

Public Announcement and Intuitionistic Epistemic Logic

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Abstract

We introduce intuitionistic public announcement logic: system for public announcement in intuitionistic logic as well as in a variant of intuitionistic epistemic logic, namely IEL with a knowledge modal operator K interpreted as “it is verified that the formula holds intuitionistically”, i.e., that it has a proof which is not necessarily specified in the process of verification. Hilbert-style axiomatisations and Kripke-style semantics for the systems are provided and soundness and completeness of the axiomatisation are discussed. The completeness theorem is proved by the standard method of translation and reduction of the question of completeness of the PA systems to the systems without PA operators.

Keywords

Intuitionistic Epistemic Logic, Public Announcement, Kripke Semantics, Axiomatisation

1. Introduction

Logic has traditionally been preoccupied with the study of correct reasoning and arguments without any reference to agents and their abilities. Nevertheless, in the late XXth century the focus of attention has partially moved towards modeling of interaction between agents. This led to development of different types of multi-agent systems giving rise to a number of new questions considering the variety of agents. It became obvious that multi-agent modeling should account for the existence of different types of agent. And indeed, we observe that in reality different agents possess unequal reasoning capacities, memory and presumptions. This is true not only of human agents, but also artificial intelligence. One can identify those characteristics that are pertinent for formal logical modeling that yield various agent types. Hence, by the variety of agents we understand the logical diversity that results from various attempts to develop formal systems capable of modelling actions, reasoning and behaviour of actual rational agents. The question is mostly studied in two branches of contemporary philosophical logic: in epistemic logic and numerous action models. The roots of the logical problem of agent diversity lie in the idea that various agents might have different capacities with respect to knowledge, beliefs, and desires, as well as use diverse rules of inference. The question has arisen


8th Workshop on Formal and Cognitive Reasoning, September 19, 2022, Trier, Germany

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 CEUR Workshop Proceedings (CEUR-WS.org)

quite recently, as it was traditionally believed that logical laws represent some universal norms using which everyone arrives at the same conclusion given the identical premises. However, if we want to model real agents, we should be able to embrace the limited character of their capacities, for instance, in case of imperfect or fallible information.

Although, intuitionistic logic has been very well studied both from technical point of view and the one on the constructive philosophy, the question of knowledge progress and belief change was undeservedly put aside. There has been proposed a vast number of different intuitionistic and constructive modal systems without rising a question of how to model the knowledge accumulation in the intuitionistic setting. Nevertheless, one should mention the paper [1] where public announcement was introduced to the epistemic logic. That being said, we insist that our approach extends that result and applied **PAL** to a, epistemic logic **IEL** that differs significantly from other know intuitionistic epistemic systems because of how *knowledge* is understood.

Another reason for choosing this logic is that intuitionistic logic enjoys several nice properties, e.g., the disjunctive property, finite model property, and it allows for an analytic calculus. Thus, it is of a particular interest to study its possible extensions. Moreover, there is a wide range of different logical games for intuitionistic logic (e.g. [2, 3, 4]) that can be extended to the modal level and provide not only technical results but also philosophical insight on the intuitionistic epistemology in general. Finally, intuitionistic logic is important for computer science due to the Curry-Howard-Lambek correspondence (or the Curry-Howard isomorphism) which states a strong connection between types in programming languages and propositions in intuitionistic logic. Thus, the research might be relevant for applications to semantics of programming languages.

How would our model change, once we get a proof of a statement? This is the question we are answering in section 2. In section 2, we propose two systems (that can be combined into one with two separate operators) that yield the same model change but capture it differently. The **IPAL**_◇ (cf. 2.1) system with ◇-type operator suggest a more conservative change model: after a truthful update $\langle !\varphi \rangle \psi$, we delete all the worlds where φ is not true. The **IPAL**_□ (cf. 2.2) system captures a more interesting relation between public announcement and non-modal formulae. In section 3 we introduce a knowledge operator into the two previously discussed systems. we prove soundness and completeness for the proposed systems.

2. PAL in Intuitionistic logic

In the present chapter we discuss the simplest dynamic extension (i.e. with the public announcement operator) of the standard *intuitionistic propositional logic* where the truth of a formula φ is understood as φ is provable. Intuitionistic logic is one of the most studied logics along with the classical logic. It gives rise to families of other logics if we impose additional restrictions or add new elements, e.g., modalities. Yet, it enjoys a number of nice properties as mentioned in section 1 The development of intuitionistic logic was originally motivated by the *philosophy of mathematics* and research in the area of *foundations of mathematics*. The general idea of *intuitionism* was proposed by

Brouwer [5], however he did not provide any formal system for the logic as his main concern was that of proofs in mathematics. Now it is common to understand the "truth" in intuitionistic logic as provability. This is called an informal Brouwer-Heyting-Kolmogorov semantic (simply **BHK**):

- a proof of $\varphi \wedge \psi$ consists in a proof of φ and a proof of ψ ;
- a proof of $\varphi \vee \psi$ consists in a proof of φ or a proof of ψ ;
- a proof of $\varphi \rightarrow \psi$ consists in a construction which given a proof of φ returns a proof of ψ ;
- $\neg\varphi$ is an abbreviation for $\varphi \rightarrow \perp$, and \perp is a proposition that has no proof.

The formal system for intuitionism was first suggested in the form of Hilbert style (i.e. axiomatic) calculus independently first by Kolmogorov [6], and then by Heyting [7, 8, 9], and Glivenko [10] (however, this was a feeble system whose implicative fragment contained only the identity law and transitivity of implication). Intuitionistic calculus has one axiom less than the classical one, namely it does not contain the principle *tertium non datur*: $\varphi \vee \neg\varphi$ (alternatively, to obtain the classical calculus, one can add the double negation elimination ($\neg\neg\varphi \rightarrow \varphi$) to the intuitionistic calculus). In 1935, Gentzen [11] introduced two new proof systems that were supposed to both better capture reasoning in mathematics and provide a simpler proof search: Natural Deduction for classical (**NK**) and intuitionistic (**NI**) logics; and Sequent Calculi **LK**, and **LI** respectively. Sequent calculus for intuitionistic logic is analytic which is an important proof-theoretical feature as it simplifies the proof search. Sequent calculi are based on rewriting rules for sequents, which are objects of the form $\Gamma \vdash \Delta$ where Γ and Δ are sequences of formulae. In the most cases (for instance, for classical and intuitionistic calculi), Γ is interpreted as a big conjunction of the formulae inside, whereas Δ represents a disjunction. In the case of intuitionistic logic, the most widespread calculi restricts Δ to only one formula. Intuitively, this can be interpreted as follows: at least one formula from Δ follows from the conjunction of formulae in Γ . Unfortunately, sequent calculi are not capable of capturing many interesting logics, so various extensions have been suggested in the last thirty years, such as *hypersequents*, *nested sequents*, *labelled calculi* etc.

As for the semantics for intuitionistic propositional logic, it has been proved by Gödel [12] that intuitionistic logic cannot be characterised by a fixed finite number of truth values. It is well-known that Intuitionistic logic *Int* can be embedded into modal logic **S4** by means of the Gödel translation. Hence, the standard semantics for intuitionistic logic is *Kripke semantics* for modal logic **S4** with monotonicity of valuations (which leads to the monotonicity of forcing). Alternatively, there is a neighbourhood semantics for *intuitionistic logic*, but our research mostly concentrates on the Kripke style one.

Finally, an important result with respect to intuitionistic logic, called *Curry-Howard isomorphism*, relates intuitionistic logic to computability theory and programming languages. Haskell Curry and William Howard noticed the correspondence between intuitionistic natural deduction and *simply typed lambda calculus*. The inference rules and axioms

for typing programs resemble the inference rules and axioms for proving formulae in intuitionistic logic. *Propositions-as-types* correspondence, and *proofs-as-programs* correspondence establish an isomorphism between types and formulae, on the one hand, and between programs and proofs, on the other hand.

We now extend standard intuitionistic propositional logic with a new operator of *public announcement*. In what follows we define the language, semantics and axioms for the system **IPAL**. In fact, there are several ways of defining *public announcement operator* for intuitionistic logic. In what follows we suggest two different operators: $\langle !\varphi \rangle$ and $[\!|\varphi]$. In classical **PAL**, these are dual operators. However, like for quantifiers in intuitionistic **FOL**, these operators are not dual any more in **IPAL**. We first describe **IPAL** $_{\langle !\varphi \rangle}$, i.e., the extension with $\langle !\varphi \rangle$ operator, and then **IPAL** $_{[\!|\varphi]}$ with the $[\!|\varphi]$ operator respectively.

Note that the presented *public announcement* operators do not only give the truthful information that the announced formula is true (i.e., there is a proof of it), but also provides us with the proof. That is so, because, as it is formally defined in the following sections, the announcement leads to *model transformation* and those worlds where the announced formula φ is not true are erased, i.e., not only worlds where $\neg\varphi$ is true, but also such $x \in W$ that $\mathcal{M}, x \not\Vdash \varphi$. As weaker announcement is the subject of further research.

2.1. System **IPAL** $_{\langle !\varphi \rangle}$

Definition 1. Let P be a countable set of propositional variables. The language $\mathcal{L}_{\mathbf{IPAL}_{\langle !\varphi \rangle}}$:

$$\varphi ::= p \mid \varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \langle !\varphi \rangle \varphi \mid \perp,$$

where $p \in P$.

Negation is an abbreviation for $\varphi \rightarrow \perp$. Formula $\langle !\varphi \rangle \psi$ operator of *public announcement* is read as "after *some* truthful public announcement of φ , it holds that ψ ", as it is essentially a \Diamond -type modal operator. It assumes that φ can indeed be truthfully announced (unlike the \Box -based operator we introduce in 2.2). Note that in the BNF form $\langle !\varphi \rangle \varphi$ we express the *type* of the formulae, which is the same for the announcement formula and the one that is following it. We introduce a Hilbert-style proof system for **IPAL** $_{\langle !\varphi \rangle}$:

Definition 2. *Intuitionistic Public Announcement Logic with $\langle !\varphi \rangle$ operator, **IPAL** $_{\langle !\varphi \rangle}$, is defined by the following axioms:*

I Axioms of propositional intuitionistic logic:

1 : $\varphi \rightarrow (\psi \rightarrow \varphi)$	6 : $\varphi \rightarrow (\varphi \vee \psi)$
2 : $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$	7 : $\psi \rightarrow (\varphi \vee \psi)$
3 : $(\varphi \wedge \psi) \rightarrow \varphi$	8 : $(\varphi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow ((\varphi \vee \psi) \rightarrow \chi))$
4 : $(\varphi \wedge \psi) \rightarrow \psi$	9 : $\perp \rightarrow \varphi$
5 : $\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$	

II Axioms for $\langle !\varphi \rangle$ operator:

$$\begin{array}{ll}
\mathbf{10} : \langle !\varphi \rangle p \leftrightarrow (\varphi \wedge p) & \mathbf{13} : \langle !\varphi \rangle (\psi \vee \chi) \leftrightarrow (\langle !\varphi \rangle \psi \vee \langle !\varphi \rangle \chi) \\
\mathbf{11} : \langle !\varphi \rangle \neg \psi \leftrightarrow (\varphi \wedge \neg \langle !\varphi \rangle \psi) & \mathbf{14} : \langle !\varphi \rangle \langle !\psi \rangle \chi \leftrightarrow \langle !\varphi \wedge \langle !\psi \rangle \psi \rangle \chi \\
\mathbf{12} : \langle !\varphi \rangle (\psi \wedge \chi) \leftrightarrow (\langle !\varphi \rangle \psi \wedge \langle !\varphi \rangle \chi) & \mathbf{15} : \langle !\varphi \rangle (\psi \rightarrow \chi) \rightarrow (\langle !\varphi \rangle \psi \rightarrow \langle !\varphi \rangle \chi)
\end{array}$$

- Inference rule: *Modis ponens* (MP).

Kripke Semantics for $\mathbf{IPAL}_{\langle \rangle}$. Kripke semantics is a triple $\mathcal{M} = \langle W, R, v \rangle$, where W is a non-empty set of *possible worlds*, R is a *partial order* on W , and $v : W \times W \rightarrow \{0, 1\}$ is the *variable valuation function*. The function v is required to be *monotonic* with respect to R , i.e., if xRy , then $v(p, x) \leq v(p, y)$ for any $p \in P$ and $x, y \in W$. Given the definition of the forcing relation, it is straightforward that the *monotonicity of forcing* is preserved as well: if $\mathcal{M}, x \Vdash \varphi$ and xRy , then $\mathcal{M}, y \Vdash \varphi$.

Definition 3. We write $\mathcal{M}, x \Vdash \varphi$ if formula φ is true in the world x of model \mathcal{M} . We define the forcing relation, denoted by \Vdash , as follows:

- $\mathcal{M}, x \not\Vdash \perp$;
- $\mathcal{M}, x \Vdash p$ iff $v(p, x) = 1$ (truth of variables is prescribed by the valuation function);
- $\mathcal{M}, x \Vdash \varphi \wedge \psi$ iff $\mathcal{M}, x \Vdash \varphi$ and $\mathcal{M}, x \Vdash \psi$;
- $\mathcal{M}, x \Vdash \varphi \vee \psi$ iff $\mathcal{M}, x \Vdash \varphi$ or $\mathcal{M}, x \Vdash \psi$;
- $\mathcal{M}, x \Vdash \varphi \rightarrow \psi$ iff for all $y \in R(x)$ either $\mathcal{M}, y \not\Vdash \varphi$ or $\mathcal{M}, y \Vdash \psi$.
- $\mathcal{M}, x \Vdash \langle !\varphi \rangle \psi$ iff $\mathcal{M}, x \Vdash \varphi$ and $\mathcal{M}_\varphi, x \Vdash \psi$, where $\mathcal{M}_\varphi = \langle W^\varphi, R^\varphi, v^\varphi \rangle$. We define $W^\varphi := \{w \in W \mid \mathcal{M}, w \Vdash \varphi\}$. If $W^\varphi = \emptyset$, then \mathcal{M}_φ is undefined. If $W^\varphi \neq \emptyset$, then $R^\varphi := R \cap (W^\varphi \times W^\varphi)$ and $v^\varphi(p) := v(p) \cap W^\varphi$.

Additionally, by $R(x) := \{y \mid xRy\}$.

Theorem 1 (Soundness). If $\vdash_{\mathbf{IPAL}_{\langle \rangle}} \varphi$, then $\Vdash \varphi$ for all $\varphi \in \mathcal{L}_{\mathbf{IPAL}_{\langle \rangle}}$.

Proof. To prove soundness, one needs to prove two following statements: (1) if φ is an axiom in $\mathbf{IPAL}_{\langle \rangle}$, then $\Vdash \varphi$; (2) that the forcing relation is closed under application of *modus ponens*. The proof of (2) is standard for *propositional intuitionistic logic*.

For (1), one needs to check all the axioms. The soundness of axioms 1–9 is due to the soundness of standard *propositional intuitionistic logic*. Thus, one needs to check only axioms 10–15. This proof is similar to 4 and is based on the forcing relation definition, thus it is left to the reader to check. \square

The proof of completeness is done the standard way for \mathbf{PAL} , i.e., via translation and reduction [13]. All formulae with public announcement operators are translated into formulae without them. This reduces the question of completeness of the \mathbf{PA} systems to the systems without \mathbf{PA} operators. We first define a *complexity function*:

Definition 4 (Complexity function). *Complexity function c assigns numbers to formulae:*

- $c(p) := 1$
- $c(\neg\varphi) := 1 + c(\varphi)$
- $c(\varphi \star \psi) := 1 + \max(c(\varphi), c(\psi))$ where $\star \in \{\wedge, \vee, \rightarrow\}$
- $c(K\varphi) := 1 + c(\varphi)$
- $c(\langle !\varphi \rangle \psi) := (4 + c(\varphi)) \cdot \psi$
- $c([\! \varphi \!] \psi) := (4 + c(\varphi)) \cdot \psi^1$

Our translation uses the equivalences that we used as axioms.

Lemma 1 ((Decreasing-Complexity Lemma). *For all $p \in P$ and all $\varphi, \psi, \chi \in \mathcal{L}$*

- $c(\psi) \leq c(\varphi)$ if $\psi \in \text{Sub}(\varphi)$ where Sub is the set of subformulae;
- $c(\langle !\varphi \rangle p) > c(\varphi \wedge p)$;
- $c(\langle !\varphi \rangle \neg\psi) > c(\varphi \wedge \neg\langle !\varphi \rangle \psi)$;
- $c(\langle !\varphi \rangle (\psi \wedge \chi)) > c(\langle !\varphi \rangle \psi \wedge \langle !\varphi \rangle \chi)$;
- $c(\langle !\varphi \rangle (\psi \vee \chi)) > c(\langle !\varphi \rangle \psi \vee \langle !\varphi \rangle \chi)$
- $c(\langle !\varphi \rangle \langle !\psi \rangle \chi) > c(\langle !\varphi \wedge !\psi \rangle \chi)$;
- $c(\langle !\varphi \rangle (\psi \rightarrow \chi)) > c(\langle !\varphi \rangle \psi \rightarrow \langle !\varphi \rangle \chi)$

Proof. Given the definition of the *complexity function*, the proof is trivial and left to the reader to the sake of space. □

We finally define the translation function:

Definition 5. *The function t by recursion on complexity $c(\varphi)$:*

- $t(p) := p$
- $t(\neg\varphi) := \neg t(\varphi)$
- $t(\varphi \star \psi) := t(\varphi) \star t(\psi)$ where $\star \in \{\wedge, \vee, \rightarrow\}$
- $t(K\varphi) := Kt(\varphi)$
- $t(\langle !\varphi \rangle p) := t(\varphi \wedge p)$
- $t(\langle !\varphi \rangle \neg\psi) := t(\varphi \wedge \neg\langle !\varphi \rangle \psi)$

¹We give the general definition that works for all of our systems.

- $t(\langle !\varphi \rangle(\psi \wedge \chi)) := t(\langle !\varphi \rangle\psi \wedge \langle !\varphi \rangle\chi)$
- $t(\langle !\varphi \rangle(\psi \vee \chi)) := t(\langle !\varphi \rangle\psi \vee \langle !\varphi \rangle\chi)$
- $t(\langle !\varphi \rangle\langle !\psi \rangle\chi) := t(\langle !\varphi \wedge !\psi \rangle\chi)$
- $t(\langle !\varphi \rangle(\psi \rightarrow \chi)) := t(\langle !\varphi \rangle\psi \rightarrow \langle !\varphi \rangle\chi)$

Lemma 2. $t(\varphi) \in \mathcal{L}_{IPC}$ for each $\varphi \in \mathcal{L}_{IPAL}_{\langle \rangle}$, i.e., $t(\varphi)$ has no announcements.

Proof. By induction on $c(\varphi)$. □

Lemma 3. $\vdash_{IPAL}_{\langle \rangle} \varphi \leftrightarrow t(\varphi)$ for all $\varphi \in \mathcal{L}_{IPAL}_{\langle \rangle}$.

Proof. By induction on $c(\varphi)$. □

Theorem 2 (Completeness). If $S4 \Vdash \varphi$, then $\vdash_{IPAL}_{\langle \rangle} \varphi$ for all $\varphi \in \mathcal{L}_{IPAL}_{\langle \rangle}$.

Proof. 1. $S4 \Vdash \varphi$ assumption;

2. $\vdash_{IPAL}_{\langle \rangle} \varphi \leftrightarrow t(\varphi)$ by lemma 3;

3. $S4 \Vdash \varphi \leftrightarrow t(\varphi)$ from 2. by soundness of $IPAL_{\langle \rangle}$;

4. $S4 \Vdash t(\varphi)$ MP from 1. and 3.;

5. $\vdash_{S4} (\varphi)$ by completeness of $S4$;

6. $\vdash_{IPAL}_{\langle \rangle} (\varphi)$ from 5. because $S4 \subseteq IPAL_{\langle \rangle}$;

7. $\vdash_{IPAL}_{\langle \rangle} \varphi$ from 2. and 6. □

2.2. System $IPAL_{\square}$

The idea behind $[\!|\varphi|\!]$ operator is that even if in the possible world in question we do not have a prove of φ , we can still reason about what other statements (formulae) will be proved once the proof of φ is given to us.

Definition 6. The language for Intuitionistic propositional logic $\mathcal{L}_{IPAL}_{\square} = \mathcal{L}_{IPC} \cup [\!|\varphi|\!]$.

Formula $[\!|\varphi|\!]\psi$ operator of *public announcement* is read as "after every public announcement of φ , it holds that ψ ". We introduce a Hilbert-style proof system for $IPAL_{\square}$:

Definition 7. Intuitionistic Public Announcement Logic with $[\!|\varphi|\!]$ operator, $IPAL_{\square}$, is defined by the following axioms:

I Axioms of propositional intuitionistic logic;

II Axioms for $[\!|\varphi|\!]$ operator:

10. $[\!|\varphi|\!]p \leftrightarrow (\neg\varphi \vee (\varphi \rightarrow p))$;

11. $[\!|\varphi]\neg\psi \leftrightarrow (\varphi \rightarrow \neg[\!|\varphi]\psi)$;
12. $[\!|\varphi](\psi \wedge \chi) \leftrightarrow ([\!|\varphi]\psi \wedge [\!|\varphi]\chi)$;
13. $[\!|\varphi](\psi \vee \chi) \rightarrow (\varphi \rightarrow ([\!|\varphi]\psi \vee [\!|\varphi]\chi))$;
14. $[\!|\varphi][\!|\psi]\chi \leftrightarrow [\!|\varphi \wedge [\!|\psi]\chi]$;
15. $[\!|\varphi](\psi \rightarrow \chi) \rightarrow ([\!|\varphi]\psi \rightarrow [\!|\varphi]\chi)$.

- *Inference rule: Modis ponens (MP).*

Kripke Semantics for \mathbf{IPAL}_{\square} . *Kripke semantics* for \mathbf{IPAL}_{\square} is defined the same way as the one for \mathbf{IPAL}_{\diamond} (in 2.1), except for the condition on public announcement operator.

Definition 8. *Instead of $\langle\!|\varphi\rangle\psi$ we have the following condition for $[\!|\varphi]$:*

- $\mathcal{M}, x \Vdash [\!|\varphi]\psi$ iff $\mathcal{M}, x \Vdash \neg\varphi$, or else for all $y \in R(x)$, such that $y \in \mathcal{M}_{\varphi}$, $\mathcal{M}_{\varphi}, y \Vdash \psi$, where $\mathcal{M}_{\varphi} = \langle W^{\varphi}, R^{\varphi}, v^{\varphi} \rangle$.

Theorem 3 (Soundness and Completeness). *If $\vdash_{\mathbf{IPAL}_{\square}} \varphi$ iff $\Vdash \varphi$ for all $\varphi \in \mathcal{L}_{\mathbf{IPAL}_{\square}}$.*

Proof. The proofs are analogous to the one for \mathbf{IPAL}_{\diamond} , theorem 2. □

3. PAI for Intuitionistic Epistemic Logic

In this section we are considering previously described public announcements in the setting of *intuitionistic epistemic logic IEL*. The system was proposed by Tudor Protopopescu and Sergei Artemov [14, 15]. Here we consider only their system **IEL** which models knowledge as opposed to the system **IEL**⁻ corresponding to belief.

IEL extends *intuitionistic logic* by a new modal operator K interpreted as *knowledge*. It is based on the intuition behind the *BHK* semantics, where *truth* of a formula φ is interpreted as *having a proof* φ . *Knowledge* $K\varphi$ is then interpreted as "*it is verified that φ holds intuitionistically*", i.e., that φ has a proof which is not necessarily specified in the process of verification (cf. [14]). The idea that knowledge corresponds to the existence of a verification for some formula in question yields specific conditions on K operator different from the standard classical account. Thus, following Artemov and Protopopescu [14, 15], one can extend the semi-formal BHK semantics to capture this understanding of knowledge as follows:

- a proof of $K\varphi$ is a conclusive evidence of verification that φ has a proof.

As a real-world example of the K operator, one can name a plenty of things: *zero-knowledge protocol*, when a prover gives us an answer that a statement in question is true without providing the proof; *testimony of an authority* which applied even in science (we may know that a theorem is true without being able to reproduce its proof); *existential generalisation* and many others. In what follows, we deliberately add K operator to the systems described above in section 2

3.1. System IEPAL

Definition 9. The language $\mathcal{L}_{\text{IEPAL}} = (\mathcal{L}_{\text{IPD}_{\langle \cdot \rangle}} \cap \mathcal{L}_{\text{IPB}_{\langle \cdot \rangle}}) \cup K\varphi$.

Definition 10. Now we introduce the axiomatisation for IEPAL:

I axioms of $\text{IPAL}_{\langle \cdot \rangle}$ and IPAL_{\square} :

II Axioms of **IEL**:

16. $K(\varphi \rightarrow \psi) \rightarrow (K\varphi \rightarrow K\psi)$;

17. $\varphi \rightarrow K\varphi$;

18. $K\varphi \rightarrow \neg\neg\varphi$;

III An axiom relating K and $\langle ! \cdot \rangle$:

20. $\langle !\varphi \rangle K\psi \leftrightarrow (\varphi \wedge K\langle !\varphi \rangle \psi)^2$;

21. $[\!]\varphi K\psi \leftrightarrow (\neg\varphi \vee K[\!]\varphi \psi)$;

IV Modus ponens rule.

The K -necessitation rule is derivable, the deduction theorem and uniform substitution hold for **IEL**, as well as negative and positive introspection (for proof and other properties of **IEL** cf. [14]). The above axioms for K are not the only possible ones. Other acceptable formulations that yield equivalent systems (without *public announcement*) can be found in [14].

Kripke Semantics for IEPAL. We take an intuitionistic model with $\langle ! \cdot \rangle$ and add a new binary *knowledge* relation:

Definition 11. A model \mathfrak{M} for IEPAL is a quadruple $\mathcal{M} = \langle W, R, v, E \rangle$ such that

1. $\langle W, R, v \rangle$ is an intuitionistic model for $\text{IPAL}_{\langle \cdot \rangle}$ ‘defined in 2.1;
2. E is a binary knowledge relation on W coordinated with the relation R :
 - a) $E(u) \subseteq R(u)$ for any state u^3 , i.e., if uEv , then uRv ;
 - b) uRv yields $E(v) \subseteq E(u)$, i.e., if uRv and vEw , then uEw ;
 - c) seriality, i.e., for all u , there is a v such that uEv . (alternatively, denoted as $E(u) \neq \emptyset$ for each $u \in W$).
3. the forcing relation is extended to epistemic assertions:
 - $\mathcal{M}, x \Vdash K\varphi$ iff for all $y \in E(x)$: $\mathcal{M}, y \Vdash \varphi$.

Theorem 4 (Soundness). If $\vdash_{\text{IEPAL}} \varphi$, then $S4V \Vdash \varphi^4$ for all $\varphi \in \mathcal{L}_{\text{IEPAL}}$.

²The theorem $\langle !\varphi \rangle K\psi \leftrightarrow (\varphi \rightarrow K\langle !\varphi \rangle \psi)$ is derivable.

³ $R(u)$ and $E(u)$ denote for some state u its the R -successors and the E -successors, respectively.

⁴ $S4V$ is an extension of $S4$ with a verification modality.

Proof.

$$\mathbf{IEPAL} = \mathbf{IEL} + \text{axioms } 10 - 21.$$

Soundness for **IEL** has been proved in [14]. We have also proved soundness for **IPAL**_◇ 1 and **IPAL**_□ 3. Thus, we need to check only the soundness of the *axiom 20*: $\langle !\varphi \rangle K\psi \leftrightarrow (\varphi \wedge K\langle !\varphi \rangle \psi)$. The case for *axiom 21* is analogous.

→-direction

Assume that $\not\models \langle !\varphi \rangle K\psi \rightarrow (\varphi \wedge K\langle !\varphi \rangle \psi)$. Then there exists a model \mathcal{M} and a world x , such that: $\mathcal{M}, x \not\models \langle !\varphi \rangle K\psi \rightarrow (\varphi \wedge K\langle !\varphi \rangle \psi)$. Then there exists an $y \in W$, $y \in R(x)$: $\mathcal{M}, y \Vdash \langle !\varphi \rangle K\psi$ and $\mathcal{M}, y \not\models (\varphi \wedge K\langle !\varphi \rangle \psi)$. Then $\mathcal{M}, y \Vdash \varphi$ and $\mathcal{M}_\varphi, y \Vdash K\psi$. Then for all $z \in E(y)$, $z \in \mathcal{M}_\varphi$: $\mathcal{M}_\varphi, z \Vdash \psi$.

From $\mathcal{M}, y \not\models (\varphi \wedge K\langle !\varphi \rangle \psi)$ by def. follows either (1) $\mathcal{M}, y \not\models \varphi$ or (2) $\mathcal{M}, y \not\models K\langle !\varphi \rangle \psi$. Consider these two cases separately;

Case 1: $\mathcal{M}, y \not\models \varphi$, this contradicts $\mathcal{M}, y \Vdash \langle !\varphi \rangle K\psi$.

Case 2: $\mathcal{M}, y \not\models K\langle !\varphi \rangle \psi$. Then there exists $w \in E(y)$: $\mathcal{M}, w \not\models \langle !\varphi \rangle \psi$. Then either (1) $\mathcal{M}, w \not\models \varphi$ or (2) $\mathcal{M}_\varphi, w \not\models \psi$. Consider these two cases separately:

Case 2.1: $\mathcal{M}, w \not\models \varphi$. This contradicts $\mathcal{M}, y \Vdash \langle !\varphi \rangle K\psi$ (by monotonicity of forcing and condition 2a for E , $\mathcal{M}, w \Vdash \varphi$).

Case 2.2: $\mathcal{M}_\varphi, w \not\models \psi$. As $w \in E(y)$ and $w \in \mathcal{M}_\varphi$, then $w \in Z$ where Z is a set of all $z \in E(y)$, $z \in \mathcal{M}_\varphi$. Thus contradiction with $\mathcal{M}_\varphi, z \Vdash \psi$.

←-direction

Assume that $\not\models (\varphi \wedge K\langle !\varphi \rangle \psi) \rightarrow \langle !\varphi \rangle K\psi$. Then there exists a model \mathcal{M} and a world x , such that: $\mathcal{M}, x \not\models (\varphi \wedge K\langle !\varphi \rangle \psi) \rightarrow \langle !\varphi \rangle K\psi$. Then there exists an $y \in W$, $y \in R(x)$: $\mathcal{M}, y \Vdash (\varphi \wedge K\langle !\varphi \rangle \psi)$ and $\mathcal{M}, y \not\models \langle !\varphi \rangle K\psi$. Then $\mathcal{M}, y \Vdash \varphi$ and $\mathcal{M}, y \Vdash K\langle !\varphi \rangle \psi$. Then for all $z \in E(y)$: $\mathcal{M}, z \Vdash \langle !\varphi \rangle \psi$. Then $\mathcal{M}, z \Vdash \varphi$ and $\mathcal{M}_\varphi, z \Vdash \psi$.

From $\mathcal{M}, y \not\models \langle !\varphi \rangle K\psi$ by def. follows either (1) $\mathcal{M}, y \not\models \varphi$ or (2) $\mathcal{M}_\varphi, y \not\models K\psi$. Consider these two cases separately:

Case 1: $\mathcal{M}, y \not\models \varphi$. Contradiction with $\mathcal{M}, y \Vdash (\varphi \wedge K\langle !\varphi \rangle \psi)$;

Case 2: $\mathcal{M}_\varphi, y \not\models K\psi$. Then there exists $w \in E(y)$, $w \in \mathcal{M}_\varphi$: $\mathcal{M}_\varphi, w \not\models \psi$. As for all $\mathcal{M}, z \Vdash \varphi$, then $w \in Z$ where Z is a set of all $z \in E(y)$. Thus, contradiction with $\mathcal{M}_\varphi, z \Vdash \psi$.

□

Theorem 5 (Completeness). *If $S4V \Vdash \varphi$, then $\vdash_{\mathbf{IEPAL}} \varphi$ for all $\varphi \in \mathcal{L}_{\mathbf{IEPAL}}$.*

Proof. The proof is analogous to the one for **IPAL**_◇, theorem 2.

□

As a corollary, we get that $S4$, **IPAL**_◇ and **IPAL**_□ are equiexpressive; and $S4V$, **IEPAL** are equiexpressive as well.

4. Summary and Conclusion

We have investigated the possibility of a dynamic extension of the two related logics: *intuitionistic propositional logic* and *epistemic intuitionistic logic* **IEL**. The epistemic

logic we have extended has a special understanding of knowledge as a weaker notion than the notion of proof. We believe that this treatment of knowledge is more applicable to the epistemology of scientific research, scientism and philosophy of science than the standard classical $S5$ knowledge (justified true belief). In general, we believe that intuitionistic logic is a useful tool for formal epistemology and our study of information update in **IEL** will serve also as a useful tool for philosophical research. The study of public announcement in **IEL** can shed light on the nature of information update in intuitionism in general.

We have added two dynamic operators of *public announcement* that are dual in the classical case but not in the intuitionistic. The first one can also deal with the announcements that have already been proved w.r.t. the possible world in question deleting only the preceding worlds. Thus, it is the most trivial update option. The second one allows us to reason about announcements in worlds where the announced formula is not yet true. As a result, we introduced four new formal systems (with both Hilbert-style axiomatisation and Kripke semantics): $\mathbf{IPAL}_{\langle \rangle}$, \mathbf{IPAL}_{\square} , $\mathbf{IEPAL}_{\langle \rangle}$, and \mathbf{IEPAL}_{\square} . We have proved soundness and completeness for all four systems. We have seen that in case of intuitionistic epistemic logic even the simplest type of information update, i.e., public announcement, leads to more complex system (at least because of the lack of duality).

4.1. Future research: further extensions

The above considered systems were related to non-pointed models, i.e., models where the current state, or world, is not specified. However, in order to further develop intuitionistic epistemology (especially knowledge progress and belief change) with the help of the **IEL** system and its dynamic extensions, we should look into applying them to the pointed models. The task of transforming Kripke models for $\mathbf{IPAL}_{\langle \rangle}$ and $\mathbf{IEPAL}_{\langle \rangle}$ is a trivial. However, for \mathbf{IPAL}_{\square} and \mathbf{IEPAL}_{\square} rise questions on how to deal with the models where the current (pointed) state x : $\mathcal{M}, x \not\models \varphi$ and φ is the announced formula. This problem does not arise in the classical $S5$ **PAL** as it treats truth differently and thus such an announcement is impossible. Our conjecture is that in \mathbf{IPAL}_{\square} and \mathbf{IEPAL}_{\square} in case of a successful announcement $[\!|\varphi|\!] \psi$ with $\mathcal{M}_{\varphi} \models \psi$ in the new model we should substitute the pointed world with the one x' that differs from the pointed one at most by the values of φ and ψ . Thus, this is the topic of the future research.

Another direction of research would be to make the announcement less strict, meaning that the announcement would give *evidence* of the truth of a formula but not the proof itself. This should lead to much different systems as it will mostly relate to the *knowledge* modality.

The most straightforward to ask is to add multiple agents to the system, which does not seem to be problematic. A more intriguing question concerns the possibility of modelling *common knowledge* in the above system **IEL** and its public announcement extensions.

Finally, the ultimate goal of the future work is to examine the possibility of other types of information updates, rather than simple announcements. It is also of a particular interest to study systems of intuitionistic belief, like \mathbf{IEL}^{-} with respect to information change.

Acknowledgments

We thank the two anonymous reviewers whose comments helped improve this paper.

The research was supported by RSF (project No.20-18-00158 “Formal philosophy of argumentation and a comprehensive methodology for finding and selecting solutions to a dispute”).

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