Construction of (L-R)-Type Numbers Based on Fuzzy Linguistic Assessments

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Abstract

An approach to constructing fuzzy numbers and fuzzy intervals based on fuzzy linguistic statements has been proposed. Mechanisms are proposed that allow for the direct construction of fuzzy numbers of the L-R type from fuzzy linguistic statements. This allows setting model parameters, fuzzy time series, forming databases in fuzzy inference systems, as well as information queries with fuzzy linguistic statements under conditions of significant uncertainty in the information environment. An example of the application of such an approach in modeling random variables of predictive parameters is given.

Keywords

fuzzy numbers and intervals, membership function, fuzzy linguistic statements, soft computing fuzzy modelling.

1. Introduction

Lately, fuzzy modelling has become one of the most active and promising areas of applied research [1-6]. Fuzzy numbers are most often used to represent fuzzy sets in fuzzy modelling. They form the basis for constructing mathematical models using linguistic variables and allow for the assignment of fuzzy magnitudes to the model and the execution of arithmetic operations.

Arithmetic operations for fuzzy numbers and intervals can be defined using the Zadeh's extension principle. However, carrying out such operations is quite labour-intensive. Therefore, in practice, operations over (L-R)-type fuzzy numbers and intervals, which reduce the volume of computations, have gained widespread acceptance. In addition, (L-R) numbers can be used to define intervals of parameter values, the precise boundaries of which are difficult to specify in conditions of uncertainty.

The issues of (L-R)-approximation have been examined by many researchers [7-11]. For example, in work [7], the use of LR-type numbers for the approximation of fuzzy numbers is considered. The proposed approximations can be generalized to most approximations in the Euclidean class. In [8], trapezoidal approximations of fuzzy numbers using quadratic programs are examined. In [9], the use of the convolution method for constructing approximations, containing fuzzy numerical sequences with useful properties for a general fuzzy number, is considered. It's shown that this method can generate differentiable approximations with finite steps for fuzzy numbers with finite non-differentiable points. The necessary and sufficient conditions of linear operators, which are preserved by interval, triangular, symmetric triangular, trapezoidal, or symmetric trapezoidal approximations of fuzzy numbers, are considered in [10]. In [11], conditional weighted LR-approximations of fuzzy numbers are examined.

Despite a wide range of research on various aspects of (L-R)-approximation, there remains a need for the development of powerful, yet simple, ways of approximating the (L-R)-numbers themselves. This situation defines the goal and main content of this article.

2. Fuzzy Linguistic Assessments

When it is necessary to evaluate a certain parameter of the model by a fuzzy magnitude or to construct a fuzzy time series, fuzzy numbers and intervals could be directly given by fuzzy linguistic

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assessments. A fuzzy linguistic assessment is understood as a numerical evaluation that is expressed by statements with the quantifiers "approximately/about" [12]:

"The value is approximately equal to c"

or (1)

"The value is approximately in the range from c to d"

These estimates can approximate (L-R)-type fuzzy numbers. Triangular numbers (c,α,β) are approximated by estimates of the first type, and trapezoidal numbers (c,d,α,β) are approximated by estimates of the second type. Since the values of c and d in these numbers are given by linguistic estimates, therefore, the task of constructing such numbers is only to determine the fuzziness coefficients α and β.

3. Calculation of fuzziness coefficients

The fuzziness coefficients determine the boundaries of the support of the fuzzy set A, that is, points at which, typically, $\mu_{\tilde{A}}(x) = 0.01$. There are several ways to determine these coefficients. The simplest, and at the same time, the most unreliable is direct expert evaluation. Since in this case the task of psychological measurements is complicated by the fact that a person, as a rule, has an uncertainty about the correctness of the estimates of these values. This difficulty is to some extent eliminated by a group expert evaluation. However, this approach has its known challenges.

Considering this, it is proposed to calculate the fuzziness coefficients based on membership functions. Fuzzy (L-R)-numbers are represented by triangular (trapezoidal) membership functions or Gaussian functions, which are most widely used in solving practical problems. Consider first triangular and trapezoidal functions (Fig. 1)

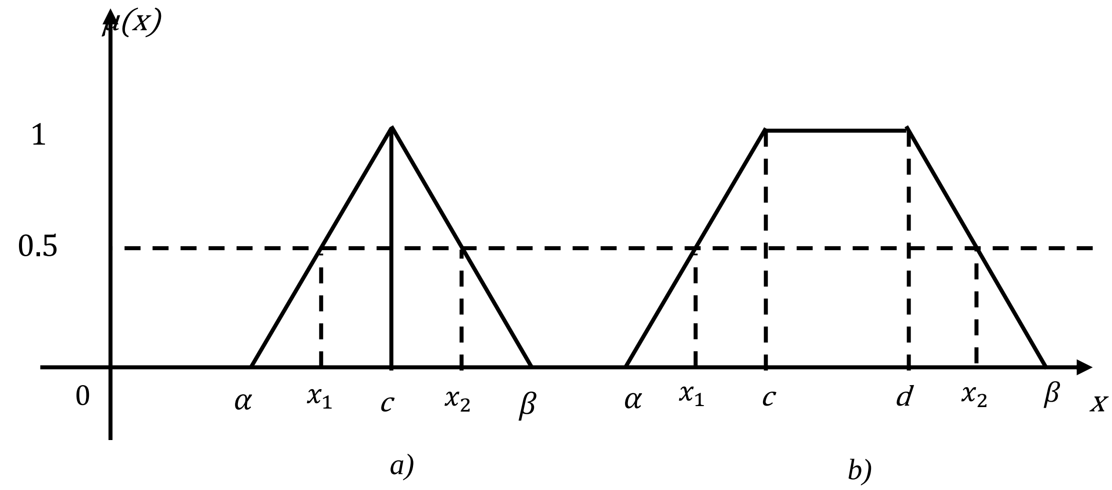

Figure 1: Membership functions: a) - triangular; b) – trapezoidal

Let's consider the triangular function. Take points $(x_1, 0.5)$ and $(c, 1)$. Draw a line $\mu = f_L(x)$ through these points. Then the value of α is calculated by the formula $\alpha = f_L^{-1}(0.01)$. Similarly, the coefficient β is calculated: $\beta = f_R^{-1}(0.01)$, where the line $\mu = f_R(x)$ passes through points $(c, 1)$ and $(x_2, 0.5)$.

Now it is necessary to determine the transition points x_1 and x_2. These points can also be determined by experts. However, this way leads to known problems. Given this, a more constructive approach is one that allows one to calculate transition points based on the distance between them. Let d is the distance between points x_1 and x_2. Then $x_1 = c - \frac{d}{2}$ and $\mathrm{x}_2 = \mathrm{c} + \frac{\mathrm{d}}{2}$.

To determine such a distance, an algorithm discussed in [13] is used. This algorithm is based on experimental data, which according to experts, reflect the transition points for numbers approximately equal to T. The results obtained are shown in Table 1 [13].

Let a fuzzy set be defined as "the number is around T ", where T is a natural number. If $T \in [1,99]$, then $b(T)$ is determined according to Table 1. Otherwise, let its least significant digit have the order of q. Possible values of q are divided into residue classes modulo 3. As a result, three classes $M_d, d \in$

$\{0,1,2\}$ are obtained, where $d = q\ mod(3)$. In this case, the value $b(T)$ also depends on the class M_d to which the number T belongs.

Table 1
The distance between the transition points

Number x	The distance between the transition points $b(x)$
1, 2, 3, 4, 6, 7, 8, 9	0,46 x
10, 20, 30, 40, 60, 70, 80, 90	$(0{,}357 - 0{,}00163x)x$
35, 45, 55, 65, 75, 85, 95	$(0{,}213\ -\ 0{,}00067x)x$
5	2,8
15	6,48
25	6,75
50	24
Other two-digit numbers	$\frac{1}{2}\left(b\left(\left[\frac{x}{10}\right]\cdot 10+5\right)+b\left(x-\left[\frac{x}{10}\right]\cdot 10\right)\right)$

Let r_q be the digit in the q place of the number T. Then:

1. if $T \in M_0$, then $b(T) = b(x)\cdot 10^{q-2}$, where $x = r_q \cdot 10$, and $b(x)$ is taken from Table 1.
2. if $T \in M_1$, then there are two options:
   a) if $r_{q+1} = 0$, then $b(T) = b(x)\cdot 10^{q-1}$, where $x = r_q$;
   b) if $r_{q+1} \neq 0$, then $b(T) = b(x)\cdot 10^{q-1}$, where $x = r_{q+1}\cdot 10 + r_q$.
3. if $T \in M_2$, then there are also two options:
   a) if $r_{q+1} = 0$, then $x = r_q \cdot 10$; $b(T) = b(x)\cdot 10^{q-2}$;;
   b) if $r_{q+1} \neq 0$, then $x = r_{q+1}\cdot 10 + r_q$; $b(T) = b(x)\cdot 10^{q-1}$..

As a result, the value $b(T)$ of will be obtained. This algorithm can also be used in cases where the number is expressed as a decimal fraction. In this case, the algorithm is applied to the mantissa of the fraction, and then its order is considered.

For trapezoidal functions, the distances $b(c)$, $b(d)$ are calculated, and the points x_1 and x_2 (Fig. 1b) are determined: $x_1 = c - \frac{b(c)}{2}$, $x_2 = d + \frac{b(d)}{2}$. Then, the equations of the lines $\mu = f_1(x)$ and $\mu = f_2(x)$ are constructed, passing respectively through the points $(x_1, 0.5)$,$(c, 1)$, $(d, 1)$, $(x_2, 0.5)$ and similarly, the coefficients α,β are calculated.

Let's now show how fuzzy numbers can be constructed using Gaussian functions (Fig.2).
The standard Gaussian function is used to define fuzzy sets $\tilde{A} \triangleq$ "the number is approximately equal to c". The form of the Gaussian function used is as follows [13]:

$$\mu_{\tilde{A}}(x) = exp(-a(x-c)^2), \tag{2}$$

where $a = -\frac{4\ ln\ 0.5}{b^2(c)}$, and $b(c)$ is the distance between the transition points. Gaussian function has an unbounded support, as it asymptotically tends to zero on both the left and right. However, in practice, the support of this function can be considered to be limited by points, at which its value is approximately equal to 0.01, which corresponds to the complete non-membership of an element in the fuzzy set $\tilde{A}$. Therefore, the coefficients α and β are found from the equation $\mu_{\tilde{A}}(x) = 0.01$. These coefficients can also be calculated approximately as follows: $\alpha = c - \frac{k\cdot b(c)}{2}$ and $\beta = c + \frac{k\cdot b(c)}{2}$, where $k \approx 2.5$ is a scaling factor [14]. The combined function describes the fuzzy set $\tilde{A} \triangleq$ "the number is approximately in the interval from ***c*** to ***d***". This function takes the form:

$$\mu_{\tilde{A}}(x) = \begin{cases} x < c, & \mu_{\tilde{B}}(x) \\ c \le x \le d, & 1 \\ x > d, & \mu_{\tilde{C}}(x) \end{cases}, \tag{3}$$

where $\mu_{\tilde{B}}(x)$ is the membership function of the fuzzy set $\tilde{B} \triangleq$ "the number is around ***c***", and $\mu_{\tilde{C}}(x)$ is the membership function of the fuzzy set $\tilde{C} \triangleq$ "the number is around ***d***". In this case, the fuzziness coefficients α and β can be found from equations $\mu_{\tilde{B}}(x) = 0.01$ and $\mu_{\tilde{C}}(x) = 0.01$. As a result, a fuzzy interval (c, d, α, β) is obtained.

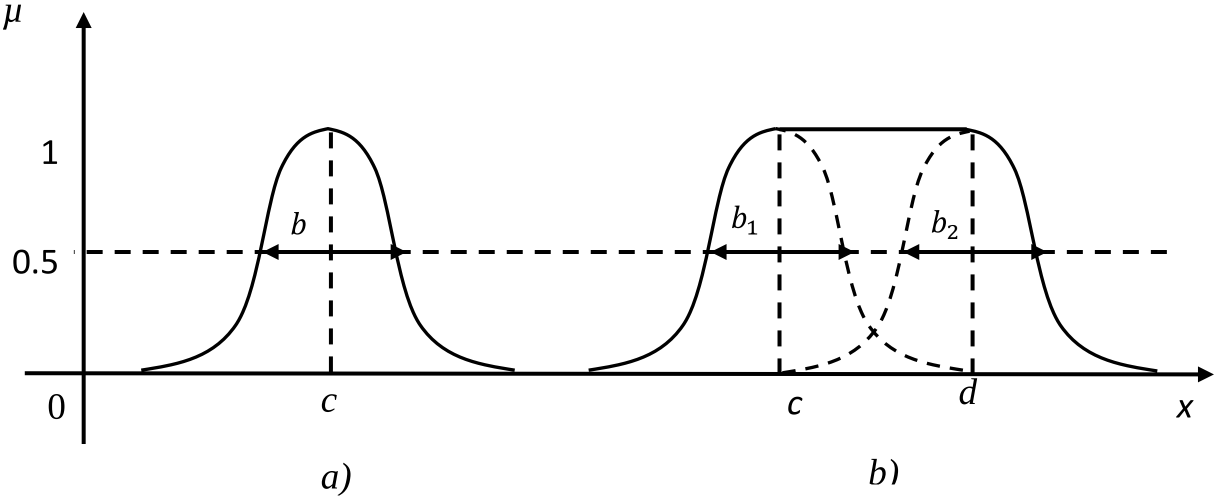

Figure 2: Gaussian membership functions: a) - standard; b) - combined (double)

Setting task parameters in the form of a fuzzy interval is a convenient form for formalizing imprecise quantities. A regular interval is often an unsatisfactory representation of the value of a certain parameter, as it is necessary to fix the boundaries of this interval. They can be overestimated or underestimated, which can raise doubts about the accuracy of the results. A fuzzy interval assessment, firstly, is psychologically easier to give, and secondly, the carrier of the fuzzy interval is guaranteed to contain the value of the parameter under consideration [15].

4. Consistency of linguistic assessments

Linguistic fuzzy assessments, of course, to some extent reduce a person's psychological uncertainty about the accuracy of their estimates under conditions of uncertainty. However, they are subjective. Therefore, when increased requirements are placed on the accuracy of the results, an obvious condition for reducing the degree of subjectivity is the conduct of a group expertise. The results of such an examination are generally considered reliable when the assessments of the experts are is good consistency.

Issues of consistency in group expertise evaluations have been discussed in many studies [16-19]. Particularly, in [19], provides a mechanism for coordinating interval estimates. The coefficient of variation is used as a measure of assessment consistency. This coefficient is determined separately for the left and right boundaries of intervals using the formula $V = \frac{s}{\bar{x}}$, where s is the sample standard deviation of evaluations; $\overline{X}$ is their average value.

Suppose it is necessary to give a fuzzy linguistic assessment of the form "the parameter value is approximately in the interval from a to b". Also, let's suppose $[a_1, b_1]$,…,$[a_k, b_k]$ are the interval [a, b], evaluations given by k experts. Then, the coefficients of variation for the boundaries a and b are determined as follows:

for the left boundaries, the formula is

$$V_L = \frac{s_L}{\bar{x}_L}, \tag{4}$$

where

$$\boldsymbol{s_L} = \sqrt{\frac{\mathbf{1}}{\boldsymbol{k-1}}\sum_{\boldsymbol{j=1}}^{\boldsymbol{k}}(\boldsymbol{a_j} - \overline{\boldsymbol{x}}_{\boldsymbol{L}})^{\mathbf{2}}\boldsymbol{r_j}},\ \bar{x}_L = \sum_{j=1}^{k} a_j r_j; \tag{5}$$

for the right boundaries, the formula is

$$V_R = \frac{s_R}{\bar{x}_R}, \tag{6}$$

where

$$s_R = \sqrt{\frac{1}{k-1}\sum_{j=1}^{k}(b_j - \bar{x}_R)^2 r_j}, \bar{x}_R = \sum_{j=1}^{k} b_j r_j. \quad (7)$$

Here, V_L, s_L, $\bar{x}_L$ and V_R, s_R, $\bar{x}_R$ are the coefficients of variation, sample standard deviations, and mean values for the estimates a_k and b_k,, respectively. And r_j is the weight coefficient of the j-th expert, where $\sum_{j=1}^{k} r_j$=1.

The practice of applying expert assessment methods shows that the results of the expertise can be considered satisfactory if $0,2 \le V \le 0,3$, and good if $V < 0,2$. These conditions can be used as a criterion for the consistency of assessments and as a basis for their refinement. At the same time, the aim of the clarification is to reduce the spread of estimates a_k, b_k. After the estimates are refined, the weighted average values of the boundaries of the interval [a, b] are calculated.

5. Practical Application

To better understand the essence of such calculations, let's show the application of the proposed approach when simulating random variables using the Monte Carlo method.

Suppose at the early stages of design, a random variable $X \triangleq$ "project implementation time". needs to be played out. First, its value range is defined. Since this random variable has a predictive nature, there is significant uncertainty about the boundaries of its values at these stages. Hence, to obtain more reliable values for the boundaries of this area, a group of experts is engaged who provide fuzzy linguistic evaluations for these boundaries (table 2).

Table 2
Experts estimates.

	Project implementation time is approximately in the range
Expert 1	From 87 to 123
Expert 2	From 90 134
Expert 3	From 93 to 145

First, let's assess the consistency of these estimates assuming equal competence of the experts. According to formulas (4)-(7) for intervals [87,123], [90,134], and [93,145], we have: $\bar{x}_L = 90$, $\bar{x}_R = 134$, $s_L = 3$, s_R=11, V_L=0.03, $V_R = 0.08$. Therefore, assuming the experts are balanced, the average values of their estimates are taken as the boundaries of the value range of the random variable. As a result, a group of agreed estimate that the "project implementation time is approximately in the interval from 90 to 134" is obtained.

This statement corresponds to the trapezoidal number $M = (c, d, \alpha, \beta)$, where c $= 90$ and $d = 134$. To determine the coefficients α and β, we first calculate $b(90)$ and $b(134)$. The distance $b(90)$ is determined according to table 1 and equals $b(90) \approx 19$, and the value of $b(134)$ is calculated using the above algorithm.

The least significant digit of the number 134 is in the ones place (q=1), therefore $r_q = r_1 = 7$, $r_{q+1} = r_2 = 3$ is the digit, the order of which is one higher than the order of the least significant digit of the number 134. When dividing q by 3 we get a remainder of 1, therefore, the number 134 belongs to the equivalence class M_1, so $d = 1$.

Since $r_{q+1} \neq 0$, then according to item 2b of this algorithm we have $x = r_{q+1} \cdot 10 + r_q = r_2 \cdot 10 + r_1 = 34$ and $b(134) = b(34)$, and $b(34)$ is calculated by the formula

$$b(34) = \frac{1}{2}\left(b\left(\left[\frac{34}{10}\right] \cdot 10 + 5\right) + b\left(34 - \left[\frac{34}{10}\right] \cdot 10\right)\right) = \frac{1}{2}(b(35) + b(4))$$

in which $b(35)$ and $b(4)$ are found from table 1: $b(35) = 6.63$ and $b(4) = 1.84$. Then $b(134) = \frac{1}{2}(6.63 + 1.84) \approx 4$. Knowing the distances $b(90)$ and $b(134)$, the coefficients α and β can be calculated using both the trapezoidal and the combined Gaussian function.

First, the trapezoidal function is used. For this function, the transition points x_1 and x_2 are calculated by the formulas: $x_1 = 90 - \frac{b(90)}{2} = 80.5$ and $x_2 = 134 + \frac{b(134)}{2}$=136. Equations of straight lines $\mu = f_1(x)$ and $\mu = f_2(x)$, passing through the points (80.5,0.5), (90,1) and (134,1), (136,0.5). are then constructed. These equations take the form $\mu = \frac{x-71}{19}$ and $\mu = \frac{138-x}{4}$, which at μ = 0.01 have roots x $x \approx 71 = \alpha$ и $x \approx 138 = \beta$ respectively. As a result, a fuzzy number $M = (90,134,71,138)$ is obtained, implying the value range of the random variable X lies within the interval [71, 138].

Now, a fuzzy number $M = (c, d, \alpha, \beta)$ is constructed using the combined Gaussian function (3), which describes the fuzzy set $\tilde{A} \triangleq$ "the number lies approximately in the interval from 90 to 134" and has the form:

$$\mu_{\tilde{A}}(x) = \begin{cases} x < 90, & \mu_{\tilde{B}}(x) \\ 90 \le x \le 134, & 1 \\ x > 134, & \mu_{\tilde{C}}(x) \end{cases},$$

where $\mu_{\tilde{B}}(x) = e^{-\frac{4\ln 0.5(x-90)^2}{19^2}}(x)$ and $\mu_{\tilde{C}}(x) = e^{-\frac{4\ln 0.5(x-134)^2}{4^2}}$ - membership functions of fuzzy sets $\tilde{B} \triangleq$ "the number is around 90" and$\tilde{C} \triangleq$ "the number is around 134". In this case, coefficients α and β are derived from the equations $\mu_{\tilde{B}}(x) = 0.01$ and $\mu_{\tilde{C}}(x) = 0.01$ and equal $\approx 66, \beta \approx 139$. Hence, the range of the random variable $X \triangleq$ "project realization time" lies within the interval [66, 139].

It's worth noting that the interval of values for the random variable X, obtained using the Gaussian function, exceeds the corresponding interval of the trapezoidal function. Therefore, such intervals will assuredly contain the values of forecast parameters, with more distant forecast horizons having increasingly blurred boundaries of their values. Consequently, the use of the Gaussian function in the construction of fuzzy numbers is preferable.

To describe random variables, the values of which are limited by a finite interval, a beta distribution is primarily used [20]. The beta distribution is parameterized by two positive parameters α and β, which define its shape. Therefore, almost all applied probability distributions can be expressed through this distribution.

The standard beta distribution on the interval $x \in [0,1]$ is given by the density function:

$$f(x) = \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \tag{8}$$

where $B(\alpha,\beta) = \int_0^1 y^{\alpha-1} \cdot (1-y)^{\beta-1} dy$ is Euler's beta function.
Meanwhile, the distribution function is expressed through the incomplete beta function:

$$F(x) = \frac{1}{B(\alpha,\beta)} \int_0^x y^{\alpha-1}(1-y)^{\beta-1}\, dy,$$

whereas this function is tabulated [21].

Considering that the function $F(x)$ is tabulated, the simulation of the random variable X in the interval [66, 139] will be carried out using the Neumann exclusion method [22]. This method is based on the following theorem.

Let the random variable ξ be defined on the interval $[a, b]$ and has a bounded density function $f(x)$ from above. Also, let r_1,r_2 be independent realizations of the random variable ξ and

$$x = a + (b-a)r_1\ , y = Mr_2,$$

where $M = \max_{a \le x \le b} f(x)$.

Then, if $y < f(x)$, then the value x is a realization of the random variable ξ. The efficiency of the exclusion method is directly proportional to the probability of the condition $y < f(x)$ being met, i.e.

$$P\{y < f(x)\} = [M(b-a)]^{-1}. \tag{9}$$

This probability allows for the desired number of realizations of the random variable ξ to determine the number of necessary model's runs. The main advantage of this method is its universality, i.e., its applicability for generating random variables that have any computable or tabularly given probability density.

To model the values of the random variable X, beta distribution will be used with parameters $\alpha = 2$, $\beta = 3$. In this case, $B(2,3) = \frac{1}{12}$. Since the quantity X is defined on the interval [66, 139], it is necessary to scale the density function (8). In general, for the interval $[a, b]$, this function has the form:

$$f(x) = \begin{cases} \frac{12}{(b-a)^4} \cdot (x-a)(b-x)^2, & a \le x \le b \\ 0, & otherwise \end{cases}.$$

In our case, on the interval [66, 139], the density function has the form:

$$f(x) = \begin{cases} \frac{12}{73^4} \cdot (x-66)(139-x)^2, & 66 \le x \le 139 \\ 0, & otherwise \end{cases}$$

and takes the maximum value of $M \approx 0.024$.

Suppose that random numbers $r_1 = 0.4$ and $r_2 = 0.7$are obtained by different generators (under the condition of independence). These numbers are scaled respectively into the interval [66, 139] and [0, 0.024]: $x = 66 + 73 \cdot 0.4 = 95$ and $y = 0.024 \cdot 0.7 = 0.016$. Then, $f(95) = 0.024$ is calculated. Since the condition is met, the value $x = 95$ is accepted as a realization of the random variable X. Otherwise, this value is discarded. In this case, the efficiency of modelling by the exclusion method, according to (9), is directly proportional to the probability of 0.57. This means that to obtain, for example, 1000 realizations of the random variable, approximately 1750 model runs must be carried out.

The considered approach can also be used in constructing, for example, linear S- and Z-shaped membership functions, which look like:

S-shaped function

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x < \alpha \\ \left(\frac{x-\alpha}{\beta-\alpha}\right), & \alpha \le x \le \beta. \\ 1, & x > \beta \end{cases}$$

Z-shaped function

$$\mu_{\tilde{A}}(x) = \begin{cases} 1, & x < \beta \\ \left(\frac{\alpha-x}{\alpha-\beta}\right), & \beta \le x \le \alpha. \\ 0, & x > \alpha \end{cases}$$

In these formulas, the parameters α and β define the bounds of the support of the fuzzy set Ã and can be determined based on the fuzzy linguistic estimation of the bounds of the corresponding interval. Let's say Ã ≜ "high pressure in the tank." Then the interval of values "high pressure" a fuzzy linguistic score may be given, such as "high pressure is approximately in the range of 70 to 90".

6. Conclusion

An approach to the construction of fuzzy numbers and intervals based on fuzzy linguistic statements has been proposed. Depending on the type of statements, triangular or trapezoidal fuzzy numbers of the L-R type are constructed. Such numbers are constructed using triangular, trapezoidal, and Gaussian membership functions. It has been demonstrated that the use of Gaussian function in the construction of fuzzy numbers is preferable, as their intervals will assuredly contain the values of forecast parameters, especially in cases when the forecasting horizon is more distant in time.

When constructing membership functions, it is proposed to use the distances between transition points, which are experimental data reflecting a person's perception of the boundaries of number classes, approximately equal to some number. This allows the construction of fuzzy numbers to be automated and provides the possibility to use this approach when setting the parameters of models, modelling random variables, presenting fuzzy time series in a verbal form, forming databases in fuzzy inference systems, information requests, as well as in many other applied aspects under conditions of significant uncertainty in the information environment.

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