,dx, \tag{30}$$

for $p_k'(x) = p_k(x)/(p_k(x)-1)$ and any $\varphi \in C_0^\infty(\mathbb{R}^2)$.

Then passing to the limit in (30) as $k \to \infty$ and utilizing property (27) and the fact that

$$\lim_{k\to\infty} \int_\Omega f_k(x)\varphi(x)\,dx = \int_\Omega f(x)\varphi(x)\,dx \quad \text{for all } \ \varphi \in L^{\alpha'}(\Omega), \tag{31}$$

we obtain

$$\liminf_{k\to\infty} \int_\Omega |f_k(x)|^{p_k(x)}\,dx \ge \int_\Omega p(x) f(x)\varphi(x)\,dx - \int_\Omega \frac{p(x)}{p'(x)}|\varphi(x)|^{p'(x)}\,dx.$$

Since the last inequality is valid for all $\varphi \in C_0^\infty(\mathbb{R}^2)$ and the set $C_0^\infty(\mathbb{R}^2)$ is dense in $L^{p'(\cdot)}(\Omega)$, it follows that this relation holds true for $\varphi \in L^{p'(\cdot)}(\Omega)$. So, taking $\varphi = |f(x)|^{p(x)-2}f(x)$, we arrive at the announced inequality (29). As an consequence of (29) and estimate (**??**), we get: $f \in L^{p(\cdot)}(\Omega)$.

To end of the proof, it remains to observe that relation (31) holds true for $\varphi \in C_0^\infty(\mathbb{R}^2)$ as well. From this the weak convergence $f_k \rightharpoonup f$ in the variable Orlicz space $L^{p_k(\cdot)}(\Omega)$ follows.

Remark A.4. *Arguing in a similar manner and using, instead of* (30)*, the estimate*

$$\liminf_{k\to\infty} \int_\Omega \frac{1}{p_k(x)} |f_k(x)|^{p_k(x)}\,dx \ge \int_\Omega p(x) f(x)\varphi(x)\,dx - \int_\Omega \frac{1}{p_k'(x)}|\varphi(x)|^{p'(x)}\,dx,$$

it is easy to show that the lower semicontinuity property (29) *can be generalized as follows*

$$\liminf_{k\to\infty} \int_\Omega \frac{1}{p_k(x)} |f_k(x)|^{p_k(x)}\,dx \ge \int_\Omega \frac{1}{p(x)}|f(x)|^{p(x)}\,dx. \tag{32}$$

We need the following result that leads to the analog of the Hölder inequality in Lebesgue spaces with variable exponents (for the details we refer to [20]).

Proposition A.6. *If $f \in L^{p(\cdot)}(\Omega)^N$ and $g \in L^{p'(\cdot)}(\Omega)^N$, then $(f,g) \in L^1(\Omega)$ and*

$$\int_\Omega (f,g)\ dx \le 2\|f\|_{L^{p(\cdot)}(\Omega)^N}\|g\|_{L^{p'(\cdot)}(\Omega)^N}. \tag{33}$$

Appendix B. Sobolev Spaces with Variable Exponent

We recall here the well-known facts concerning the Sobolev spaces with variable exponent. Let $p(\cdot)$ be a measurable exponent function on Ω such that $1 < \alpha \le p(x) \le \beta < \infty$ a.e. in Ω, where α and β are given constants. We associate with it the so-called Sobolev-Orlicz space

$$W^{1,p(\cdot)}(\Omega) := \left\{ y \in W^{1,1}(\Omega)\ :\ \int_\Omega \left[|y(x)|^{p(x)} + |\nabla y(x)|^{p(x)} \right]\,dx < +\infty \right\}$$

and equip it with the norm $\|y\|_{W_0^{1,p(\cdot)}(\Omega)} = \|y\|_{L^{p(\cdot)}(\Omega)} + \|\nabla y\|_{L^{p(\cdot)}(\Omega;\mathbb{R}^2)}$.

It is well-known that, in general, unlike classical Sobolev spaces, smooth functions are not necessarily dense in $W = W_0^{1,p(\cdot)}(\Omega)$. Hence, with the given variable exponent $p = p(x)$ ($1 < \alpha \leq p \leq \beta$) it can be associated another Sobolev space,

$$H = H^{1,p(\cdot)}(\Omega) \text{ as the closure of the set } C^\infty(\overline{\Omega}) \text{ in } W^{1,p(\cdot)}(\Omega)\text{-norm.}$$

Since the identity $W = H$ is not always valid, it makes sense to say that an exponent $p(x)$ is regular if $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$.

The following result reveals an important property ensuring the regularity of exponent $p(x)$.

Proposition B.1. *Assume that there exists $\delta \in (0,1]$ such that $p \in C^{0,\delta}(\overline{\Omega})$. Then the set $C^\infty(\overline{\Omega})$ is dense in $W^{1,p(\cdot)}(\Omega)$, and, therefore, $W = H$.*

Proof. Let $p \in C^{0,\delta}(\overline{\Omega})$ be a given exponent. Since

$$\lim_{t\to 0} |t|^\delta \log(|t|) = 0 \quad \text{with an arbitrary } \delta \in (0,1], \tag{34}$$

it follows from the Hölder continuity of $p(\cdot)$ that

$$|p(x) - p(y)| \leq C|x-y|^\delta \leq \left[\sup_{x,y\in\Omega} \frac{|x-y|^\delta}{\log(|x-y|^{-1})}\right] \omega(|x-y|), \quad \forall\, x,y \in \Omega,$$

where $\omega(t) = C/\log(|t|^{-1})$, and $C > 0$ is some positive constant.

Then property (34) implies that $p(\cdot)$ is a log-Hölder continuous function. So, to deduce the density of $C^\infty(\overline{\Omega})$ in $W^{1,p(\cdot)}(\Omega)$ it is enough to refer to Theorem 13.10 in [21].

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