Multi Context Model Counting

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Abstract

Contextual reasoning is the result of the composition of local reasoning in each context via a set of inference rules, called *bridge rules*. In the past it has been argued that organizing knowledge in multiple related contexts (modules) provides many advantages. Logical inference and satisfiability in multi context systems have been widely studied. However,there are other important inference tasks such as model counting and weighted model counting that one want to perform on an MCS. In this note we concentrate on Model Counting task that can provide a good base for probabilistic reasoning in Multi Context Systems. The paper proposes a method that computes model counting for a multi context system in terms of the combination of model counting in each context.

1. Introduction

Multi Context Systems (MCSs) [1, 2, 3, 4, 5, 6] are logical frameworks that allow modelling knowledge distributed amongst a set theories called *contexts*. Each theory is specified in a (possibly different) logical language, called *local language*. The fact that a formula ϕ holds in a context c is expressed by the labelled formula $c : \phi$. The connections between knowledge of different contexts are modeled by the so called *bridge rules*, which are inference rules with premises and conclusions in different contexts. An interpretation of an MCS maps every context in a set of interpretations of the local language, called *local interpretations*. An MCS interpretation is a model if local interpretations satisfy local theories and the combination of local interpretations satisfy the bridge rules. *Model counting for MCS* is the task of determining the cardinality of the set of models of an MCS.

Model counting for logical system is obtaining increasing attentions due to its important role in the modern approaches of AI [7], where logical reasoning are blended with some form of quantitative inference such as for instance probability. The aim of this paper is to investigate about model counting method for MCS that exploit the intrinsic modular structure available in an MCS. To the best of our knowledge this is the first attempt to solve the model counting problem for MCSs. The paper provides theoretical result that proves how model counting in MCS can be obtained by a suitable combination of algorithms for model counting in each single context.

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2. Multi context systems

Let C be a set. Every element of C is called *context*. A Multi Context System on a family of logical languages $\boldsymbol{L} = \{L_c\}_{c \in C}$, is a structure $MC = \left\langle \{\Phi_c\}_{c \in C}, BR \right\rangle$ where Φ_c is a set of closed formulas of the language L_c and BR is a set of *bridge rules*, i.e., expressions of the form

$$c_1 : \phi_1, \ldots, c_n : \phi_n \Rightarrow c_{n+1} : \phi_{n+1} \tag{1}$$

where ϕ_i is a sentential formula of L_{c_i}. A model for an MCS $\left\langle \{\Phi_c\}_{c \in C}, BR \right\rangle$ is a structure $\boldsymbol{\Omega} = \{\Omega_c\}_{c \in C}$ where

1. Ω_c is a set of interpretations of L_c such that for all $\omega \in \Omega_c$, $\omega \models \Phi_c$ (written as $\Omega_c \models \Phi_c$);
2. For every bridge rule of the form (1) in BR, if $\Omega_{c_i} \models \phi_i$ for all $i = 1, \ldots, n$, then $\Omega_{c_{n+1}} \models \phi_{n+1}$

By considering each $\phi \in \Phi_c$ as a rule with zero premises, an MCS can be seen as a set of bridge rules BR. In the following we consider this simpler form of an MCS.

Example 1. *Alice and Bob are two agents who have beliefs about two propositions p and q. Bob can query Alice on her beliefs about p and q, Alice can have partial or contraddictory beliefs about p and q, therefore, it is possible that Alice provides no answer or contraddictory answers to Bob's queries. Bob has the following beliefs about Alice. He believes that, if Alice is reliable (r), then Alice's answers are correct, and therefore he will also believe in what Alice says. This simple scenario can be formulated with an MCS MC_{ab} with two contexts $C = \{a, b\}$; where the local languages L_a and L_b are s propositional languages on the propositions $\{p, q\}$ and $\{p, q, r\}$ respectively, The two contexts are connected by the following bridge rules:*

$$a : p \Rightarrow b : r \to p \qquad\qquad a : \neg p \Rightarrow b : r \to \neg p \tag{2}$$
$$a : q \Rightarrow b : r \to q \qquad\qquad a : \neg q \Rightarrow b : r \to \neg q \tag{3}$$

The above bridge rules simulate all the query-answering interactions between Alice and Bob. Notice that if Alice provides contraddictory answers, e.g., $a : p$ and $a : \neg p$ then Bob will believe that she is not reliable ($b : \neg r$. A model for MC_{ab} is a pair $\langle \Omega_a, \Omega_b \rangle$, where Ω_a and Ω_b represent the final belief state of the two agents after Bob have done all the possible queries to Alice. We are interested in counting the number of final states of such a simple system, i.e., the number of models of MC_{ab}.

3. Multi context model counting

Suppose that for every language context c, there is an oracle mc_c that returns the number of models $mc_c(\Phi)$[1] for every set of L_c-sentences Φ. We are interested in finding a way to solve the model counting problem for BR using such oracles.

Let $\boldsymbol{B}$ be the set of labelled formulas $c : \phi$ contained in in some bridge rule of BR. Let $\boldsymbol{F} \subseteq \boldsymbol{B}$ be any subset of $\boldsymbol{B}$ closed under BR. Let $\mathbb{F}$ be the set of all such $\boldsymbol{F}$'s. Let's define $mc(\boldsymbol{F})$ as the number of $\boldsymbol{\Omega} = \{\Omega_c\}_{c \in C}$, where Ω_c is a set of models for L_c, such that:

[1]When it is clear from the context we will omit the index to the context and use the simpler notation $mc(\phi)$.

1. $\Omega_c \models \phi$ for every $c : \phi \in \boldsymbol{F}$ and
2. $\Omega_c \not\models \phi$ for every $c : \phi \in \bar{\boldsymbol{F}} = \boldsymbol{B} \setminus \boldsymbol{F}$.

Notice that $\boldsymbol{\Omega}$ satisfies conditions 1 and 2 if and only if $\boldsymbol{\Omega}$ is a model of BR.

Proposition 1. *$\boldsymbol{\Omega}$ cannot satisfy conditions 1 and 2 for two distinct $\boldsymbol{F}$ and $\boldsymbol{F}'$.*

Proof: If $\boldsymbol{F} \neq \boldsymbol{F}'$ there is a labelled formula $c : \phi$ such that either $c : \phi \in \boldsymbol{F} \cap \bar{\boldsymbol{F}}'$ or $c : \phi \in \bar{\boldsymbol{F}} \cap \boldsymbol{F}'$. If $\boldsymbol{\Omega}$ satisfies condition 1 and 2 for both $\boldsymbol{F}$ and $\boldsymbol{F}'$ then $\Omega_c \models \phi$ and $\Omega_c \not\models \phi$, which is a contradiction. □

Lemma 1. *For every $\boldsymbol{F} \subseteq \boldsymbol{B}$:*

$$mc(\boldsymbol{F}) = \prod_c \sum_{G \subseteq \bar{F}_c} (-1)^{|G|} 2^{mc(F_c \cup G)}$$

where, for every set of labelled formulas $\boldsymbol{X}$, X_c denotes the set $\{\phi \mid c : \phi \in \boldsymbol{X}\}$.

Proof: For a given $\boldsymbol{F}$, an MC interpretation $\boldsymbol{\Omega} = \{\Omega_c\}_{c \in C}$ satisfies all the $c : \phi$ in $\boldsymbol{F}$ and does not satisfy all the $c : \phi \in \bar{\boldsymbol{F}}$ if and only if it satisfies the following two conditions for every $c \in C$:

1. for all $\phi \in F_c$, $\omega \models \phi$ for all $\omega \in \Omega_c$;
2. for all $\phi \in \bar{F}_c$, $\omega \models \neg\phi$ for some $\omega \in \Omega_c$.

Therefore we have to count how many such a Ω_c exist for every c. For this purpose we use the following result:

Corollary 1 ([8] section 4.2). *Let X be a set of objects and let $\mathcal{Y} = \{Y_1, \ldots, Y_m\}$ be a set of subsets of X. For every $\mathcal{Q} \subseteq \mathcal{Y}$, let $N(\supseteq \mathcal{Q})$ be the count of objects in X that belong to all the subsets $Y_i \in \mathcal{Q}$, i.e., $N(\supseteq \mathcal{Q}) = \left|\{\bigcap_{Y_i \in Q} Y_i\}\right|$. For every $0 \leq l \leq m$, let $s_l = \sum_{|\mathcal{Q}|=l} N(\supseteq \mathcal{Q})$ and let e_0 be count of objects that do not belong to any of the Y_i in $\mathcal{Y}$, then*

$$e_0 = \sum_{l=0}^{m} (-1)^l s_l \tag{4}$$

In our case, let X be the set of all the subsets of models of F_c. I.e. $X = \{\Omega \subseteq \Omega(L_c) \mid \Omega \models F_c\}$, where $\Omega(L_c)$ is the set of all interpretations of L_c. Let $\mathcal{Y} = \{Y_\phi\}_{\phi \in \bar{F}_c}$, where $Y_\phi = \{\Omega \in X \mid \Omega \models \phi\}$. With this definition we have that e_o is the number of subsets of X (models of F_c) that do not satisfy none of the formulas in $\bar{F}_c$. To apply Corollary 1 we need to calculate s_l for every $0 \leq l \leq |\bar{F}_c|$. By definition of s_l we have:

$$s_l = \sum_{|\mathcal{Q}|=l} N(\supseteq \mathcal{Q}) = \sum_{\substack{G \subseteq \bar{F}_c \\ |G|=l}} \left| \bigcap_{\phi \in G} Y_\phi \right|$$

Notice that $\Omega \in \bigcap_{\phi \in G} Y_\phi$ if and only if $\Omega \models F_c \wedge G$. This implies that $\left|\bigcap_{\phi \in G} Y_\phi\right|$ is the number of subsets of the set of models that satisfiey $F_c \wedge G$, i.e.,

$$s_l = \sum_{\substack{G \subseteq \bar{F}_c \\ |G|=l}} 2^{mc(F_c \wedge G)}$$

From which we conclude that the number of sets of models that satisfies F_c and do not satisfy $\bar{F}_c$ is:

$$e_0 = \sum_{G \subseteq \bar{F}_c} (-1)^{|G|} 2^{mc(F_c \cup G)}$$

Notice that every model of $\boldsymbol{F}$ that does not satisfy the formulas in $\bar{F}$ can be obtained by selecting for every c a set of models that satisfy F_c and do not satisfy $\bar{F}_c$. Since we have $\sum_{G \subseteq \bar{F}_c} (-1)^{|G|} 2^{mc(F_c \cup G)}$ of such sets of models, we can conclude that

$$mc(\boldsymbol{F}) = \prod_c \sum_{G \subseteq \bar{F}_c} (-1)^{|G|} 2^{mc(F_c \cup G)} \tag{5}$$

$$\tag{6}$$

□

Theorem 1.

$$mc(BR) = \sum_{\boldsymbol{F} \in \mathbb{F}(BR)} \prod_c \sum_{G \subseteq \bar{F}_c} (-1)^{|G|} 2^{mc(F_c \cup G)} \tag{7}$$

Proof: For every model $\boldsymbol{\Omega}$ of BR there is an $\boldsymbol{F}$ such that $\Omega \models \boldsymbol{F}$ and for every $c : \phi \in \bar{\boldsymbol{F}}$ $\Omega_c \not\models \phi$. Furthermore, Proposition 1 guarantees that $\boldsymbol{\Omega}$ cannot be a model of two distinct $\boldsymbol{F}$'s. This allows us to infer that

$$\begin{aligned} mc(BR) &= \sum_{\boldsymbol{F} \in \mathbb{F}(BR)} mc(\boldsymbol{F}) \\ &= \sum_{\boldsymbol{F} \in \mathbb{F}(BR)} \prod_c \sum_{G \subseteq \bar{F}_c} (-1)^{|G|} 2^{mc(F_c \cup G)} \end{aligned}$$

□

Notice that the set $\mathbb{F}(BR)$ can be computed by starting from any subset of $\boldsymbol{B}$ and by applying bridge rules until a fixpoint is reached. This operation takes at most $|BR|$ steps. In the worse case at every step only one element of BR is fired. Furthermore one can compute and cash $mc(X_c)$ for all the subset $X_c \subseteq B_c$. The complexity of this is fully determined by mc_c and it is not influenced by the complexity of the model counting in the other contexts. Therefore the complexity of the entire process is just the sum of the complexity of computing model count in c for all the subsets of B_c.

Example 2 (continuation of Example 1). *Let us apply Theorem* **??** *to a simplified version of Example 1, where L_a contains the only proposition p and L_b r and p we consider only bridge rules* (2). *The set $\boldsymbol{B}$ is equal to:*

$$\boldsymbol{B} = \{a : p,\ a : \neg p,\ b : r \rightarrow p,\ b : r \rightarrow \neg p\}$$

The set $\mathbb{F}$ of subsets $\boldsymbol{F}$ of $\boldsymbol{B}$ closed under the bridge rules (2) *and* (3)*, are the following:*

$$\begin{aligned}
\boldsymbol{F}_0 &= \{\} \\
\boldsymbol{F}_1 &= \{b : r \rightarrow p\} \\
\boldsymbol{F}_2 &= \{b : r \rightarrow \neg p\} \\
\boldsymbol{F}_3 &= \{b : r \rightarrow p,\ b : r \rightarrow \neg p\} \\
\boldsymbol{F}_4 &= \{a : p,\ b : r \rightarrow p\} \\
\boldsymbol{F}_5 &= \{b : r \rightarrow p,\ b : r \rightarrow \neg p\} \\
\boldsymbol{F}_6 &= \{a : \neg p,\ b : r \rightarrow \neg p\} \\
\boldsymbol{F}_7 &= \{a : \neg p,\ b : r \rightarrow p,\ b : r \rightarrow \neg p\} \\
\boldsymbol{F}_8 &= \{a : p,\ a : \neg p,\ b : r \rightarrow p,\ b : r \rightarrow \neg p\}
\end{aligned}$$

Using formula (5) *one can cmput $mc(\boldsymbol{F}_i)$ and then sum all the result, obtaining $mc((2))$. Let us for instance compute $mc(\boldsymbol{F}_3)$ and $mc(\boldsymbol{F}_4)$*

$$\begin{aligned}
mc(\boldsymbol{F}_3) &= (2^{mc_a(\top)} - 2^{mc_a(p)} - 2^{mc_a(\neg p)} + 2^{mc_a(p \wedge \neg p)}) \cdot 2^{mc_b((r \rightarrow p) \wedge (r \rightarrow \neg p))} \\
&= 2^2 - 2^1 - 2^1 + 2^0) \cdot 2^2 && = 4 \\
mc(\boldsymbol{F}_4) &= (2^{mc_a(p)} - 2^{mc_a(p \wedge \neg p)}) \cdot (2^{mc_b(r \rightarrow p)} - 2^{mc_b(r \rightarrow p \wedge r \rightarrow \neg p)}) \\
&= (2^1 - 2^0) \cdot (2^3 - 2^2) && = 4
\end{aligned}$$

4. Conclusion and future directions

In this short note, we proved a formula to compute model counting for MC systems that is based on model counting for each single context. This initial idea can be generalised in a number of directions. The first direction concerns the generalisation to MC weighted model counting. Weighted model counting is tightly connected to probabilistic reasoning (see e.g., [9]), this will open the opportunity of doing contextual probabilistic inference. A second research direction can be obtained by exploiting the correspondence between modal logic and MC systems proved in [2] and develop a context based approach to model counting and probabilistic inference for modal logic. Finally one could extend the result of this note to distributed first order logic [5] and distributed description logics [6]. Further generalisation involves more complex bridge rules including for instance negated labelled formulas and disjunction of labelled formulas.

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