Method of a Point Localization in a Polygon in Relation to Ecological and Geographical Problems

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Abstract
The problem of point localization in an arbitrary polygon is considered in relation to the problems of geophysics and ecology. An analytical solution to the problem is obtained using mathematical analysis based on the Cauchy's integral formula from the theory of functions of a complex variable. On the basis of the obtained solution a program was developed for data sampling according to a given zoning of water areas, the further development of which will make it possible to assess the ecological state of the marine environment of Ukraine.

Keywords
Cauchy's integral formula, polygon, point localization

1. Introduction

The zoning of the Black Sea, especially its northwestern part, according to various ecological and geographical processes or performed according to a certain parameter, has been devoted to a large number of works, in particular, this can be found in the literature review of [1]. The relevance of the problem of zoning of sea areas is due to the need to solve various theoretical and practical problems: modeling and forecasting the variability of climatic and ecological processes, environmental quality management, planning environmental measures, justification of especially valuable territories, etc. [1]. When the boundaries of the regions are determined, an equally urgent problem arises about the distribution of the coordinates of the points of the monitoring study or satellite observations and the corresponding parameters over the given regions for subsequent analysis. That is, the problem arises about the point localization relative to a certain area, as a rule, given by a polygon on a plane. In general, this problem belongs to one of the directions of computational geometry, known as geometric search, in which it is customary to distinguish two main problems: localization problems and regional search problems. This task belongs to the first type. It is relevant not only for solving ecological and geographical problems. It is related to information theory, computer graphics, robotics, data protection and compression, information security, video surveillance, etc.

2. Brief description of the known methods

Several algorithms are known for solving the point localization problem in a polygon, the main of them are ray tracing (counting the number of intersections), summing angles, point localization for convex and star polygons, stripe method, chain method, triangulation detail method, trapezoid method and others [2]-[3].

Ray tracing is one of the easiest ways to determine whether a point belongs to an arbitrary simple polygon. The algorithm is based on the idea of counting the number of intersections of a ray outgoing from a given point in the direction of the horizontal axis, with the sides of a polygon. If it is even, the point does not belong to the polygon. A problem arises in the algorithm when a ray crosses the vertex of a polygon or an edge

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that partially coincides with the ray. The method is simple, but generally not recommended.

The summing angle method is also simple. The algorithm is based on the idea of calculating the sum of the signed angles formed at a given point by the endpoints of each edge of a given polygon. If the sum is close to zero, the point is outside the polygon, if not, then it is inside. The number of turns can be calculated by finding the closest multiple of 2π. The problem with this scheme is that it includes the square root, inverse cosine, division, point, and cross product for each edge tested. However, this method is very impractical, as it requires calculating expensive operations for each edge (inverse trigonometric functions, square roots, division), and was even called the "worst algorithm in the world" for this problem [4].

Point localization for convex and star polygons can be determined using binary search. For a convex polygon, segments are drawn from an arbitrary point inside it to the vertices of the polygon. Due to its convexity, the polygon will be split into disjoint wedges. Wedges are ordered by polar angle counterclockwise. Then a binary search is applied for the desired wedge. After the wedge is found, it is checked whether the point lies inside the wedge or outside. The target point lies inside the wedge if and only if a right turn is performed. If the turn is left, then the specified point is outside the polygon. A star polygon contains at least one arbitrary point, such that the segment from it to the desired point lies entirely inside the polygon for any vertex from the polygon. This method is quite laborious to implement.

One of the reasons for the wide variety of methods for solving this problem is that each known method, along with its advantages, has its own disadvantages and limitations, which are usually quite significant. Namely, this is the occurrence of exceptional situations, slowness, requirement of convexity, complexity in software implementation.

3. Mathematical method of a point localization in a polygon

In [3], [5]-[7] a new approach is considered based on the Cauchy's integral theorem and formula from the theory of functions of a complex variable. This approach was also discussed on the forums [8] - [10]. The following value is calculated:

$$K = \int_{\Delta} \frac{dz}{z - z_0}, \tag{1}$$

where Δ is the contour of a given polygon M, $z_0 = x_0 + iy_0$ is a given point on the complex plane.

From the Cauchy's formula (1) it follows that:

$$K = \begin{cases} 0,\ if\ z_0 \notin M \\ 2\pi i,\ if\ z_0 \in M \\ \infty,\ \ if\ z_0 \in \Delta \end{cases}$$

However, the final expression obtained in the indicated works [3], [5]-[7] using the MATEMATICA package turned out to be incorrect, possibly due to the cumbersome calculations or a technical error. To verify this, it is enough to consider an example from these works and check if the point with coordinates (1; 1) falls into the triangular region defined by the points (0; 0), (0; 1), and (1; 0). In this work, the correct expressions for calculating the value of K are obtained using mathematical analysis, without using special mathematical packages.

The integral in (1) is represented as the sum of two curvilinear integrals of the second kind:

$$K = \int_{\Delta} \frac{(x - x_0)dx + (y - y_0)dy}{(x - x_0)^2 + (y - y_0)^2} + \\ +i \int_{\Delta} \frac{(x - x_0)dy - (y - y_0)dx}{(x - x_0)^2 + (y - y_0)^2}, \tag{2}$$

Let us calculate the first integral on the right-hand side of expression (2) using Green's formula:

$$\int_{\Delta} P(x,y)dx + Q(x,y)dy = \\ = \iint_{M} \left(\frac{\partial P(x,y)}{\partial y} - \frac{Q(x,y)}{\partial x} \right) dxdy$$

where Δ is the contour of a polygon M,

$$P(x,y) = \frac{(x - x_0)}{(x - x_0)^2 + (y - y_0)^2},$$

$$Q(x,y) = \frac{(y - y_0)}{(x - x_0)^2 + (y - y_0)^2}.$$

Let's find the partial derivatives:

$$\frac{\partial P(x,y)}{\partial y} = \frac{-2(x - x_0)(y - y_0)}{[(x - x_0)^2 + (y - y_0)^2]^2},$$

$$\frac{\partial Q(x,y)}{\partial x} = \frac{-2(x - x_0)(y - y_0)}{[(x - x_0)^2 + (y - y_0)^2]^2}.$$

Thus, $\frac{\partial P(x,y)}{\partial y} \equiv \frac{\partial Q(x,y)}{\partial x}$, so, the first integral in (2) is equal to zero and expression (2) will now take the following form:

$$K = i\int\limits_{\Delta}\frac{(x-x_0)dy-(y-y_0)dx}{(x-x_0)^2+(y-y_0)^2}, \qquad (3)$$

To find the integral in (3), consider one side (segment) AB of the contour Δ of an arbitrary polygon M on the plane with coordinates $A(x_1, y_1)$ and $B(x_2, y_2)$. The equation of the straight line passing through these two points (segment) has the form:

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1},$$

whence we have the following expressions:

$$y=\frac{y_2-y_1}{x_2-x_1}(x-x_1)+y_1,\ \ dy=\frac{y_2-y_1}{x_2-x_1}dx,$$

$$x=\frac{x_2-x_1}{y_2-y_1}(y-y_1)+x_1,\ \ dx=\frac{x_2-x_1}{y_2-y_1}dy.$$

Thus, the integral for side AB will be equal to the following sum:

$$i\int\limits_{AB}\frac{(x-x_0)dy-(y-y_0)dx}{(x-x_0)^2+(y-y_0)^2}= = i\frac{y_{21}}{x_{21}}\cdot \cdot\int\limits_{x_1}^{x_2}\frac{(x-x_0)dx}{(x-x_0)^2+\left(\frac{y_{21}}{x_{21}}(x-x_1)+y_{10}\right)^2} - -i\frac{x_{21}}{y_{21}}\cdot \cdot\int\limits_{x_1}^{x_2}\frac{(y-y_0)dy}{(y-y_0)^2+\left(\frac{x_{21}}{y_{21}}(y-y_1)+x_{10}\right)^2}, \qquad (4)$$

where $x_{21}=x_2-x_1$, $y_{21}=y_2-y_1$, $x_{10}=x_1-x_0$, $y_{10}=y_1-y_0$.

In fact, it is necessary to solve one integral, since in the resulting sum (4) the integrands have the same form. We introduce the following substitutions to simplify further calculations: $t=x-x_0,\ dt=dx,\ y_{20}=y_2-y_0,$ $x_{20}=x_2-x_0$.

We will consider the first integral in the last sum. Let's substitute the substitutions:

$$\int\limits_{x_1}^{x_2}\frac{(x-x_0)dx}{(x-x_0)^2+\left(\frac{y_{21}}{x_{21}}(x-x_1)+y_{10}\right)^2}= =\int\limits_{x_1}^{x_2}\frac{tdt}{t^2+\left(\frac{y_{21}}{x_{21}}(t-x_{10})+y_{10}\right)^2}$$

After transforming the denominator of the integrand, we obtain the integral:

$$\frac{x_{21}^2}{x_{21}^2+y_{21}^2}\int\limits_{x_1}^{x_2}\frac{tdt}{\left(t+\frac{y_{21}h}{x_{21}^2+y_{21}^2}\right)^2+\frac{x_{21}^2h^2}{(x_{21}^2+y_{21}^2)^2}},$$

where $h=(y_{10}x_{21}-y_{21}x_{10})$, and, therefore, its solution, the form of which after a series of transformations, will be:

$$\frac{1}{2}\frac{x_{21}^2}{x_{21}^2+y_{21}^2}ln\frac{(x_{21}x_{20}+y_{21}y_{20})^2+h^2}{(x_{10}x_{21}+y_{21}y_{10})^2+h^2} + +\frac{y_{21}}{x_{21}}\frac{x_{21}^2}{x_{21}^2+y_{21}^2}\left(arctg\frac{x_{10}x_{21}+y_{21}y_{10}}{y_{10}x_{21}-y_{21}x_{10}} - -arctg\frac{x_{21}x_{20}+y_{21}y_{20}}{y_{10}x_{21}-y_{21}x_{10}}\right), \qquad (5)$$

Substituting (5) into (4), we obtain solution (4) for the side AB, and the logarithms will be canceled:

$$i\int\limits_{AB}\frac{(x-x_0)dy-(y-y_0)dx}{(x-x_0)^2+(y-y_0)^2}= = i\left(arctg\frac{x_{10}x_{21}+y_{21}y_{10}}{y_{10}x_{21}-y_{21}x_{10}} - -arctg\frac{x_{21}x_{20}+y_{21}y_{20}}{y_{10}x_{21}-y_{21}x_{10}}\right)$$

As a result, we obtain an expression for calculating the value of K in (1) for the entire polygon M, where the summation extends to all its sides:

$$K = i\sum_{k}^{n}\left(arctg\frac{x_{k+1,k}x_{k0}+y_{k+1,k}y_{k,0}}{y_{k0}x_{k+1,k}-y_{k+1,k}x_{k,0}} - -arctg\frac{x_{k+1,k}x_{k+1,0}+y_{k+1,k}y_{k+1,0}}{y_{k,0}x_{k+1,k}-y_{k+1,k}x_{k,0}}\right), \qquad (6)$$

It's useful to note that in [8] a solution to this problem was also obtained, but in a different form, because the method of shifting coordinates is used there.

For the test triangular region from [3], [5]-[7], given by the coordinates: (0; 0), (1; 0) and (0; 1), we have the following expression:

$$K = arctg\frac{x_0}{y_0}+arctg\frac{1-x_0}{y_0}+ +arctg\frac{x_0-y_0-1}{x_0+y_0-1}-arctg\frac{x_0-y_0+1}{x_0+y_0-1}+ +arctg\frac{1-y_0}{x_0}+arctg\frac{y_0}{x_0}$$

It is easy to check that the point (1; 1) does not belong to the given triangular region, the point (0.25; 0.25) belongs to the region, and the point (0.5; 0.5) belongs to the boundary.

The considered mathematical method makes it possible to solve the formulated problem of a point localization with respect to curvilinear polygonal regions. So, in studies [3], [5] an example is given in which a curvilinear boundary given by an equation $y = x^2$ is considered (Figure 1).

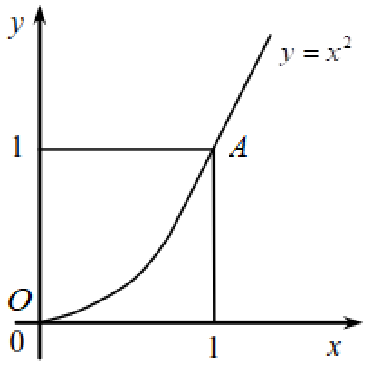

Figure 1: Example with curved side OA

From expression (3), we can obtain the following integral of the function of a complex variable for a parabola from 0 to 1:

$$\int_{OA} \frac{dz}{z - z_0} = i \int_{OA} \frac{(x - x_0)dy - (y - y_0)dx}{(x - x_0)^2 + (y - y_0)^2} =$$

$$= \left| \begin{matrix} y = x^2 \\ dy = 2xdx \end{matrix} \right| =$$

$$= i \int_0^1 \frac{2x(x - x_0) - (x^2 - y_0)}{(x - x_0)^2 + (x^2 - y_0)^2} dx =$$

$$= i \int_0^1 \frac{x^2 - 2xx_0 + y_0}{(x - x_0)^2 + (x^2 - y_0)^2} dx$$

Solving the resulting integral, one can obtain an expression for a given curvilinear side.

The obtained solution (6) of the problem of a point localization in a polygon is quite simple and can be implemented in any suitable programming language, for example, C ++, C #, Fortran, Python. Unfortunately, the author of this work is not aware of another similar analytical method for solving this problem.

Applied to ecological and geographical problems, a program was developed in the C ++ language in the visual object-oriented programming environment Borland C++Builder6, which allows to distribute the coordinates of points and the corresponding values of hydrological and hydrochemical parameters based on the results of monitoring or satellite observations for further analysis of the ecological state of sea waters in specified areas. Within the framework of the EMBLAS+ project, relatively homogeneous water areas (sea water bodies), presented in Figures 2 and 3, were identified to carry out a basic assessment of the ecological state of the marine environment in Ukraine [11].

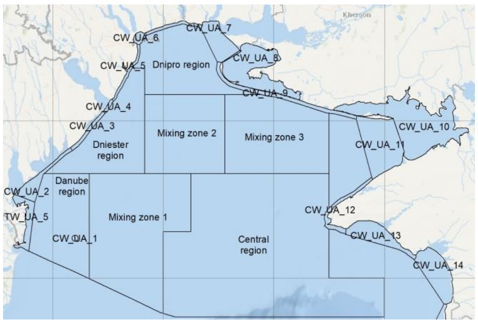

Figure 2: Schematic map of the zoning of the northwestern part of the Black Sea

Figure 3: Schematic map of the zoning of the Azov Sea

The coordinates of the peaks (longitude and latitude) of each water area (polygon) were entered into text files. Also, text files of satellite observation data on temperature, chlorophyll, transparency (longitude, latitude, parameter), as well as monitoring observations were formed in advance. The program first loads the coordinates of the vertices of the polygon. Then the observational data is loaded, a sample is performed over a given area, and the results are written to a new text file.

Further work will be aimed at clarifying the boundaries of these areas, especially coastal water bodies. developing a software package based on the developed program, which includes a database of basic hydrological and hydrochemical parameters for each water body. The database will store a table of polygons with the following columns: identifier, name and coordinates of the polygon. Taking into account the fact that there are a large number of zoning options for the Black and Azov Seas according to various characteristics, it is assumed that the database will be able to provide the user with the ability to store several zoning options. Such a software package will make it possible to assess the long-term dynamics of both average annual and monthly average changes in the parameters of the marine environment, the ecological state, eutrophication trends and the quality of waters of specific water bodies using the complex indicators E-TRIX and BEAST. The author hopes that the results obtained in this work will also find application in geographic information systems.

4. Conclusions

As a result of the study, the following conclusions were made:

- An analytical solution to the problem of point localiziation in an arbitrary polygon is obtained by methods of mathematical analysis based on the Cauchy's integral formula
- As noted in [3], [5]-[7], this method can be used for various curvilinear boundaries, as well as when the point is near the boundary of the region. Competing methods lead to the need to compare practically equal numbers, while in this method one has to compare values that differ significantly in magnitude: $0, 2\pi, \infty$
- In addition, this method does not require the convexity of the polygon, unlike other known methods of computational geometry
- A program has been developed to localize the coordinates of points in the areas of specific water bodies of the seas of Ukraine
- Prospects for further developments and research in problems of geophysics and ecology are considered

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