On the Isomorphism Problem of Concept Algebras

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Abstract. Weakly dicomplemented lattices are bounded lattices equipped with two unary operations to encode a negation on *concepts*. They have been introduced to capture the equational theory of concept algebras [12]. They generalize Boolean algebras. Concept algebras are concept lattices, then complete lattices, with a weak negation and a weak opposition. A special case of the representation problem for weakly dicomplemented lattices, posed in [4] is whether complete weakly dicomplemented lattices are isomorphic to concept algebras. In this contribution we give a negative answer to this question (Theorem 3). We also provide a new proof of a well known result due to M.H. Stone [8], saying that *each Boolean algebra is a field of sets* (Corollary 4).

1 Weak dicomplementation.

Definition 1. *A **weakly dicomplemented lattice** is a bounded lattice L equipped with two unary operations $\triangle$ and $\triangledown$ called **weak complementation** and **dual weak complementation**, and satisfying for all $x, y \in L$ the following equations:*

(1) $x^{\triangle\triangle} \leq x$,   *(1') $x^{\triangledown\triangledown} \geq x$,*
(2) $x \leq y \implies x^{\triangle} \geq y^{\triangle}$,   *(2') $x \leq y \implies x^{\triangledown} \geq y^{\triangledown}$,*
(3) $(x \wedge y) \vee (x \wedge y^{\triangle}) = x$,   *(3') $(x \vee y) \wedge (x \vee y^{\triangledown}) = x$.*

*We call $x^{\triangle}$ the **weak complement** of x and $x^{\triangledown}$ the **dual weak complement** of x. The pair $(x^{\triangle}, x^{\triangledown})$ is called the **weak dicomplement** of x and the pair $(\triangle, \triangledown)$ a **weak dicomplementation** on L. The structure $(L, \wedge, \vee, \triangle, 0, 1)$ is called a **weakly complemented lattice** and $(L, \wedge, \vee, \triangledown, 0, 1)$ a **dual weakly complemented lattice**.*

The following properties are easy to verify: (i) $x \vee x^\triangle = 1$, (ii) $x \wedge x^\triangledown = 0$, (iii) $0^\triangle = 1 = 0^\triangledown$, (iv) $1^\triangle = 0 = 1^\triangledown$, (v) $x^\triangledown \leq x^\triangle$, (vi) $(x \wedge y)^\triangle = x^\triangle \vee y^\triangle$, (vii) $(x \vee y)^\triangledown = x^\triangledown \wedge y^\triangledown$, (viii) $x^{\triangle\triangle\triangle} = x^\triangle$, (ix) $x^{\triangledown\triangledown\triangledown} = x^\triangledown$, and (x) $x^{\triangle\triangledown} \leq x^{\triangle\triangle} \leq x \leq x^{\triangledown\triangledown} \leq x^{\triangledown\triangle}$.

Example 1.

(a) The natural examples of weakly dicomplemented lattices are Boolean algebras. For $(B, \wedge, \vee, ^-, 0, 1)$ a Boolean algebra, $(B, \wedge, \vee, ^-, ^-, 0, 1)$ (the complementation is duplicated, i.e. $x^\triangle := \bar{x} =: x^\triangledown$) is a weakly dicomplemented lattice.

(b) Each bounded lattice can be endowed with a **trivial weak dicomplementation** by defining $(1, 1)$, $(0, 0)$ and $(1, 0)$ as the dicomplement of 0, 1 and of each $x \notin \{0, 1\}$, respectively.

Definition 2. *Let $(P, \leq)$ be a poset and $f : P \to P$ be a map. f is a **closure operator** on P if for all $x, y \in P$, $x \leq f(y) \iff f(x) \leq f(y)$. This is equivalent to $x \leq f(x)$, $x \leq y \implies f(x) \leq f(y)$ and $f(f(x)) = f(x)$. Usually we will write a closure operator on a set X to mean a closure operator on the powerset $(\mathcal{P}(X), \subseteq)$ of X. Dually, f is a **kernel operator** on P if for all $x, y \in P$, $x \geq f(y) \iff f(x) \geq f(y)$. As above, we will say that f is a kernel operator on X to mean a kernel operator on $(\mathcal{P}(X), \subseteq)$.*

For a weakly dicomplemented lattice $(L, \wedge, \vee, ^\triangle, ^\triangledown, 0, 1)$, the maps $x \mapsto x^{\triangle\triangle}$ and $x \mapsto x^{\triangledown\triangledown}$ are resp. kernel and closure operators on L. If f is a closure operator (resp. a kernel operator) on a lattice L, then $f(L)$ (with the induced order) is a lattice. Recall that for any closure operator h on L we it holds $h(h(x) \wedge h(y)) = h(x) \wedge h(y)$ as well as $h(h(x) \vee h(y)) = h(x \vee y)$, and for any kernel operator k on L it holds $k(k(x) \wedge k(y)) = k(x \wedge y)$ and $k(k(x) \vee k(y)) = k(x) \vee k(y)$. We denote by P^d the dual poset of $(P, \leq)$, i.e. $P^d := (P, \geq)$. Then f is a kernel operator on P iff f is a closure operator on P^d.

Proposition 1. *Let h be a closure operator and k a kernel operator on a set X. For $A \subseteq X$ define $A^{\triangle_h} := h(X \setminus A)$ and $A^{\triangledown_k} := k(X \setminus A)$.*

(i) *$(h\mathcal{P}(X), \cap, \vee^h, ^{\triangle_h}, h\emptyset, X)$, with $A \vee^h B := h(A \cup B)$, is a weakly complemented lattice.*

(i') *$(k\mathcal{P}(X), \wedge_k, \cup, ^{\triangledown_k}, \emptyset, kX)$, with $A \wedge_k B := k(A \cap B)$, is a dual weakly complemented lattice.*

(ii) *If $h\mathcal{P}(X)$ is isomorphic to $k\mathcal{P}(Y)$, then h and k induce weakly dicomplemented lattice structures on $h\mathcal{P}(X)$ and on $k\mathcal{P}(Y)$ that are extensions of those in (i) and (i') above respectively.*

Proof. For (i), let h be a closure operator on X; $(h\mathcal{P}(X), \cap, \vee^h, h\emptyset, X)$ is a bounded lattice. So we should only check the equations $(1) - (3)$ in Definition 1. For $x \in h\mathcal{P}(X)$, we have $x^{\triangle\triangle} = h(X \setminus h(X \setminus x)) \subseteq h(X \setminus (X \setminus x)) = h(x) = x$, and (1) is proved. For $x_1 \leq x_2$ in $h\mathcal{P}(X)$, we have $x_1 \subseteq x_2$ and $h(X \setminus x_1) \supseteq h(X \setminus x_2)$,

and (2) is proved. Now we consider $x, y \in h\mathcal{P}(X)$. Trivially $(x \cap y) \vee^h (x \cap y^{\triangle_h}) \leq x$. In addition,

$$(x \cap y) \vee^h (x \cap y^{\triangle_h}) = (x \cap y) \vee^h (x \cap h(X \setminus y)) = h((x \cap y) \cup (x \cap h(X \setminus y)))$$

$$\supseteq h((x \cap y) \cup (x \cap (X \setminus y))) = h(x) = x.$$

(i') is proved similarly.

For (ii) we will extend the structures of (i) and (i') to get weakly dicomplemented lattices. By (i), $(h\mathcal{P}(X), \cap, \vee^h, {}^{\triangle_h}, h\emptyset, X)$ is a weakly complemented lattice. Let φ be an isomorphism from $h\mathcal{P}X$ to $k\mathcal{P}X$. We define ∇_φ on $h\mathcal{P}(X)$ by: $x^{\nabla_\varphi} := \varphi^{-1}(\varphi(x)^{\nabla_k})$. Then

$$x^{\nabla_\varphi \nabla_\varphi} = \left(\varphi^{-1}\left(\varphi(x)^{\nabla_k}\right)\right)^{\nabla_\varphi} = \varphi^{-1}\left(\varphi\left(\varphi^{-1}\left(\varphi(x)^{\nabla_k}\right)\right)^{\nabla_k}\right) = \varphi^{-1}\left(\varphi(x)^{\nabla_k \nabla_k}\right),$$

and $x^{\nabla_\varphi \nabla_\varphi} \geq \varphi^{-1}(\varphi(x)) = x$. For $x \leq y$ in $h\mathcal{P}X$ we have $\varphi(x) \leq \varphi(y)$ implying $\varphi(x)^{\nabla_k} \geq \varphi(y)^{\nabla_k}$ and $x^{\nabla_\varphi} = \varphi^{-1}(\varphi(x)^{\nabla_k}) \geq \varphi^{-1}(\varphi(y)^{\nabla_k}) = y^{\nabla_\varphi}$. For x, y in $h\mathcal{P}X$, we have

$$(x \vee y) \wedge (x \vee y^{\nabla_\varphi}) = (x \vee y) \wedge (x \vee \varphi^{-1}(\varphi(y)^{\nabla_k}))$$
$$= \varphi^{-1}\left((\varphi(x) \vee \varphi(y)) \wedge (\varphi(x) \vee \varphi(y)^{\nabla_k})\right)$$
$$= \varphi^{-1}(\varphi(x)) = x.$$

Therefore $(h\mathcal{P}(X), \cap, \vee^h, {}^{\triangle_h}, {}^{\nabla_\varphi}, h\emptyset, X)$ is a weakly dicomplemented lattice. Similarly $(k\mathcal{P}(X), \wedge^k, \cup, {}^{\triangle_\varphi}, {}^{\nabla_k}, \emptyset, kX)$ with $x^{\triangle_\varphi} := \varphi(\varphi^{-1}(x)^{\triangle_h})$ is a weakly dicomplemented lattice.

Proposition 2. *Let h be a closure operator on X and k a kernel operator on Y such that $h\mathcal{P}(X)$ is isomorphic to $k\mathcal{P}(Y)$. Let φ be an isomorphism from $h\mathcal{P}(X)$ to $k\mathcal{P}(Y)$. We set $L := \{(x, y) \in h\mathcal{P}(X) \times k\mathcal{P}(Y) \mid y = \varphi(x)\}$. L has a weakly dicomplemented lattice structure induced by h and k.*

Proof. By Lemma 1 $(h\mathcal{P}(X), \cap, \vee^h, {}^{\triangle_h}, h\emptyset, X)$ is a weakly complemented lattice and $(k\mathcal{P}(X), \wedge_k, \cup, {}^{\nabla_k}, \emptyset, kX)$ a dual weakly complemented lattice. For every $y \in k\mathcal{P}(Y)$ there is a unique $x \in h\mathcal{P}(X)$ such that $y = \varphi(x)$. For (a, b) and (c, d) in L, we have $a \leq c \iff b \leq d$. We define a relation $\leq$ on L by: $a \leq c \iff : (a, b) \leq (c, d) : \iff b \leq d$. Then $h\mathcal{P}(X) \overset{\pi_1}{\cong} L \overset{\pi_2}{\cong} k\mathcal{P}(Y)$ where π_i is the ith projection. Thus $(L, \leq)$ is a bounded lattice. Moreover $(a, b) \wedge (c, d) = (a \cap c, \varphi(a \cap c))$ and $(a, b) \vee (c, d) = (\varphi^{-1}(b \cup d), b \cup d)$. For $(x, y) \in L$, we define $(x, y)^{\triangle} := (x^{\triangle_h}, \varphi(x^{\triangle_h}))$ and $(x, y)^{\nabla} := (\varphi^{-1}(y^{\nabla_k}), y^{\nabla_k})$. We claim that $(L, \wedge, \vee, {}^{\triangle}, {}^{\nabla}, 0, 1)$ is a weakly dicomplemented lattice. In fact,

$$(x, y)^{\triangle\triangle} = (x^{\triangle_h}, \varphi(x^{\triangle_h}))^{\triangle} = (x^{\triangle_h \triangle_h}, \varphi(x^{\triangle_h \triangle_h})) \leq (x, \varphi(x)) = (x, y).$$

If $(x, y) \leq (z, t)$ in L, we have $x \leq z$ and $y \leq t$, implying $x^{\triangle_h} \geq z^{\triangle_h}$ and $\varphi(x^{\triangle_h}) \geq \varphi(z^{\triangle_h})$; thus $(x, y)^{\triangle} = (x^{\triangle_h}, \varphi(x^{\triangle_h})) \geq (z^{\triangle_h}, \varphi(z^{\triangle_h})) = (z, t)^{\triangle}$.

These prove (1) and (2) of Definition 1. It remains to prove (3). Let (x, y) and (z, t) in L;

$$
\begin{aligned}
((x,y) \wedge (z,t)) \vee ((x,y) \wedge (z,t)^\triangle) &= (x \cap z, \varphi(x \cap z)) \vee ((x,y) \wedge (z^{\triangle_h}, \varphi(z^{\triangle_h}))) \\
&= (x \cap z, \varphi(x \cap z)) \vee (x \cap z^{\triangle_h}, \varphi(x \cap z^{\triangle_h})) \\
&= (\varphi^{-1}(\varphi(x \cap z) \cup \varphi(x \cap z^{\triangle_h})), \varphi(x \cap z) \cup \varphi(x \cap z^{\triangle_h})) \\
&= ((x \cap z) \vee^h (x \cap z^{\triangle_h}), \varphi((x \cap z) \vee^h (x \cap z^{\triangle_h}))) \\
&= (x, \varphi(x)).
\end{aligned}
$$

and (3) is proved.

The advantage of the weakly dicomplemented lattice L in Lemma 2 is that, in addition to extending the weakly and dual weakly complemented lattice structures induced by h and k, it also keeps track of the closure and kernel systems.

Definition 3. *Let L be a bounded lattice and $x \in L$. The element $x^* \in L$ (resp. $x^+ \in L$) is the pseudocomplement (resp. dual pseudocomplement) of x if*

$$
x \wedge y = 0 \iff y \le x^* \quad (\text{resp. } x \vee y = 1 \iff y \ge x^+) \text{ for all } y \in L.
$$

A double p-algebra is a lattice in which every element has a pseudocomplement and a dual pseudocomplement.

Example 2. Boolean algebras are double p-algebras. Finite distributive lattices are double p-algebras. N_5 is a double p-algebra that is not distributive. All distributive double p-algebras are weakly dicomplemented lattices.

The following result give a class of "more concrete" weakly dicomplemented lattices.

Proposition 3. *Let L be a finite lattice. Denote by $J(L)$ the set of join irreducible elements of L and by $M(L)$ the set of meet irreducible elements of L respectively. Define two unary operations $\triangle$ and $\triangledown$ on L by*

$$
x^\triangle := \bigvee \{a \in J(L) \mid a \not\le x\} \text{ and } x^\triangledown := \bigwedge \{m \in M(L) \mid m \not\ge x\}.
$$

Then $(L, \wedge, \vee, ^\triangle, ^\triangledown, 0, 1)$ is a weakly dicomplemented lattice. In general, for $G \supseteq J(L)$ and $H \supseteq M(L)$, the operations $^{\triangle_G}$ and $^{\triangledown_H}$ defined by

$$
x^{\triangle_G} := \bigvee \{a \in G \mid a \not\le x\} \quad \text{and} \quad x^{\triangledown_H} := \bigwedge \{m \in H \mid m \not\ge x\}
$$

turn $(L, \wedge, \vee, ^{\triangle_G}, ^{\triangledown_H}, 0, 1)$ into a weakly dicomplemented lattice.

Proof. Let $G \supseteq J(L)$, $b \in G$ and $x \in L$. Then $b \not\le \bigvee \{a \in G \mid a \not\le x\}$ implies $b \le x$; i.e. $b \not\le x^{\triangle_G} \implies b \le x$. Thus $x^{\triangle_G \triangle_G} = \bigvee \{b \in G \mid b \not\le x^{\triangle_G}\} \le x$ and (1) is proved. For $x \le y$ we have $\{a \in G \mid a \not\le x\} \supseteq \{a \in G \mid a \not\le y\}$ implying $x^{\triangle_G} \ge y^{\triangle_G}$, and (2) is proved. For (3), it is enough to prove that for $a \in J(L)$, $a \le x \iff a \le (x \wedge y) \vee (x \wedge y^{\triangle_G})$, since $J(L)$ is $\bigvee$-dense in L. Let $a \le x$. We have $a \le y$ or $a \le y^{\triangle_G}$. Then $a \le x \wedge y$ or $a \le x \wedge y^{\triangle_G}$. Thus $a \le (x \wedge y) \vee (x \wedge y^{\triangle_G})$. The reverse inequality is obvious. $(1') - (3')$ are proved similarly.

Example 3 above is a special case of concept algebras. Before we introduce concept algebras, let us recall some notions from Formal Concept Analysis (FCA). The reader is refered to [5]. Formal Concept Analysis was born in the eighties from the formalization of the notion of *concept* [10]. Traditional philosophers considered a **concept** to be determined by its extent and its intent. The extent consists of all objects belonging to the concept while the intent is the set of all attributes shared by all objects of the concept. In general, it may be difficult to list all objects or attributes of a concept. Therefore a specific *context* should be fixed to enable formalization. A **formal context** is a triple (G, M, I) of sets such that $I \subseteq G \times M$. The members of G are called **objects** and those of M **attributes**. If $(g, m) \in I$, then the object g is said to have m as an attribute. For subsets $A \subseteq G$ and $B \subseteq M$, A' and B' are defined by

$$A' := \{m \in M \mid \forall g \in A \ g \, I \, m\} \quad \text{and} \quad B' := \{g \in G \mid \forall m \in B \ g \, I \, m\}.$$

A **formal concept** of the formal context (G, M, I) is a pair (A, B) with $A \subseteq G$ and $B \subseteq M$ such that $A' = B$ and $B' = A$. The set A is called the **extent** and B the **intent** of the concept (A, B). $\mathfrak{B}(G, M, I)$ denotes the set of all formal concepts of the formal context (G, M, I). For concepts (A, B) and (C, D), (A, B) is called a **subconcept** of (C, D) provided that $A \subseteq C$ (which is equivalent to $D \subseteq B$). In this case, (C, D) is a **superconcept** of (A, B) and we write $(A, B) \leq (C, D)$. The relation **subconcept-superconcept** encodes the **hierarchy on concepts**, namely, that a concept is more general if it contains more objects, and equivalently, if it is determined by less attributes.

Theorem 1 ([10]). *The concept lattice $\mathfrak{B}(G, M, I)$ is a complete lattice in which infimum and supremum are given by:*

$$\bigwedge_{t \in T} (A_t, B_t) = \left(\bigcap_{t \in T} A_t, \left(\bigcup_{t \in T} B_t \right)'' \right) \ and \ \bigvee_{t \in T} (A_t, B_t) = \left(\left(\bigcup_{t \in T} A_t \right)'', \bigcap_{t \in T} B_t \right).$$

A complete lattice L is isomorphic to $\mathfrak{B}(G, M, I)$ iff there are mappings $\tilde{\gamma} : G \to L$ and $\tilde{\mu} : M \to L$ such that $\tilde{\gamma}(G)$ is supremum-dense, $\tilde{\mu}(M)$ is infimum-dense and $g \, I \, m \iff \tilde{\gamma}g \leq \tilde{\mu}m$ for all $(g, m) \in G \times M$.

$(\mathfrak{B}(G, M, I); \leq)$ is called the **concept lattice** of the context (G, M, I). All complete lattices are (copies of) concept lattices. We adopt the notations below for $g \in G$ and $m \in M$:

$$g' := \{g\}', \quad m' := \{m\}', \quad \gamma g := (g'', g') \quad \text{and} \quad \mu m := (m', m'').$$

The concept γg is called **object concept** and μm **attribute concept**. The sets γG is supremum-dense and μM infimum-dense in $\mathfrak{B}(G, M, I)$. We usually assume our context clarified, meaning that $x' = y' \implies x = y$ for all $x, y \in G \cup M$. If γg is supremum-irreducible we say that the **object** g is **irreducible**. An **attribute** m is said **irreducible** if the attribute concept μm is infimum-irreducible. A **formal context** is called **reduced** if all its objects and attributes are irreducible. For every finite lattice L there is, up to isomorphism, a unique reduced context

$\mathbb{K}(L) := (J(L), M(L), \leq)$ such that $L \cong \mathfrak{B}(\mathbb{K}(L))$. We call it **standard context** of L. The meet and join operations in the concept lattice can be used to formalize the conjunction and disjunction on concepts [6]. To formalize a negation two operations are introduced as follow:

Definition 4. *Let (G, M, I) be a formal context and (A, B) a formal concept. We define*

$$its \text{ \textbf{weak negation} } by \quad (A, B)^{\triangle} := \left((G \setminus A)'' , (G \setminus A)'\right)$$

$$and \text{ } its \text{ \textbf{weak opposition} } by \quad (A, B)^{\triangledown} := \left((M \setminus B)' , (M \setminus B)''\right).$$

$\mathfrak{A}(\mathbb{K}) := \left(\mathfrak{B}(\mathbb{K}); \wedge, \vee, ^{\triangle}, ^{\triangledown}, 0, 1\right)$ *is called the* **concept algebra** *of the formal context $\mathbb{K}$, where $\wedge$ and $\vee$ denote the meet and the join operations of the concept lattice.*

These operations satisfy the equations in Definition 1 (cf. [12]). In fact concept algebras are typical examples of weakly dicomplemented lattices. One of the important and still unsolved problems in this topic is to find out the equational theory of concept algebras; that is the set of all equations valid in all concept algebras. Is it finitely generated? i.e. is there a finite set $\mathcal{E}$ of equations valid in all concept algebras such that each equation valid in all concept algebras follows from $\mathcal{E}$? We start with the set of equations defining a weakly dicomplemented lattice and have to check whether they are enough to represent the equational theory of concept algebras. This representation problem can be split:

strong representation: Describe weakly dicomplemented lattices that are isomorphic to concept algebras.
equational axiomatization: Find a set of equations that generate the equational theory of concept algebras.
concrete embedding: Given a weakly dicomplemented lattice L, is there a context $\mathbb{K}_{\triangledown}^{\triangle}(L)$ such that L can be embedded into the concept algebra of $\mathfrak{A}\left(\mathbb{K}_{\triangledown}^{\triangle}(L)\right)$?

We proved (see [4] or [3]) that finite distributive weakly dicomplemented lattices are isomorphic to concept algebras. However we cannot expect all weakly dicomplemented lattices to be isomorphic to concept algebras, since concept algebras are first of all complete lattices. In Section 3 we will show that being complete is not enough for weakly dicomplemented lattices to be isomorphic to concept algebras. Before that we show in Section 2 that weakly dicomplemented lattices generalize Boolean algebras.

2   Weakly Dicomplemented Lattices with Negation

Example 1 states that duplicating the complementation of a Boolean algebra leads to a weakly dicomplemented lattice. Does the converse hold? The finite case is easily obtained [Corollary 1]. Major parts of this section are taken from [4]. We will also describe weakly dicomplemented lattices whose Boolean part is the intersection of their skeletons (definitions below).

Definition 5. *A weakly dicomplemented lattice is said to be* **with negation** *if the unary operations coincide, i.e., if $x^\triangledown = x^\triangle$ for all x. In this case we set $x^\triangle =: \bar{x} := x^\triangledown$.*

Lemma 1. *A weakly dicomplemented lattice with negation is uniquely complemented.*

Proof. $x^{\triangle\triangle} \le x \le x^{\triangledown\triangledown}$ implies that $x = \bar{\bar{x}}$. Moreover, $x \wedge \bar{x} = 0$ and $\bar{x}$ is a complement of x. If y is another complement of x then

$$x = (x \wedge y) \vee (x \wedge \bar{y}) = x \wedge \bar{y} \implies x \le \bar{y}$$

$$x = (x \vee y) \wedge (x \vee \bar{y}) = x \vee \bar{y} \implies x \ge \bar{y}$$

Then $\bar{y} = x$ and $\bar{x} = y$. L is therefore a uniquely complemented lattice.

It can be easily seen that each uniquely complemented atomic lattice is a copy of the power set of the set of its atoms, and therefore distributive. Thus

Corollary 1. *The finite weakly dicomplemented lattices with negation are exactly the finite Boolean algebras.*

Of course, the natural question will be if the converse of Lemma 1 holds. That is, can any uniquely complemented lattice be endowed with a structure of a weakly dicomplemented lattice with negation? The answer is yes for distributive lattices. If the assertion of Corollary 1 can be extended to lattices in general, the answer will unfortunately be no. In fact R. P. Dilworth proved that each lattice can be embedded into a uniquely complemented lattice [**?**]. The immediate consequence is the existence of non-distributive uniquely complemented lattices. They are however infinite. If a uniquely complemented lattice could be endowed with a structure of weakly dicomplemented lattice, it would be distributive. This cannot be true for non distributive uniquely complemented lattices.

Lemma 2. *Each weakly dicomplemented lattice with negation L satisfies the de Morgan laws.*

Proof. We want to prove that $\overline{x \wedge y} = \bar{x} \vee \bar{y}$.

$$(\bar{x} \vee \bar{y}) \vee (x \wedge y) \ge \bar{x} \vee (x \wedge \bar{y}) \vee (x \wedge y) = \bar{x} \vee x = 1$$

and

$$(\bar{x} \vee \bar{y}) \wedge (x \wedge y) \le (\bar{x} \vee \bar{y}) \wedge x \wedge (\bar{x} \vee y) = \bar{x} \wedge x = 0.$$

So $\bar{x} \vee \bar{y}$ is a complement of $x \wedge y$, hence by uniqueness it is equal to $\overline{x \wedge y}$. Dually we have $\overline{x \vee y} = \bar{x} \wedge \bar{y}$.

Now for the distributivity we can show that

Lemma 3. $\overline{x \wedge (y \vee z)}$ *is a complement of* $(x \wedge y) \vee (x \wedge z)$.

Proof. Since in every lattice the equation $x \wedge (y \vee z) \geq (x \wedge y) \vee (x \wedge z)$ holds, we have that $\overline{x \wedge (y \vee z)} \leq \overline{(x \wedge y) \vee (x \wedge z)}$; so we have to show only that $\overline{x \wedge (y \vee z)} \vee (x \wedge y) \vee (x \wedge z) = 1$. Using the de Morgan laws and axiom (3) several times we obtain:

$$\overline{x \wedge (y \vee z)} \vee (x \wedge y) \vee (x \wedge z) = \bar{x} \vee (\bar{y} \wedge \bar{z}) \vee (x \wedge y) \vee (x \wedge z)$$
$$= \bar{x} \vee (\bar{y} \wedge \bar{z} \wedge x) \vee (\bar{y} \wedge \bar{z} \wedge \bar{x}) \vee (x \wedge y \wedge z)$$
$$\vee (x \wedge y \wedge \bar{z}) \vee (x \wedge z \wedge \bar{y})$$
$$= \bar{x} \vee (\bar{y} \wedge \bar{z} \wedge \bar{x}) \vee (x \wedge y \wedge z) \vee (x \wedge y \wedge \bar{z})$$
$$\vee (x \wedge \bar{y} \wedge z) \vee (x \wedge \bar{y} \wedge \bar{z})$$
$$= \bar{x} \vee (\bar{y} \wedge \bar{z} \wedge \bar{x}) \vee (x \wedge y) \vee (x \wedge \bar{y})$$
$$= \bar{x} \vee (\bar{y} \wedge \bar{z} \wedge \bar{x}) \vee x$$
$$= 1.$$

Thus $\overline{x \wedge (y \vee z)}$ is a complement of $(x \wedge y) \vee (x \wedge z)$.

Since the complement is unique we get the equality

$$x \wedge (y \vee z) = \overline{\overline{x \wedge (y \vee z)}} = (x \wedge y) \vee (x \wedge z).$$

Thus weakly dicomplemented lattices generalize Boolean algebras in the following sense

Theorem 2. *Boolean algebras with the complementation duplicated*[3] *are weakly dicomplemented lattices. If $^\triangle = ^\triangledown$ in a weakly dicomplemented lattice L, then $(L, \wedge, \vee, ^-, 0, 1)$, with $\bar{x} := x^\triangle = x^\triangledown$ for all $x \in L$, is a Boolean algebra.*

As the equality $x^\triangle = x^\triangledown$ not always holds, we can look for a maximal subset with this property.

Definition 6. *For any weakly dicomplemented lattice L, we will call $B(L) := \{x \in L \mid x^\triangle = x^\triangledown\}$ the* **subset of elements with negation***.*

As in Definition 5 we denote by $\bar{x}$ the common value of $x^\triangle$ and $x^\triangledown$. We set $L^\triangle := \{a^\triangle \mid a \in L\} = \{a \in L \mid a^{\triangle\triangle} = a\}$ and call it the **skeleton** of L, as well as $L^\triangledown := \{a^\triangledown \mid a \in L\} = \{a \in L \mid a^{\triangledown\triangledown} = a\}$ and call it the **dual skeleton** of L.

Corollary 2. $(B(L), \wedge, \vee, ^-, 0, 1)$ *is a Boolean algebra that is a subalgebra of the skeleton and the dual skeleton.*

Proof. From $x^\triangle = x^\triangledown$ we get $x^{\triangle\triangle} = x^{\triangledown\triangle}$ and $x^{\triangle\triangledown} = x^{\triangledown\triangledown}$. Thus

$$x^{\triangle\triangledown} = x^{\triangle\triangle} = x = x^{\triangledown\triangledown} = x^{\triangledown\triangle}$$

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[3] see Example 1

and $B(L)$ is closed under the operations $\triangle$ and ∇. We will prove that $B(L)$ is a subalgebra of L. We consider x and y in $B(L)$. We have

$$(x \wedge y)^\triangle = x^\triangle \vee y^\triangle = x^\nabla \vee y^\nabla \le (x \wedge y)^\nabla \le (x \wedge y)^\triangle \text{ and}$$

$$(x \vee y)^\nabla = x^\nabla \wedge y^\nabla = x^\triangle \wedge y^\triangle \ge (x \vee y)^\triangle \ge (x \vee y)^\nabla.$$

Thus $x \wedge y$ and $x \vee y$ belong to $B(L)$. $B(L)$ is a weakly dicomplemented lattice with negation, and is by Theorem 2, a Boolean algebra.

While proving Corollary 2 we show that $B(L)$ is a subalgebra of L. It is the largest Boolean algebra that is a subalgebra of the skeletons and of L. We call it the **Boolean part** of L. The inclusion $B(L) \subseteq L^\triangle \cap L^\nabla$ can be strict (see Fig. 1). It would be nice to find under which conditions the Boolean part is the intersection of the skeleton and dual skeleton?

Lemma 4. *If L is a finite distributive lattice with $\nabla = *$ (pseudocomplementation) and $\triangle = +$ (dual pseudocomplementation), then $B(L)$ is the set of complemented elements of L.*

Proof. Let L be a finite distributive lattice with $\nabla = *$ and $\triangle = +$. We denote by $C(L)$ the set of complemented elements of L. Of course $B(L) \subseteq C(L)$. Let $x \in C(L)$. From the distributivity there is a unique elements $z \in L$ such that $x \vee z = 1$ and $x \wedge z = 0$. Then $z \le x^\nabla \le x^\triangle \le z$, and $x \in B(L)$.

Even in this case, the Boolean part can still be strictly smaller than the intersection of the skeletons (see Fig. 1 below).
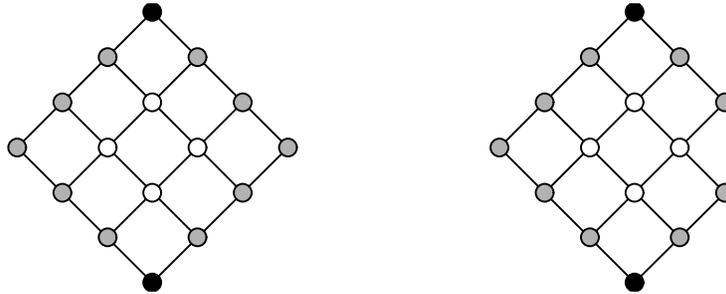
Fig. 1. Examples of dicomplementations. For L_1, the elements c, b and a are each image (of their image). The operation $\triangle$ is the dual of ∇. We have $B(L_1) = \{0, 1\}$, $L_1^\triangle = \{0, 1, c, d, e, c^\triangle, d^\triangle, e^\triangle\}$, $L_1^\nabla = \{0, 1, c, a, b, c^\nabla, a^\nabla, b^\nabla\}$ and $C(L_1) = \{0, 1, c, a^\nabla\}$. Thus $B(L_1) \subsetneq C(L_1) = L_1^\triangle \cap L_1^\nabla$. For L_2, $\triangle = +$ and $\nabla = *$. $L_2^\triangle = \{0, 1, c, c^\triangle\}$, $L_2^\nabla = \{0, 1, c, c^\nabla\}$, $B(L_2) = \{0, 1\} = C(L_2) \subsetneq \{0, 1, c\} = L_2^\triangle \cap L_2^\nabla$.

Lemma 5. $B(L) = L^\triangle \cap L^\nabla$ *iff* $x^{\triangle\triangle} = x^{\nabla\nabla} \implies x^{\triangle\nabla} = x^{\nabla\triangle}$.

Proof. ($\Rightarrow$). Let $x \in L$ such that $x^{\triangle\triangle} = x^{\triangledown\triangledown}$. Then $x \in L^{\triangle} \cap L^{\triangledown} = B(L)$ and implies $x^{\triangle} = x^{\triangledown}$. Therefore $x^{\triangle\triangledown} = x^{\triangledown\triangledown} = x = x^{\triangle\triangle} = x^{\triangledown\triangle}$.

($\Leftarrow$). Let $x \in L^{\triangle} \cap L^{\triangledown}$. Then $x^{\triangle\triangle} = x = x^{\triangledown\triangledown}$ and implies $x^{\triangle} = x^{\triangledown\triangle\triangle} \leq x^{\triangledown}$. Thus $x^{\triangle} = x^{\triangledown}$, and $x \in B(L)$.

3   Strong representation problem

We start this section by a negative result, namely by showing that *completeness is not enough for weakly dicomplemented lattices to be (copies of) concept algebras.*

Theorem 3. *There is no formal context whose concept algebra is isomorphic to a complete atomfree Boolean algebra.*

Proof. Let B be a complete and atomfree Boolean algebra. By Theorem 1, there is a context (G, M, I) such that $\mathfrak{B}(G, M, I) \cong B$ (lattice isomorphism). Without loss of generality, we can assume that (G, M, I) is a subcontext of $(B, B, \leq)$. We claim that there are $g, h \in G$ with $0 < h < g < 1$. In fact, for an element $g \in G \subseteq B$ with $0 \neq g$ there is $a \in B$ such that $0 < a < g$, since B is atomfree. Moreover G is $\bigvee$-dense in B and then $0 \neq a = \bigvee\{x \in G \mid x \leq a\}$, implying that there $\{x \in G \mid 0 < x \leq a\} \neq \emptyset$. Thus we can choose $h \in G$ with $0 < h \leq a < g$. In the concept algebra of $(G, M, \leq)$ we have $h^{\triangle} = \bigvee\{x \in G \mid x \not\leq h\} \geq g > h$. From $h \vee h^{\triangle} = 1$ we get $h^{\triangle} = 1 \neq h'$ (the complement of h in B).

Theorem 3 says that an atomfree Boolean algebra is not isomorphic to a concept algebra. However it can be embedded into a concept algebra. The corresponding context is constructed via ultrafilters. A general construction was presented in [4].

Definition 7. *A **primary filter** is a (lattice) filter that contains w or $w^{\triangle}$ for all $w \in L$. Dually, a **primary ideal** is an ideal that contains w or $w^{\triangledown}$ for all $w \in L$. $\mathfrak{F}_{\mathrm{pr}}(L)$ denotes the set of all primary filters and $\mathfrak{I}_{\mathrm{pr}}(L)$ the set of primary ideals of L.*

For Boolean algebras, a proper filter F is primary iff it is an ultrafilter, iff it is a prime filter ($x \vee y \in F \implies x \in F$ or $y \in F$). The following result, based on Zorn's lemma provides the sets of $\mathbb{K}_{\triangledown}^{\triangle}$.

Theorem 4 ("Prime ideal theorem"). *For every filter F and every ideal I such that $F \cap I = \emptyset$ there is a primary filter G containing F and disjoint from I. Dually, for every ideal I and every filter F such that $I \cap F = \emptyset$ there is a primary ideal J containing I and disjoint from F.*

Corollary 3. *If $x \not\leq y$ in L, then there exists a primary filter F containing x and not y.*

For $x \in L$, we set

$$\mathcal{F}_{\mathrm{x}} := \{F \in \mathfrak{F}_{\mathrm{pr}}(L) \mid x \in F\} \quad \text{and} \quad \mathcal{I}_{\mathrm{x}} := \{I \in \mathfrak{I}_{\mathrm{pr}}(L) \mid x \in I\}.$$

The **canonical context** of a weakly dicomplemented lattice L is the formal context

$$\mathbb{K}_{\triangledown}^{\triangle}(L) := (\mathfrak{F}_{\mathrm{pr}}(L), \mathfrak{I}_{\mathrm{pr}}(L), \square) \quad \text{with } F \square I : \Longleftrightarrow F \cap I \neq \emptyset.$$

The derivation in $\mathbb{K}_{\triangledown}^{\triangle}(L)$ yields, $\mathcal{F}_{\mathrm{x}}' = \mathcal{I}_{\mathrm{x}}$ and $\mathcal{I}_{\mathrm{x}}' = \mathcal{F}_{\mathrm{x}}$ for every $x \in L$. Moreover, the map

$$i : L \to \mathfrak{B}\left(\mathbb{K}_{\triangledown}^{\triangle}(L)\right)$$
$$x \mapsto (\mathcal{F}_{\mathrm{x}}, \mathcal{I}_{\mathrm{x}})$$

is a bounded lattice embedding with $i(x^{\triangledown}) \leq i(x)^{\triangledown} \leq i(x)^{\triangle} \leq i(x^{\triangle})$. If the first and last inequalities above were equalities, we would get a weakly dicomplemented embedding into the concept algebra of $\mathbb{K}_{\triangledown}^{\triangle}(L)$. This would give a solution to the representation problem of weakly dicomplemented lattices.

Theorem 5. *If L is a Boolean algebra, then the concept algebra of $\mathbb{K}_{\triangledown}^{\triangle}(L)$ is a complete and atomic Boolean algebra into which L embeds.*

Proof. If B is a Boolean algebra, then a proper filter F of L is primary iff it is an ultrafilter, and a proper ideal J is primary iff it is maximal. Thus $\mathfrak{F}_{\mathrm{pr}}(L)$ is the set of ultrafilters of L and $\mathfrak{I}_{\mathrm{pr}}(L)$ the set of its maximal ideals. In addition, the complement of an ultrafilter is a maximal ideal and vice-versa. For $F \in \mathfrak{F}_{\mathrm{pr}}(L)$, $L \setminus F$ is the only primary ideal that does not intersect F, and for any $J \in \mathfrak{I}_{\mathrm{pr}}(L)$, $L \setminus J$ is the only primary filter that does not intersect J. Thus the context $\mathbb{K}_{\triangledown}^{\triangle}(L)$ is a copy of $(\mathfrak{F}_{\mathrm{pr}}(L), \mathfrak{F}_{\mathrm{pr}}(L), \neq)$. The concepts of this context are exactly pairs (A, B) such that $A \cup B = \mathfrak{F}_{\mathrm{pr}}(L)$ and $A \cap B = \emptyset$. Thus $\mathfrak{B}(\mathbb{K}_{\triangledown}^{\triangle}(L)) \cong \mathcal{P}(\mathfrak{F}_{\mathrm{pr}}(L))$ and each subset A of $\mathfrak{F}_{\mathrm{pr}}(L)$ is an extent of $\mathbb{K}_{\triangledown}^{\triangle}(L)$. It remains to prove that the lattice embedding

$$i : L \to \mathfrak{B}\left(\mathbb{K}_{\triangledown}^{\triangle}(L)\right)$$
$$x \mapsto (\mathcal{F}_{\mathrm{x}}, \mathcal{I}_{\mathrm{x}})$$

is also a Boolean algebra embedding. If $i(x^{\triangle}) \neq i(x)^{\triangle}$ then there is $F \in \mathcal{F}_{\mathrm{x}^{\triangle}} \setminus (\mathfrak{F}_{\mathrm{pr}}(L) \setminus \mathcal{F}_{\mathrm{x}})'' = \mathcal{F}_{\mathrm{x}^{\triangle}} \setminus (\mathfrak{F}_{\mathrm{pr}}(L) \setminus \mathcal{F}_{\mathrm{x}}) = \emptyset$, which is a contradiction. Similarly $i(x^{\triangledown}) = i(x)^{\triangledown}$. Therefore B embeds into the complete and atomic Boolean algebra $\mathcal{A}\left(\mathbb{K}_{\triangledown}^{\triangle}(L)\right)$ which is a copy of $\mathcal{P}\left(\mathfrak{F}_{\mathrm{pr}}(L)\right)$.

The above result is a new proof to a well-known result (Corollary 4) due to Marshall Stone [8]. The advantage here is that the proof is very simple and does not require any knowledge from topology. Recall that a field of subsets of a set X is a subalgebra of $\mathcal{P}(X)$, .i.e. a family of subsets of X that contains $\emptyset$ and X, and that is closed under union, intersection, and complementation.

Corollary 4 ([8]). *Each Boolean algebra embeds into a field of sets.*

We conclude this section by an example. Consider the Boolean algebra $F\mathbb{N}$ of finite and cofinite subsets of $\mathbb{N}$. It is not complete. But $\mathcal{P}(\mathbb{N})$ is a complete and atomic Boolean algebra containing $F\mathbb{N}$. By Theorem 5 $\mathfrak{A}(\mathbb{K}_{\triangledown}^{\triangle}(B))$ is also a complete and atomic Boolean algebra into which $F\mathbb{N}$ embeds. The atoms of $F\mathbb{N}$ are $\{n\}, n \in \mathbb{N}$. These generate its principal ultrafilters. $F\mathbb{N}$ has exactly one non-principal ultrafilter U (the cofinite subsets). Thus $|F\mathbb{N}| = |\mathbb{N}| + 1 = |\mathbb{N}|$. We can find a bijection let say f between the atoms of $\mathcal{P}(\mathbb{N})$ and the atoms of $\mathfrak{A}(\mathbb{K}_{\triangledown}^{\triangle}(F\mathbb{N}))$. f induces an isomorphism $\hat{f} : \mathcal{P}(\mathbb{N}) \to \mathfrak{A}(\mathbb{K}_{\triangledown}^{\triangle}(F\mathbb{N}))$. Is there a universal property for $\mathfrak{A}(\mathbb{K}_{\triangledown}^{\triangle}(B))$ of Boolean algebras. For example *is $\mathfrak{A}(\mathbb{K}_{\triangledown}^{\triangle}(B))$ the smallest complete and atomic Boolean algebra into which B can be embedded*?

4 Conclusion

Weakly dicomplemented lattices with negation are exactly Boolean algebras (Thm. 2). Even if they are not always isomorphic to concept algebras (Thm. 3), they embed into concept algebras (Thm. 5). Finite distributive weakly dicomplemented lattices are isomorphic to concept algebras [3]. Extending these results to finite weakly dicomplemented lattices in one sense and to distributive weakly dicomplemented lattices in the other are the next tasks. Finding a kind of universal property to characterize the construction in Thm. 5 is a natural question to be addressed.

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