Analysis of Colouring Properties of Proper (2,3)-poles

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Abstract

Snarks, that is 2-connected cubic graphs admitting no 3-edge-colouring, provide a promising family of cubic graphs with respect to many widely-open conjectures. They are often constructed by joining several building blocks which can be regarded as cubic "graphs" with dangling edges allowed, formally called multipoles. Colouring properties of multipoles with at most five dangling edges relevant for constructions of snarks are almost completely characterised. The remaining uncharacterised class of such multipoles are so-called proper (2,3)-poles that can be obtained by severing an edge and removing a vertex from a snark.

Therefore, in our work, we analyse the colouring properties of proper (2,3)-poles. To conduct our analysis, we explore using a computer all proper (2,3)-poles resulting from nontrivial snarks with at most 28 vertices. This encompasses a total of 3,247 snarks and 3,476,400 proper (2,3)-poles. In our research, we provide various structures that can be utilized to expand the colourability of proper (2,3)-poles. In the core of our work, we provide theorems regarding the colouring properties of proper (2,3)-poles, specifically necessary and sufficient conditions for these properties. Additionally, we present the data and observations from the analysis.

Keywords

snark, multipole, edge-colouring, Tait colouring, colouring set

1. Introduction

The study of *snarks*, that is 2-connected cubic graphs that are not 3-edge-colourable, is important since they are smallest possible counterexamples for several open problems. One such conjecture is the Cycle Double Cover Conjecture stating that each bridgeless graph has a family of cycles, such that each edge appears in exactly two of the cycles. To exclude trivial cases, some additional properties are required from snarks like the following.

Let G be a cubic graph and S an edge-cut of size n. If at least two components of $G-S$ contain a cycle, S is said to be an *n-edge-c-cut*. Generally, these edge-cuts are called *c-cuts*. A cubic graph G is *cyclically n-edge-connected* if there is no c-cut with less than n edges. *Cyclic edge-connectivity* of a cubic graph G having at least one c-cut is the smallest number of edges of a c-cut of G. Another measurement of nontriviality is the *girth* of a graph G which is the minimum length of a cycle in G. If G does not contain a cycle, we set the girth to ∞.

Many authors (e.g. [1, 2]) require that snarks have girth at least 5 and are cyclically 4-edge-connected. We call such snarks *non-trivial* and the remaining ones *trivial*. On the other hand, some authors allow snarks to contain bridges [3].

According to our definition, the Petersen graph is the smallest snark and it satisfies every standard definition of a snark. Other notable snarks are the Blanuša snarks [4], or the infinite family of flower snarks discovered by R. Isaacs [5]. The Isaacs snarks are denoted by J_k, where k is an odd integer $k \geq 3$.

Determining whether a given graph is a snark is an NP-complete problem [6]. However, when creating a cubic graph from some smaller building blocks, of which we know their colouring properties, we also know the colouring properties of the result. This may also include if it is a snark or not. These building blocks are called multipoles (e.g. see [2]) and can be described as an extended type of a graph that allows dangling edges. Multipoles are useful for various constructions of snarks [3, 7] or also for their structural analysis [8, 9].

This paper is structured as follows. First, we develop the theory needed for our work. In Section 2, we define multipoles and all notions related to them, and in Section 3, we develop the theory for describing colouring properties of multipoles. Section 4 is specifically devoted to multipoles with five dangling edges while it explains the relation of proper (2,3)-poles to them. In Section 5, we introduce several classes of proper (2,3)-poles with respect to their colouring properties. The rest of the paper focuses on our results. Section 6 describes methods of our computer-assisted analysis of proper (2,3)-poles constructed from small non-trivial snarks. In Section 7, we provide multipoles that can be used to change colouring classes of proper (2,3)-poles. Section 8 contains theoretical results on colouring properties of proper (2,3)-poles. At the end, in Section 9, we summarise the results of our computer-assisted analysis and in Section 10 we provide problems for further research.

Definitions not provided in our work can be found in [10]. We only clarify that all graphs considered in this

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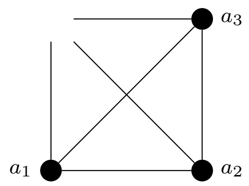

Figure 1: Example of a multipole with both types of edges.

work are undirected, and while we permit multiple edges, loops are not allowed. The distance between two vertices x and y in a graph G, denoted by $d_G(x, y)$, is defined as the length of a shortest path between x and y in G. If no such path exists, we set $d_G(x, y) = \infty$. The distance $d_G(x, ab)$ between a vertex x and an edge ab is defined as the smallest value between $d_G(x, a)$ and $d_G(x, b)$.

2. Multipoles

A *multipole* is a pair $M = (V, E)$ of distinct finite sets of vertices V and edges E, where every edge $e \in E$ has two edge ends, which may or need not be incident with a vertex. The concept of multipoles was introduced in [11].

A *link* is an edge incident with two distinct vertices and a *dangling edge* is an edge with only one end incident with a vertex. An illustration showcasing both types of edges can be seen in Figure 1: there is a link a_1a_2 and a dangling edge from a_1. We do not consider other types of edges. The edge ends not incident with any vertex are called *semiedges.* The semiedges in a multipole are endowed with a linear order and we denote the tuple of its semiedges as $S(M)$.

A multipole M with $S(M) = (a_1, \cdots, a_n)$ can also be denoted as $M(a_1, \cdots, a_n)$. In our work, we will consider *cubic* multipoles, i.e. multipoles where every vertex is incident with precisely three edges. A multipole with n semiedges is also called a *n-pole.*

Usually, it is convenient to partite $S(M)$ into pairwise disjoint tuples $S_1, \cdots, S_n$ called *connectors.* A multipole M with n connectors $S_1, \cdots, S_n$, where S_i has c_i semiedges for each i from 1 to n, is denoted by $M(S_1, \cdots, S_n)$ and is also called a $(c_1, \cdots, c_n)$-pole.

Now, we describe the process of joining two multipoles together. The *junction* of two distinct semiedges e and f corresponding to edges e' and f', respectively, is a new link joining the remaining two edge ends of e' and f' different from e and f. The junction of two connectors $S = (e_1, \cdots, e_n)$ and $T = (f_1, \cdots, f_n)$ consists of n individual junctions of semiedges e_i and f_i for i from 1 to n. Similarly, the junction of two $(c_1, \cdots, c_n)$-poles $M(S_1, \cdots, S_n)$ and $N(T_1, \cdots, T_n)$ consists of n individual junctions of connectors S_i and T_i, for i from 1 to n. The *partial junction* of M and N is a junction of some semiedges $(a_{i_1}, \cdots, a_{i_k})$ and $(b_{j_1}, \cdots, b_{j_k})$, where $k \leq n$ and $k \leq m$. In contrast to a normal junction of multipoles, which results in a graph, the partial junction can still result in a multipole.

Let G be a graph, ab its edge, and v its vertex. By *severing* the edge ab in G, we mean removing ab and adding a dangling edge to the vertices a and b. Similarly, *removing* the vertex v involves the removal of v along with all of its incident edges, followed by adding a dangling edge to all of the formerly neighbouring vertices of v. If we obtain a multipole by removing some vertices and severing some edges in a graph, there is a default way to divide the resulting semiedges into connectors. When we remove a vertex, all semiedges formerly incident with the vertex are in a new connector. Similarly, when we sever an edge, the two new semiedges are in a new connector.

To properly denote the multipoles resulting from a graph by removing some vertices and severing some edges, we will denote such multipoles as $R(G; V; E)$, where G is the former graph, V is the set of removed vertices, and E is the set of severed edges. For example, a multipole resulting from a snark G by removing vertex v and severing edge ab is denoted by $R(G; \{v\}; \{ab\})$ and consists of two connectors, one with two semiedges and one with three. In the case where a set contains only one element, we can represent it without brackets, resulting in this case in the notation $R(G; v; ab)$.

3. Multipole Colouring

When considering 3-edge-colourings it is convenient to regard the colours 1, 2, 3 as $(0, 1)$, $(1, 0)$, $(1, 1)$, respectively. In other words, we use the set of non-zero elements of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$, which we will denote as $\mathbb{K}$. Using the colours from $\mathbb{K}$, it is easy to see that each 3-edge-colouring of cubic graphs corresponds to a nowhere zero $(\mathbb{Z}_2 \times \mathbb{Z}_2)$-flow and vice versa. Since each non-zero element from this group is self-inverse, we need not assign an orientation to the edges.

These colourings are also called *Tait colourings.* These are widely used in many articles about snarks and 3-edge-colourability in general mainly because of their relation to nowhere zero flows. From now on, we only say *colouring* or *colourable* instead of 3-edge-colouring or 3-edge-colourable.

When discussing multipoles, the definition of edge-colouring is the same. The colour of an edge end is the colour of its respective edge. The *colouring set* of a k-pole $M(e_1, \cdots, e_k)$ is the set $\mathrm{Col}(M)$ defined as

$$\{(\phi(e_1), \cdots, \phi(e_k)) \mid \phi \text{ is a Tait colouring of } M\}.$$

For a connector $S = (e_1, \cdots, e_n)$ of M, the *flow through* S is the value $\phi_*(S) = \sum_{i=1}^{n} \phi(e_i)$. A connector S of a multipole M is called *proper* if $\phi_*(S) \neq 0$ for each Tait colouring ϕ of M. A multipole is called *proper* if each of its connectors is proper.

In general, for any tuple of semiedges $S = (e_1, \cdots, e_n)$ and colouring ϕ, we use the notation $\phi(S)$ to represent the tuple $(\phi(e_1), \cdots, \phi(e_n))$.

The fact that we can regard a colouring of a multipole as a flow has a valuable consequence that will be indispensable in our work. It is commonly known as the Parity Lemma, introduced and proved by B. Descartes in 1948.

Lemma 1 (Parity Lemma [12]). *Let M be a k-pole, and let k_1, k_2 and k_3 be the numbers of semiedges of colour $(0, 1)$, $(1, 0)$ and $(1, 1)$, respectively. Then $k_1 \equiv k_2 \equiv k_3 \mod 2$.*

By applying the Parity Lemma, we can conclude that any cubic graph with a bridge is not colourable. Another corollary of this lemma is that the minimum number of vertices that must be removed from a snark to obtain a colourable multipole is two [9]. The same applies to severing edges. The smallest number of edges to be severed in a snark to obtain a colourable multipole is two. If only one edge is severed, the resulting multipole contains two semiedges, both of which must have the same colour for it to be colourable because of the Parity Lemma. That would mean the former graph resulting from the junction of these two semiedges is not a snark since the colouring of the multipole could be extended to the colouring of the graph.

A key aspect when studying the colouring properties of multipoles derived from snarks is the removability of pairs of vertices or edges. Let G be a snark. A pair of its distinct vertices $\{u, v\}$ is called *removable* if $R(G; \{u, v\}; \emptyset)$ is not colourable; otherwise, it is called *unremovable*. Similarly, for edges, a pair of distinct edges $\{ab, cd\}$ is called *removable* if $R(G; \emptyset; \{ab, cd\})$ is colourable. Otherwise, it is called *unremovable*.

If we sever two adjacent edges, it is equivalent to the removal of a single vertex with regard to colourability. Therefore these pairs of edges are trivially removable because at least two vertices are needed to be removed from a snark to obtain a colourable multipole.

4. Colouring properties of 5-poles

To explore multipoles effectively, it is best to start with the simplest ones and gradually move towards more complex ones. That's why we begin by examining the k-poles starting from the smallest k. Specifically, the smallest k for which it is interesting to explore the colourability of k-poles arising from snarks with k-edge-cuts is 4. The 1-poles are trivially uncolourable. For a 2-pole to be colourable, both of its semiedges must have the same colour. Similarly, for a 3-pole, all three of its semiedges must have pairwise different colours. Also, the colouring properties of 4-poles are already widely explored since their colouring properties are limited as well [8]. The analysis of the colouring properties of 5-pole comes from the following ideas of P. J. Cameron, A. G. Chetwynd and J. J. Watkins [13].

By the Parity Lemma, in each colouring ϕ of a 5-pole M, three semiedges of M have the same colour. We call these edges *sociable* in ϕ. The remaining two semiedges of M, called *solitary* in ϕ, are coloured by the remaining two colours. Because the colouring set of M is closed under a permutation of colours, it only matters which semiedges of M may be solitary. Using this, we can visualize the colouring set of any 5-pole in the following way.

For a 5-pole $M(e_1, e_2, e_3, e_4, e_5)$ we denote by R_M a graph with the vertex set $V = \{e_1, e_2, e_3, e_4, e_5\}$, in which for each $e_i, e_j \in V$, the graph contains an edge e_ie_j if and only if the semiedges e_i and e_j are solitary in some colouring of M. The graph R_M is called the *colouring graph of M*. The colouring graph R_M of any 5-pole has no vertex of degree one [1]. We say that a 5-pole T *allows solitary cycle* $e_1e_2 \cdots e_n$ if its colouring graph R_T contains the cycle $e_1e_2 \cdots e_n$.

From a case analysis in [1], it follows that if M is a 5-pole that can be completed to a snark by performing a junction with some colourable 5-pole, then colouring graph R_M of M is a subgraph of:

- a 5-cycle, or
- the graph formed by two disjoint triangles sharing a single vertex, or
- the complement of C_3.

Accordingly, 5-poles whose colouring graphs are subgraphs of the mentioned graphs are called *superpentagons*, *negators* and *proper* $(2, 3)$*-poles*, respectively. Note that the introduced three classes of the 5-poles are not disjoint.

The colouring graph of a superpentagon is either empty or the 5-cycle. The colouring graph of a negator is empty, consists of one triangle or consists of two edge-disjoint triangles with a common vertex.

Máčajová and Škoviera characterised which negators have which colouring graphs [3]. For proper $(2, 3)$-poles, the situation is more diverse and their colouring graphs have not been studied yet.

5. Proper (2,3)-poles

As it follows from the definitions in Section 2, a proper $(2, 3)$-pole $T(A, B)$ is a multipole having two proper

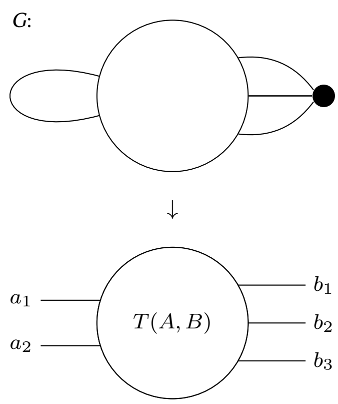

Figure 2: Creation of a proper $(2, 3)$-pole from a snark G.

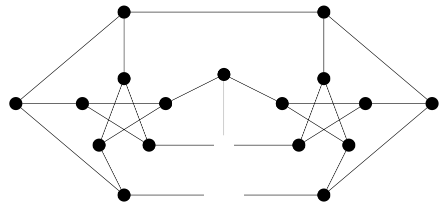

Figure 3: Example of an uncolourable proper (2,3)-pole.

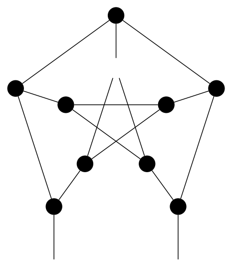

Figure 4: Example of a perfect proper (2,3)-pole.

connectors $A = (a_1, a_2)$ and $B = (b_1, b_2, b_3)$. Therefore, the colouring set of each proper (2,3)-pole is a subset of the set

$$\{(a_1, a_2, b_1, b_2, b_3) \in \mathbb{K}^5 \mid a_1 + a_2 = b_1 + b_2 + b_3 \neq 0\}.$$

We will refer to this set only as C from now on. Note that in this definition, the terms a_1 to b_3 denote the colours of the respective semiedges. The non-equality to zero is evident because they are proper, and the equality of flow through both connectors follows from the Parity lemma.

This property implies that each proper $(2, 3)$-pole T can be completed to a snark by the junction of the semiedges in the connector of size two and joining the semiedges from the connector of size three to a new vertex. Conversely, for each snark, the result after removing a vertex and severing an edge not incident with it will always be a proper (2,3)-pole [9].

A visualisation of this process can be seen in Figure 2.

Now, we describe all possible colouring sets of proper (2,3)-poles on their colouring graphs. Since there is no rule on which semiedges are labelled b_1, b_2, b_3 after removing a vertex from a snark, we will divide the colouring sets into classes, in which the colouring sets represent colourings up to a permutation of colours and a permutation of labels b_1, b_2, b_3.

However, not each graph on these vertices represents a possible colouring set. First, the colouring graph must have no pendant vertex, as mentioned above. The second restriction is that no edge is between two vertices from $\{b_1, b_2, b_3\}$. Suppose there was an edge between b_1 and b_2 in some proper (2,3)-pole T. This would mean that T allows a colouring such that b_1, b_2 and b_3 have pairwise different colours and a_1, a_2 have the same colour. Thus T could be extended to the former graph, which would be colourable and therefore not a snark, which contradicts the assumption.

This means there are only 12 different colouring sets of proper (2,3)-poles, which can be divided into classes in the following way. We denote the classes by a number representing how many vertices from $\{b_1, b_2, b_3\}$ are connected to $\{a_1, a_2\}$ with an edge in the colouring graph, followed by A if the colouring allows an edge between a_1 and a_2, or B otherwise. Resulting are six colouring classes: 0B (uncolourable), 1A, 2B, 2A, 3B and 3A. Proper (2,3)-poles from Class 3A, that is those whose colouring set coincides with C, are also called *perfect*. For each of them we have found an example, in this article we provide an example of an uncolourable proper (2,3)-pole in Figure 3 resulting from the second Blanuša snark and a perfect one in Figure 4 resulting from the Petersen graph.

The colouring graphs for each class can be observed in Figure 5.

6. Methods of Analysis

All of the results in this chapter come from our analysis conducted on several proper (2,3)-poles. For this reason, we have created a simple program in C++ that helps us get the desired results. The logic behind representing graphs in the program and some basic operations on them is done by the `ba_graph` library [14]. As input, our program receives a list of snarks in graph6 format [15], parses them, and performs the following operations on each.

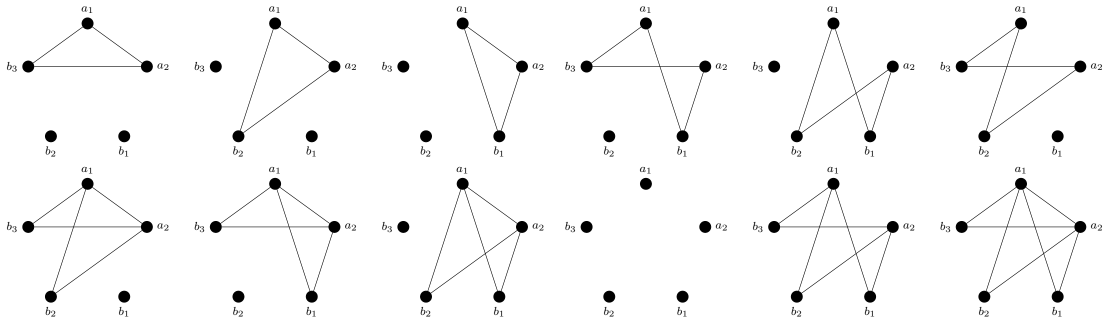

Figure 5: Colouring graphs for class 1A (first three), 2B (last three in the first row), 2A (first three in the second row), 0B, 3B and 3A.

Since the proper (2,3)-poles are multipoles resulting from a snark by removing one vertex and severing one edge, this is exactly what the program does: for each vertex v and edge e, where e is not incident with v, it removes v, severes e, and thus creates a proper (2,3)-pole. Thus, we have multiple proper (2,3)-poles from one snark.

Let T be the proper (2,3)-pole resulting from snark G, after removing the vertex v and severing the edge xy, $x \neq y \neq v \neq x$. We compute or observe the following properties for each proper (2,3)-pole:

- the resulting multipole in graph6 format;
- which edge and vertex were removed from the former snark;
- in which colouring class it is (see Section 5);
- the distance between the removed vertex and severed edge;
- how many pairs of vertices from $\{v, x\}$, $\{v, y\}$ are removable;
- how many pairs of edges $\{xy, va\}$, $\{xy, vb\}$, $\{xy, vc\}$ are removable, where a, b and c are neighbours of v.

For the colouring classes, we observe whether the multipole permits four colourings: those in which the solitary semiedges are a_1 and b_1, a_1 and b_2, a_1 and b_3, and a_1 and a_2, respectively. For instance, if it allows a colouring where the solitary semiedges are a_1 and b_1, it also allows a colouring with the solitary semiedges a_2 and b_1.

The sets of removable pairs of edges and vertices are computed for the original snark, and then for the resulting multipole it is simply checked whether those pairs are contained in the respective sets.

For each graph on input, these results are then saved in a separate file containing a row for each proper (2,3)-pole originating from it.

The source code of the program can be found at [16].

7. Obtaining perfect proper (2,3)-poles

It may be convenient to modify some proper (2,3)-poles by adding some vertices and edges to obtain perfect proper (2,3)-pole. If we know in which colouring class the proper (2,3)-pole is, then we can make a junction with the specific constructions provided in this chapter to obtain perfect colouring, of course, only if the former (2,3)-pole is colourable.

7.1. Extending class 1A to 2B

Let $T(A, B)$ be a proper (2,3)-pole, whose colouring class is 1A and let it allow a solitary cycle $b_i a_1 a_2$ for some $b_i \in B$. Now let $T'(A, B')$, $B' = (b'_1, b'_2, b'_3)$ be a proper (2,3)-pole obtained by the partial junction of T with the 6-pole M shown in Figure 6. In this partial junction we connect the connector B and the connector (c_1, c_2, c_3) such that it contains a junction of b_i and c_1. We prove that the result is a proper (2,3)-pole from colouring class 2B, which allows solitary cycle $b'_2 a_1 b'_3 a_2$.

Without loss of generality, let the semiedge b_i be b_1. Let this colouring of T be ϕ. Using a colouring ϕ_1 of M where $\phi_1(b'_1, b'_2, b'_3) = (3, 3, 2)$ and a colouring ϕ_2 of M where $\phi_2(b'_1, b'_2, b'_3) = (3, 2, 3)$, in both cases $\phi_1(c_1, c_2, c_3) = \phi_2(c_1, c_2, c_3) = (2, 1, 1)$. After the mentioned partial junction of T and M, we can colour the rest of T' by the colouring ϕ. We can see that the colouring graph of T' allows a solitary cycle $b'_2 a_1 b'_3 a_2$. No more colourings can be obtained since, as it can be seen, b'_2 and b'_3 must have different colours, so one of them is always solitary and we cannot obtain classes $2A$, $3B$, and perfect.

It is the smallest such 6-pole, considering the number of vertices, which extends class 1A to 2B.

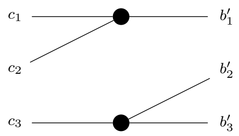

Figure 6: A 6-pole used to create colouring class 2B from 1A.

7.2. Extending class 2B to 2A

Let $T(A, B)$ be a proper (2,3)-pole whose colouring class is 2B. Let $b_i, b_j, i \neq j$ be the two semiedges from B, for which there exists a solitary cycle $a_1 b_i a_2 b_j$. Now let $T'(A, B')$, $B' = (b'_1, b'_2, b'_3)$ be a proper (2,3)-pole obtained by the partial junction of T with the 6-pole on Figure 7 by the junction of semiedges b_i to c_1, b_j to c_2 and the last semiedge to c_3. The result T' is from a colouring class $2A$ and allows solitary cycle $a_1 b'_1 a_2 b'_2 a_1$. The proof is similar to the one in extending class 1A to 2B.

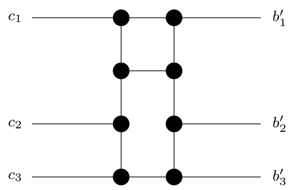

Figure 7: A 6-pole used to extend multiple colouring classes.

It must be noted that this construction produces proper (2,3)-poles which may not be contained in a nontrivial snark, since it contains a quadrilateral. If we would need to extend some proper (2,3)-pole to obtain a specific colouring class and require the extended proper (2,3)-pole to be contained in a nontrivial snark, we would need to use other, more complex constructions.

7.3. Extending proper (2,3)-poles from class 2A to perfect

To extend a proper (2,3)-pole from the colouring class 2A to a perfect one, the same 6-pole can be used as before, just with a different junction. Let $T(A, B)$ be a proper (2,3)-pole whose colouring class is 2B. Let $b_i, b_j, i \neq j$ be the two semiedges from B, for which there exists a solitary cycle $a_1 b_i a_2 a_1 b_j a_2$. Now let $T'(A, B')$, $B' = \{b'_1, b'_2, b'_3\}$ be a proper (2,3)-pole obtained by the junction of T with the 6-pole in Figure 7 by the junction of semiedges b_i to c_2, b_j to c_3 and the last semiedge to c_1. The result T' is a perfect proper (2,3)-pole, which can be proved similarly to before.

7.4. Extending proper (2,3)-poles from class 3B to perfect

Let $T(A, B)$ be a proper (2,3)-pole whose colouring class is 3B. Let $T'(A, B')$, $B' = (b'_1, b'_2, b'_3)$ be a proper (2,3)-pole obtained by the partial junction of T with the 6-pole on Figure 7, performing the junction of B to (c_1, c_2, c_3). Then T' is perfect, which can be proved similarly to before.

It is possible to incrementally modify each colourable proper (2,3)-pole to obtain a perfect one. For example, from class 1A it is possible to get class 2B, then 2A and finally perfect. It is evident that extending uncolourable multipoles to obtain colourable is impossible.

8. Theorems

Definition 1. Let M and N be multipoles. We say that M is a submultipole of N, denoted by $M \subseteq N$ if a multipole J exists such that N is a partial junction of M and J.

In other words, M is a submultipole of N if it can be extended to it by adding vertices, edges and semiedges and connecting them. The following lemma applies to the colourings of submultipoles; thus is essential when proving some propositions in this chapter.

Lemma 2. *Let M and N be multipoles such that $M \subseteq N$. If N is colourable, then M is colourable as well.*

Proof. Since $M \subseteq N$, so N is a result of the junction of M and some multipole J, there is an edge cut X splitting N into M and J. Let ϕ be the colouring of N. After removing the edge cut X, the exact colouring can be applied to colour M. □

This also means that if M is uncolourable, N is uncolourable as well.

Let M and N be multipoles, both constructed from a snark G. Since we often consider the intersection $E(M) \cap E(N)$, we clarify that:

- A link ab of G is included in the intersection if and only if it is present in both multipoles.
- A dangling edge originating from a vertex a which originated from an edge ab of G is included in the intersection if and only if it is present in both multipoles.

Since when creating a proper (2,3)-pole from a snark, we are removing a vertex and severing an edge, we cannot look at the removable vertices per se since only one vertex is removed. However, we may look at the end vertices of the severed edge.

Proposition 1. *Let G be a snark, v its vertex, ab its edge where $a \neq v$ and $b \neq v$ and $T(A, B)$ a proper (2,3)-pole $R(G; v; ab)$. If at least one of the pairs $\{v, a\}$ and $\{v, b\}$ is removable, then $T(A, B)$ is uncolourable.*

Proof. Let the removable pair be $\{v, a\}$, meaning that $R(G; \{v, a\}; \emptyset)$ is uncolourable. We see that $R(G; \{v, a\}; \emptyset) \subseteq T(A, B)$, so because of Lemma 2 the proper (2,3)-pole $T(A, B)$ is uncolourable. □

It must be noted, though, that the converse implication does not hold. There are several uncolourable proper (2,3)-poles, resulting from a snark, in which both of the pairs of vertices are unremovable. One of them is the mentioned example in Figure 3.

Another interesting attribute in the question of colourability is the edge removability. Since the removed vertex in the creation of a proper $(2, 3)$-pole has three neighbours, we can look at the removability of all three in pairs, along with the severed edge in the creation. One interesting proposition is also connected to the uncolourable proper (2,3)-poles.

Proposition 2. *Let G be a snark, v its vertex, ab its edge where $a \neq v$ and $b \neq v$ and $T(A, B)$ a proper (2,3)-pole $R(G; v; ab)$. Let x, y, z be the neighbouring vertices of v in G. $T(A, B)$ is uncolourable if and only if all three pairs $\{ab, vx\}, \{ab, vy\}, \{ab, vz\}$ are removable.*

Proof. Suppose on the contrary that $T(A, B)$ is uncolourable and at least one pair from $\{ab, vx\}$, $\{ab, vy\}$, $\{ab, vz\}$ is unremovable, say $\{ab, vx\}$. This means that $R(G; \emptyset; \{ab, vx\})$ is colourable. However, $T(A, B)$ is a submultipole of $R(G; \emptyset; \{ab, vx\})$ and since $T(A, B)$ is uncolourable, $R(G; \emptyset; \{ab, vx\})$ must also be uncolourable because of Lemma 2, leading to a contradiction. Therefore, if $T(A, B)$ is uncolourable, all three pairs $\{ab, vx\}, \{ab, vy\}, \{ab, vz\}$ are removable.

Now for the proof of the second implication, suppose that all three edge pairs are removable and $T(A, B)$ is colourable. Let the semiedge b_1 be from the dangling edge from x, b_2 from y and b_3 from z. By the definition of colouring classes, it is evident that $T(A, B)$ must allow a colouring, among others, where the solitary semiedges are a_1 and some semiedge b_i, say b_1. It is now possible to use this colouring, say ϕ, to colour $R(G; \emptyset; \{ab, vy\})$. Let us denote $R(G; \emptyset; \{ab, vy\})$ by R. We define a colouring ϕ' of R as follows: For each edge $e \in E(R) \cap E(T(A, B))$, the colour $\phi'(e)$ is equal to $\phi(e)$. The only edges not in this intersection are vx, vz and the dangling edge from v, let us denote it by d. We can set $\phi'(vx) = \phi(b_1)$, $\phi'(vz) = \phi(b_3)$. These two colours are different, since in ϕ, the semiedge b_1 is solitary and b_3 sociable. Thus we can colour the last edge, d, with the colour different from $\phi'(vx)$ and $\phi'(vz)$. Since $\{ab, vy\}$ is removable, R is uncolourable, leading to a contradiction. □

Before the following proposition, we shall prove that for these pairs of edges, it cannot happen that exactly two are removable.

Lemma 3. *Let G be a snark, v its vertex, ab its edge where $a \neq v$ and $b \neq v$. Let x, y, z be the neighbouring vertices of v in G. It is not possible that exactly two of the pairs $\{ab, vx\}, \{ab, vy\}, \{ab, vz\}$ are removable.*

Proof. We prove that if one of the pairs is unremovable, then at least one of the remaining pairs is unremovable as well. Let $\{ab, vx\}$ be unremovable, meaning that $R = R(G; \emptyset; \{ab, vx\})$ is colourable, let the colouring be ϕ. Let a_1, a_2 be the semiedges resulting from severing the edge ab and $c_1,\ c_2$ from severing vx, such that c_1 is part of the edge from x and c_2 of the edge from v. Because of the Parity Lemma and the fact that G is a snark, a_1 must have a different colour than a_2, and $c_1,\ c_2$ must be coloured with the same colours as them, also different from each other. The colours in ϕ of edges incident with v are all different, meaning that one of the edges, say vy, is coloured by the same colour as c_1. It cannot be the dangling edge, since $\phi(c_1) \neq \phi(c_2)$.

Now we can colour $R' = R(G; \emptyset; \{ab, vy\})$ with a colouring ϕ'. For each edge $e \in E(R) \cap E(R')$, $\phi'(e) = \phi(e)$. The only edges from R' not in this intersection are the edge vx, the dangling edge from y and the dangling edge from v. Let us denote the dangling edges by $d,\ e$, respectively. We will colour them the following way: $\phi'(vx) = \phi(c_1),\ \phi'(d) = \phi(vy),\ \phi'(e) = \phi(c_2)$. Since $\phi(c_1) \neq \phi(c_2)$, all three colours of edges incident with v in ϕ' will indeed be different. This means, that the pair $\{ab, vy\}$ is also unremovable.

The statement that exactly two of the pairs $\{ab, vx\}, \{ab, vy\}, \{ab, vz\}$ are removable is equivalent to the statement that exactly one of them is unremovable. However, we have proved that this is impossible, since the presence of an unremovable pair implies the existence of another unremovable pair. □

Proposition 3. *Let G be a snark, v its vertex, ab its edge where $a \neq v$ and $b \neq v$ and $T(A, B)$ a proper $(2, 3)$-pole $R(G; v; ab)$. Let x, y, z be the neighbouring vertices of v in G. The proper (2,3)-pole $T(A, B)$ is from the class 1A if and only if exactly one of the pairs $\{ab, vx\}, \{ab, vy\}, \{ab, vz\}$ is removable.*

Proof. Suppose that $T(A, B)$ is from the class $1A$ and not exactly one of the pairs $\{ab, vx\}, \{ab, vy\}, \{ab, vz\}$ is removable. If all three pairs were removable, then by Proposition 2, $T(A, B)$ would be uncolourable. Also, there cannot be exactly two removable, as we have proved in Lemma 3. That means we can only explore the cases where none of the pairs is removable. Let the semiedge b_1 be from the dangling edge from x, b_2 from y and b_3 from z.

Let $T(A, B)$ allow a solitary cycle $a_1 b_1 a_2$, implying it allows a colouring where the solitary semiedges are a_1 and b_1. Therefore $T(A, B)$ does not allow a colouring where one of the solitary semiedges is b_2 or b_3.

Let $R = R(G; \emptyset; \{ab, vx\})$ and let us denote the semiedges resulting from severing the edge vx by c_1, c_2, such that c_1 is part of the dangling edge from x and c_2 of the dangling edge from v. Since each pair is unremovable, a colouring ϕ of R exists. We see that $\phi(vy) \neq \phi(vz)$ and $T(A, B)$ is a submultipole of R. We can now construct a colouring ϕ' of $T(A, B)$ the following way. For each edge $e \in E(T(A, B)) \cap E(R)$, $\phi'(e) = \phi(e)$. The only edges from $T(A, B)$ not in this intersection are the dangling edges containing b_2 and b_3. We will colour them with $\phi'(b_2) = \phi(vy)$ and $\phi'(b_3) = \phi(vz)$. However, since $\phi(vy) \neq \phi(vz)$, the colour of b_2 is different from the colour of b_3, thus one of them is solitary in this colouring along with a_1 or a_2. This leads to a contradiction, since we suppose that $T(A, B)$ does not allow a colouring where one of the solitary semiedges is b_2 or b_3.

Now we can prove the second implication. Assume that exactly one of the pairs $\{ab, vx\}$, $\{ab, vy\}$, $\{ab, vz\}$ is removable, let it be $\{ab, vx\}$, meaning that the pairs $\{ab, vy\}$ and $\{ab, vz\}$ are unremovable. As in the proof of the previous implication, let the semiedge b_1 be from the dangling edge from x, b_2 from y and b_3 from z. Based on Proposition 2, $T(A, B)$ is colourable, so we can explore which semiedges from b_1, b_2, b_3 can be in its solitary cycle.

Suppose b_2 is in the solitary cycle of $T(A, B)$, implying the existence of a colouring ϕ_2 in which the solitary semiedges are a_1 and b_2. This would mean that $\phi_2(b_2) \neq \phi_2(b_1)$, $\phi_2(b_2) \neq \phi_2(b_3)$, $\phi_2(b_1) = \phi_2(b_3)$. From this colouring we can now construct a colouring ϕ_x of $R_x = R(G; \emptyset; \{ab, vx\})$: for each edge $e \in E(R_x) \cap E(T(A, B))$ the colour will be $\phi_x(e) = \phi_2(e)$. The only edges from R_x not included in the intersection are vy, vz and the dangling edge from v, let us denote it as d. We can then set $\phi_x(vy) = \phi_2(b_2)$, $\phi_x(vz) = \phi_2(b_3)$ and $\phi_x(d)$ as the remaining colour, different from the two colours already set for vy and vz. Since $\{ab, vx\}$ is removable, the assumption that b_2 is in the solitary cycle leads to a contradiction.

Now suppose b_3 is in the solitary cycle of $T(A, B)$, implying the existence of a colouring ϕ_3 in which the solitary semiedges are a_1 and b_3. This would mean that $\phi_3(b_3) \neq \phi_3(b_1)$, $\phi_3(b_3) \neq \phi_3(b_2)$, $\phi_3(b_1) = \phi_3(b_2)$. From this colouring we can now also construct a colouring ϕ_x of R_x as before: for each edge $e \in E(R_x) \cap E(T(A, B))$ the colour will be $\phi_x(e) = \phi_3(e)$. The only edges from R_x not included in the intersection are vy, vz and the dangling edge from v, let us denote it as d. We can then set $\phi_x(vy) = \phi_3(b_2), \phi_x(vz) = \phi_3(b_3)$ and $\phi_x(d)$ as the other colour from the two colours already set for vy and vz. Since $\{ab, vx\}$ is removable, the assumption that b_3 is in the solitary cycle also leads to a contradiction.

Because $T(A, B)$ is colourable and as we have shown, b_2 and b_3 cannot be in its solitary cycle, it must contain b_1, implying the existence of colouring where the solitary pairs are a_1, a_2; a_1, b_1; a_2, b_2; which coincides with the colouring class 1A. □

Based on this we can provide an interesting corollary for the other classes, implied by Proposition 2, Lemma 3 and Proposition 3.

Corollary 1. *Let G be a snark, v its vertex, ab its edge where $a \neq v$ and $b \neq v$ and $T(A, B)$ a proper $(2, 3)$-pole $R(G; v; ab)$. Let x, y, z be the neighbouring vertices of v in G. $T(A, B)$ is perfect or from the class 2A, 2B or 3B, if and only if all three of the pairs $\{ab, vx\}$, $\{ab, vy\}$, $\{ab, vz\}$ are unremovable.*

Proposition 4. *Let G be a snark, v its vertex, ab its edge where $a \neq v$, $b \neq v$ and the distance between ab and v is 1, that means a or b is a neighbour of v. Let $T(A, B)$ be a proper $(2, 3)$-pole $R(G; v; ab)$. Then $T(A, B)$ is either uncolourable or its colouring set is from the class 1A.*

Proof. Since the distance between ab and v is 1, at least one of the vertices a, b is the neighbour of v; let it be a. Now there are two dangling edges from the vertex a; let their semiedges be a_1 and b_1. Because of this, in each colouring of $T(A, B)$, the colours of a_1 and b_1 are different. This means that $T(A, B)$ does not allow colourings where the solitary semiedges are a_2 with b_2, or a_2 with b_3. It is evident that the only possible colouring classes are 1A and uncolourable. □

9. Data and observations

To clarify how we got the propositions or how the data looks, we provide statistics about the explored snarks and their resulting proper (2,3)-poles. In Table 1 is an example of the output table for the Petersen graph.

As an input, we have used all non-trivial snarks with at most 28 vertices, which is 3,247 snarks. There are precisely 3,476,400 proper (2,3)-poles resulting from them. Most of these results are perfect proper (2,3)-poles. The proportions are in Table 2.

By analyzing proper (2,3)-poles, we found that 8.59% of them are uncolourable, while 91.41% are colourable. Based on Corollary 1 and its converse implication, we examined the distribution of colouring classes when all of the mentioned pairs are unremovable. This analysis led to the observations presented in Table 3. We see that most

graph6	edge	vertex	colourings_class	distance	removable_vertices	removable_edges
MAMBHB@_?OA?@??O?	1, 5	0	perfect	2	0	0
MAkBHB@_?GA?@??O?	1, 6	0	1A	1	0	1
...						

Table 1
Example of the output table.

class	percentage	total number
perfect	66.13%	2,299,022
1A	20.73%	720,660
uncolourable	8.59%	298,720
2B	3.2%	111,139
3B	0.68%	23,630
2A	0.67%	23,229

Table 2
Proportion of colouring classes in explored proper (2,3)-poles.

of the proper (2,3)-poles are perfect, but the numbers are also the same as in Table 2. The equivalence between all proper (2,3)-poles from the classes perfect, 2B, 3B and 2A; and having all three pairs of edges unremovable is proved in Corollary 1.

class	percentage	total number
perfect	93.57%	2,299,022
2B	4.52%	111,139
3B	0.96%	23,630
2A	0.95%	23,229

Table 3
Proportion of colouring classes for all three unremovable pairs of edges.

Another interesting observation is that no proper (2,3)-pole from the explored ones has precisely two of the mentioned pair of edges removable. We have proved this in Lemma 3. The proportions can be seen in Table 4

removable edges	percentage	total number
0	70.68%	2,457,020
1	20.73%	720,660
3	8.59%	298,720

Table 4
Proportion of number of removable edges.

Among the 3,247 explored snarks, only five produce only colourable proper (2,3)-poles. One is the Petersen graph, then one with 20 vertices, two with 22 and one with 28 vertices. The ones with 20 and 28 vertices are the Isaacs snarks J_5 and J_7 respectively. The two snarks with 22 vertices are the Loupekine snarks. Because of Proposition 1, each of the five mentioned snarks contains no pair of removable vertices. Such snarks are called *bicritical*.

Since removable pair of vertices affect colouring properties, we have explored all bicritical snarks with at most 32 vertices – precisely 278 of them. There are 306,396 proper (2,3)-poles resulting from them. The proportions of their colouring classes is in Table 5.

class	percentage	total number
perfect	79.24%	242,784
1A	19.85%	60,830
uncolourable	0.59%	1,802
2A	0.32%	968
3B	$< 0.01\%$	10
2B	$< 0.01\%$	2

Table 5
Proportion of colouring classes in explored proper (2,3)-poles from bicritical snarks.

As before, we can look at the proportions of the colouring classes, but only for the proper (2,3)-poles with all three of the mentioned edge pairs unremovable. The results are in Table 6. We see, that almost every such proper (2,3)-pole is perfect, however there is a small number of ones from the classes 2A, 3B, 2B. This may be an interesting observation for the further research about the sufficient conditions for a proper (2,3)-pole resulting from a bicritical snark to be perfect. However, a proper (2,3)-pole constructed from a bicritical snark is not always perfect.

class	percentage	total number
perfect	99.6%	242,784
2A	0.4%	968
3B	$< 0.01\%$	10
2B	$< 0.01\%$	2

Table 6
Proportion of colouring classes for all three unremovable pairs of edges in proper (2,3)-poles from bicritical snarks.

10. Problems

During the writing of our work, several problems arose, which may be interesting for further research.

Problem 1. Suppose we only consider multipoles resulting from snarks by severing an edge and removing a vertex with a distance of more than 1, since adjacent edges are trivially removable. Is a proper (2,3)-pole constructed from a snark without any removable pair of edges always perfect?

There are only four such snarks from the ones we have explored: the Petersen graph, the Isaacs snarks J_5 and J_7, and the Double Star snark. All of the proper (2,3)-poles resulting from these graphs, with the distance between the removed vertex and the severed edge more than 1 are indeed perfect. However we have not proved this statement, thus it can be explored in further research.

Problem 2. Construct multipoles used to extend colourings of proper (2,3)-poles, which allow the resulting proper (2,3)-poles to be contained in a nontrivial snark.

As mentioned in Section 7, one of the 6-poles contains a quadrilateral, so each snark of which it is a part of is trivial.

Problem 3. Construct an infinite family of snarks, that produce only colourable proper (2,3)-poles.

We have found several snarks producing only colourable proper (2,3)-poles: the Petersen graph, the Isaacs snarks J_5 and J_7 and the two Loupekine snarks of order 22. This may be helpful when exploring infinite families of snarks producing only colourable proper (2,3)-poles.

Problem 4. If we construct a proper (2,3)-pole from a bicritical snark in such a way, that the distance between the severed edge and the removed vertex is more than one, and both are a part of a 5-cycle (not necessarily the same), is the result always perfect?

If a counterexample is found, an additional requirement of being cyclically 5-edge-connected could be imposed for the snark.

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