# Conservativity for a hierarchy of Euler and Venn reasoning systems

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**Abstract**

This paper introduces a hierarchy of Euler and Venn diagrammatic reasoning systems in terms of their expressive powers in topological-relation-based formalization. At the bottom of the hierarchy is the Euler diagrammatic system introduced in Mineshima-Okada-Sato-Takemura [13, 12], which is expressive enough to characterize syllogistic reasoning in terms of unification and deletion rules. At the top of the hierarchy is a Venn diagrammatic system such as Swoboda-Allwein's Euler/Venn diagrammatic system [23]. In order to understand the hierarchy uniformly, we introduce an algebraic structure, which also provides another description of our unification rule of Euler diagrams. We prove that each system S' of the hierarchy is conservative over any lower system S with respect to validity—in the sense that S' is an extension of S, and the semantic consequence relations of S and S' are equivalent for diagrams of S. Furthermore, we prove that a region-based Venn diagrammatic system is conservative over our topological-relation-based Euler diagrammatic system with respect to provability.

## 1 Introduction

Euler diagrams were introduced by Euler (1768) [1] to represent logical relations among the terms of a syllogism by topological relations among circles. Given two Euler diagrams which represent the premises of a syllogism, the syllogistic inference can be naturally replaced by the task of manipulating the diagrams, in particular of unifying the diagrams and extracting information from them. For example, the well-known syllogism named "Barbara," i.e., *All $A$ are $B$ and All $B$ are $C$; therefore All $A$ are $C$*, can be represented diagrammatically as in Fig. 1.

Another well-known diagrammatic representation system for syllogistic reasoning is Venn diagrams. In Venn diagrams a novel syntactic device, namely *shading*, to represent emptiness plays a central role in place of the topological relations of Euler diagrams. Because of their expressive power and their uniformity in formalizing the manipulation of combining diagrams simply as the superposition of shadings, Venn diagrams have been very well studied. Cf. Venn-I, II systems of Shin [19], Spider

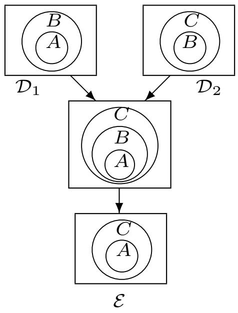



Fig. 1 Barbara with Euler diagrams

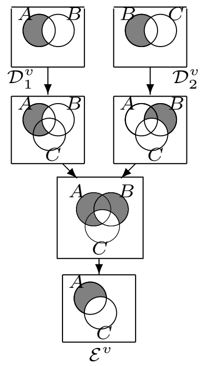



Fig. 2 Barbara with Venn diagrams

diagrams SD1 and SD2 of [9], [14], etc. For a recent survey, see [20]. However, the development of systems of Venn diagrams is obtained at the cost of clarity of Euler diagrams. As Venn [25] himself already pointed out, when more than three circles are involved, Venn diagrams fail in their main purpose of affording intuitive and sensible illustration. (For some discussions on visual disadvantages of Venn diagrams, see [8, 5]. See also [18] for our cognitive psychological experiments comparing linguistic, Euler diagrammatic, and Venn diagrammatic representations.) Recently, *Euler diagrams with shading* were introduced to make up for the shortcoming of Venn diagrams: E.g., Euler/Venn diagrams of [23, 24]; Spider diagrams ESD2 of [14] and SD3 of [10]. However, their abstract syntax and semantics are still defined in terms of regions, where shaded regions of Venn diagrams are considered as “missing” regions. That is, the idea of the *region-based* Euler diagrams is essentially along the same line as Venn diagrams.

We may point out the following complications of region-based formalization of diagrams:

1. In region-based diagrams, logical relations among circles are represented not simply by topological relations, but by the use of shading or missing regions, which makes the translations of categorical sentences uncomfortably complex. For example, *All $A$ are $B$* is expressed by a region-based diagram through a translation to the statement *There is nothing which is $A$ but not $B$* as seen in $\mathcal{D}_1^v$ of Fig. 2.
2. The inference rule of *unification*, which plays a central role in Euler diagrammatic reasoning, is defined by way of the superposition of Venn diagrams. For example, when we unify two region-based Euler diagrams as in $\mathcal{D}_1$ and $\mathcal{D}_2$ of Fig. 1, they are first transformed into Venn diagrams $\mathcal{D}_1^v$ and $\mathcal{D}_2^v$ of Fig. 2, respectively; then, by superposing the shaded regions of $\mathcal{D}_1^v$ and $\mathcal{D}_2^v$, and by deleting the circle $B$, the Venn diagram $\mathcal{E}^v$ is obtained, which is transformed into the region-based Euler diagram $\mathcal{E}$. In this way, processes of deriving conclusions are

often made complex, and hence less intuitive, in the region-based framework.

In contrast to the studies in the tradition of region-based diagrams, we proposed a novel approach in [13, 12] to formalize Euler diagrams in terms of topological relations. Our system has the following features and advantages:

1. Our diagrammatic syntax and semantics are defined in terms of *topological relations*, inclusion and exclusion relations, between two diagrammatic objects. This formalization makes the translations of categorical sentences natural and intuitive. Furthermore, our formalization makes it possible to represent a diagram by a simple ordered (or graph-theoretical) structure.

2. Our *unification* of two diagrams is formalized directly in terms of topological relations without making a detour to Venn diagrams. Thus, it can directly capture the inference process as illustrated in Fig. 1. We formalize the unification in the style of Gentzen's natural deduction, a well-known formalization of logical reasoning in symbolic logic, which is intended to be as close as possible to actual reasoning (Gentzen [3]). This makes it possible to compare our Euler diagrammatic inference system directly with natural deduction system. Through such comparison, we can apply well-developed proof-theoretical approaches to diagrammatic reasoning. See [13] for such proof-theoretical analyses.

From a perspective of proof-theory, the contrast between the standpoints of the region-based framework and the topological-relation-based framework can be understood as follows: At the level of representation, the contrast is analogous to the one between disjunctive (dually, conjunctive) normal formulas and implicational formulas; at the level of reasoning, the contrast is analogous to the one between resolution calculus style proofs and natural deduction style proofs.

From a perspective of cognitive psychology, our system is designed not just as an alternative of usual linguistic/symbolic representations; we make the best use of advantages of diagrammatic representations so that inherent definiteness or specificity of diagrams can be exploited in actual reasoning. See [18] for our experimental result, which shows that our Euler diagrams are more effective than Venn diagrams or linguistic representations in syllogism solving tasks.

We roughly review our topological-relation-based Euler diagrammatic representation system EUL in Section 2. (We also review our inference system GDS in Appendix A.) Although EUL is weaker in its expressive power than usual Venn diagrammatic systems (e.g. Shin's Venn-II [19], which is equivalent to the monadic first order logic in its expressive power), EUL is expressive enough to characterize basic logical reasoning such as syllogistic reasoning, see [12]. Our EUL-diagrams can be abstractly seen as algebraic (or graph-theoretical) structure, where inclusion relations between diagrammatic objects are reflexive transitive ordering relations, and exclusion relations are irreflexive symmetric relations. Based on this observation, in Section 3, we introduce EUL-structure, which provides another description and a verification of our unification rule of Appendix A. In Section 4, based on the EUL-structure, we introduce a hierarchy of Euler and Venn diagrammatic reasoning systems as seen in Fig. 3.
The most elementary system EUL considers only circles and points as diagrammatic objects; EUL is extended by considering intersection regions $X \cap Y$, union regions

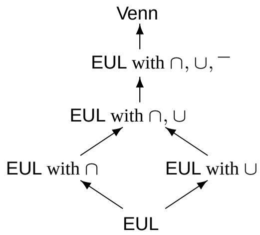



Fig. 3 Hierarchy of Euler and Venn systems

$X \cup Y$, and complement regions $\overline{X}$ as diagrammatic objects, respectively, as well as linking of points; Venn diagrams can be put at the top of the hierarchy of these extended systems. The algebraic structure thus obtained for Venn diagrams is essentially the directed acyclic graph of Swoboda-Allwein [23]. We prove that each system $\mathsf{S}'$ of the hierarchy is conservative over any lower system $\mathsf{S}$ with respect to validity—in the sense that $\mathsf{S}'$ is an extension of $\mathsf{S}$, and the semantic consequence relations of $\mathsf{S}$ and $\mathsf{S}'$ are equivalent for diagrams of $\mathsf{S}$. Moreover, we prove that a region-based Venn diagrammatic system is conservative over our topological-relation-based Euler diagrammatic system with respect to provability. We also give a procedure to transform a topological-relation-based EUL-diagram through an EUL-structure to a semantically equivalent region-based Venn diagram.

# 2 A diagrammatic representation system (EUL) for Euler circles and its set-theoretical semantics

In this section, we review our diagrammatic representation system EUL of [13, 12].

## 2.1 Diagrammatic syntax of EUL

The following definition of diagrams is slightly different from that of [13, 12] in that (1) we regard inclusion relation $\sqsubset$ as reflexive in this paper; (2) we exclude from EUL-diagrams only the empty diagram, on which no topological relation holds.

**Definition 2.1** An EUL**-diagram** is a plane ($\mathbb{R}^2$) with a finite number, at least one, of **named simple closed curves** (denoted by $A, B, C, \dots$) and **named points** (denoted by $a, b, c, \dots$), where each named simple closed curve or named point has a unique and distinct name. EUL-diagrams are denoted by $\mathcal{D}, \mathcal{E}, \mathcal{D}_1, \mathcal{D}_2, \dots$.
An EUL-diagram consisting of at most two objects is called a **minimal diagram**. Minimal diagrams are denoted by $\alpha, \beta, \gamma, \dots$.

In what follows, a named simple closed curve is called a *named circle*. Moreover, named circles and named points are collectively called *objects*, and denoted by $s, t, u, \dots$. We use a rectangle to represent a plane for an EUL-diagram.[1]

[1] Several Euler diagrammatic representation systems impose some additional conditions for well-formed diagrams. E.g., at most two circles meet at a single point, no tangential meetings or concurrency etc. Cf. e.g., [22]. However, for simplicity of the definition, those are all considered to be well-formed in EUL.

We define the following binary topological relations between diagrammatic objects[2]:

**Definition 2.2** $\mathsf{EUL}$**-relations** are the following binary relations between diagrammatic objects:

| | |
|---|---|
| $A \sqsubset B$ | "the interior of $A$ is *inside of* the interior of $B$," |
| $A \vdash\!\dashv B$ | "the interior of $A$ is *outside of* the interior of $B$," |
| $A \bowtie B$ | "there is an intersection between the interior of $A$ and the interior of $B$," |
| $b \sqsubset A$ | "$b$ is *inside of* the interior of $A$," |
| $b \vdash\!\dashv A$ | "$b$ is *outside of* the interior of $A$," |
| $a \vdash\!\dashv b$ | "$a$ is *outside of* $b$ (i.e. $a$ is not located at the point of $b$)." |

We call $\bowtie$-relation *crossing* relation.

$\mathsf{EUL}$-relations $\vdash\!\dashv$ and $\bowtie$ are symmetric, while $\sqsubset$ is not. In this paper, we consider $\sqsubset$-relation as reflexive by allowing $s \sqsubset s$ for each object $s$.

**Proposition 2.3** *Let $\mathcal{D}$ be an $\mathsf{EUL}$-diagram. For any distinct objects $s$ and $t$ of $\mathcal{D}$, exactly one of the $\mathsf{EUL}$-relations $s \sqsubset t, t \sqsubset s, s \vdash\!\dashv t, s \bowtie t$ holds.*

Observe that, by Proposition 2.3, for a given $\mathsf{EUL}$-diagram $\mathcal{D}$, the set of $\mathsf{EUL}$-relations holding on $\mathcal{D}$ is uniquely determined. We denote the set by $\mathsf{rel}(\mathcal{D})$. We also denote by $pt(\mathcal{D})$ the set of named points of $\mathcal{D}$, by $cr(\mathcal{D})$ the set of named circles of $\mathcal{D}$, and by $ob(\mathcal{D})$ the set of objects of $\mathcal{D}$.

Although in this section, $ob(\mathcal{D}) = pt(\mathcal{D}) \cup cr(\mathcal{D})$, in Section 4, diagrammatic objects are extended, in addition to named circles and points, by introducing intersection, union, and complement regions respectively.

The following properties, as well as Proposition 2.3, characterize $\mathsf{EUL}$-diagrams.

**Lemma 2.4** *Let $\mathcal{D}$ be an $\mathsf{EUL}$-diagram. Then for any objects (named circles or points) $s, t, u \in ob(\mathcal{D})$, we have the following:*

1. (Transitivity) *If $s \sqsubset t, t \sqsubset u \in \mathsf{rel}(\mathcal{D})$, then $s \sqsubset u \in \mathsf{rel}(\mathcal{D})$.*
2. ($\vdash\!\dashv$-downward closedness) *If $s \vdash\!\dashv t, u \sqsubset s \in \mathsf{rel}(\mathcal{D})$, then $u \vdash\!\dashv t \in \mathsf{rel}(\mathcal{D})$.*
3. (Point determinacy) *For any point $x$ of $\mathcal{D}$, exactly one of $x \sqsubset s$ and $x \vdash\!\dashv s$ is in $\mathsf{rel}(\mathcal{D})$.*
4. (Point minimality) *For any point $x$ ($\not\equiv s$) of $\mathcal{D}$, $s \sqsubset x \notin \mathsf{rel}(\mathcal{D})$.*

Equivalence between $\mathsf{EUL}$-diagrams is defined as follows. (See [13] for a more detailed explanation.)

**Definition 2.5** When any two objects of the same name appear in different diagrams (planes), we identify them up to isomorphism. Any $\mathsf{EUL}$-diagrams $\mathcal{D}$ and $\mathcal{E}$ such that $ob(\mathcal{D}) = ob(\mathcal{E})$ are **syntactically equivalent** when $\mathsf{rel}(\mathcal{D}) = \mathsf{rel}(\mathcal{E})$.

[2] We follow Gergonne [4] for our notations on topological relations $\sqsubset$ and $\vdash\!\dashv$.

**Example 2.6 (Equivalence of diagrams)** For example, diagrams $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, and $\mathcal{D}_4$ of Fig. 4 are equivalent since $\mathsf{rel}(\mathcal{D}_1) = \mathsf{rel}(\mathcal{D}_2) = \mathsf{rel}(\mathcal{D}_3) = \mathsf{rel}(\mathcal{D}_4) = \{A \bowtie B, A \bowtie C, B \bowtie C, a \vdash\!\dashv A, a \sqsubset B, a \vdash\!\dashv C\}$. In the description of a set of relations, we usually omit the reflexive relation $s \sqsubset s$ for each object $s$.

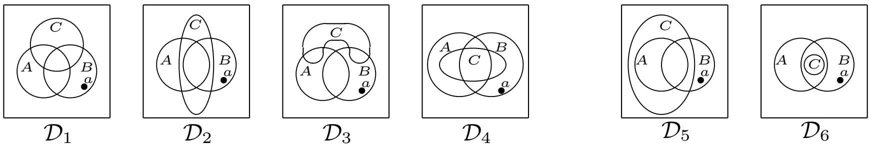



Fig. 4 Equivalence of $\mathsf{EUL}$-diagrams.

On the other hand, $\mathcal{D}_1$ and $\mathcal{D}_5$ (resp. $\mathcal{D}_1$ and $\mathcal{D}_6$) are not equivalent since different $\mathsf{EUL}$-relations hold on them: $A \sqsubset C$ holds on $\mathcal{D}_5$ in place of $A \bowtie C$ of $\mathcal{D}_1$ (resp. $C \sqsubset A$ and $C \sqsubset B$ hold on $\mathcal{D}_6$ in place of $A \bowtie C$ and $C \bowtie B$ of $\mathcal{D}_1$). Cf. Example 4.5 and 4.7 of Section 4, where $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$, and $\mathcal{D}_4$ are distinguished.

Our equation of diagrams may be explained in terms of a kind of "continuous transformation (deformation)" of named circles, which does not change any of the $\mathsf{EUL}$-relations in a diagram. (See [13] for an explanation.)

In what follows, the diagrams which are syntactically equivalent are identified, and they are referred by a single name.

**Remark 2.7 (Expressive power of $\mathsf{EUL}$)** Our equation of diagrams in the basic system $\mathsf{EUL}$ may seem to be counterintuitive since seemingly distinctive diagrams $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4$ of Example 2.6 are identified. [3]However, this slightly rough equation makes the description of unification of diagrams much simpler; see Appendix A. Furthermore, it is shown that $\mathsf{EUL}$ is expressive enough to characterize basic logical reasoning such as syllogistic reasoning; see [12]. In Section 4, we consider some extensions of $\mathsf{EUL}$, where $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$, and $\mathcal{D}_4$ are distinguished by regarding intersection and union regions respectively as diagrammatic objects. See, in particular, Examples 4.5 and 4.7. Note that, by introducing new diagrammatic objects in a representation system, $\mathsf{EUL}$-relations for these new objects are augmented, so that the system becomes more expressive. At the level of diagrammatic syntax, this means that more fine-grained distinctions between diagrams are made possible.

## 2.2 Set-theoretical semantics of $\mathsf{EUL}$

Our semantics is distinct from usual ones, e.g., [6, 8, 24, 10] in that diagrams are interpreted in terms of binary relations. In order to interpret the $\mathsf{EUL}$-relations $\sqsubset$ and $\vdash\!\dashv$ uniformly as the subset relation and the disjointness relation, respectively, we regard each point of $\mathsf{EUL}$ as a special circle which does not contain, nor cross, any other objects.

[3]This is also pointed out in Fish-Flower [2] as an drawback of the relation-based approach.

**Definition 2.8** A **model** $M$ is a pair $(U, I)$, where $U$ is a non-empty set (the domain of $M$), and $I$ is an interpretation function which assigns to each named circle or point a non-empty subset of $U$ such that $I(a)$ is a singleton for any named point $a$, and $I(a) \neq I(b)$ for any points $a, b$ of distinct names.

Note that we assign a non-empty set to each named circle. This condition is essential for our completeness. See the paragraph on the constraint for consistency in Appendix A and footnote 7 there.

**Definition 2.9** Let $\mathcal{D}$ be an EUL-diagram. $M = (U, I)$ is a **model of** $\mathcal{D}$, written as $M \models \mathcal{D}$, if the following **truth-conditions** (1) and (2) hold: For all objects $s, t$ of $\mathcal{D}$, (1) $I(s) \subseteq I(t)$ if $s \sqsubset t$ holds on $\mathcal{D}$, and (2) $I(s) \cap I(t) = \emptyset$ if $s \vdash\!\dashv t$ holds on $\mathcal{D}$.

Note that when $s$ is a named point $a$, for some $e \in U$, $I(a) = \{e\}$, and the above $I(a) \subseteq I(t)$ of (1) is equivalent to $e \in I(t)$. Similarly, $I(a) \cap I(t) = \emptyset$ of (2) is equivalent to $e \notin I(t)$.

**Remark 2.10 (Semantic interpretation of $\bowtie$-relation)** By Definition 2.9, the EUL-relation $\bowtie$ does not contribute to the truth-condition of EUL-diagrams. Informally speaking, $s \bowtie t$ may be understood as $I(s) \cap I(t) = \emptyset$ *or* $I(s) \cap I(t) \neq \emptyset$, which is true in any model. Cf. also Remark 2.7.

**Definition 2.11** An EUL-diagram $\mathcal{E}$ is a **semantically valid consequence** of EUL-diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$, written as $\mathcal{D}_1, \ldots, \mathcal{D}_n \models \mathcal{E}$, when the following holds: For any model $M$, if $M \models \mathcal{D}_1$ and ... and $M \models \mathcal{D}_n$, then $M \models \mathcal{E}$.

See Appendix A and [13] for our Generalized Diagrammatic Syllogistic inference system GDS, whose completeness holds with respect to the semantics of this section.

# 3 EUL-structure

In this section, we introduce an algebraic structure called EUL-structure for EUL-diagrams.

**Definition 3.1** An **EUL-structure** $(D, p(D), \sqsubset, \vdash\!\dashv)$ is a partially ordered structure, where $D$ is a set whose cardinality $\#D \geq 1$, and $p(D) \subseteq D$:

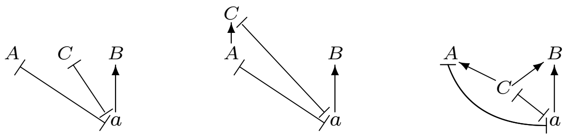
1. $\sqsubset$ is a reflexive transitive ordering relation on $D$.
2. $\vdash\!\dashv$ is an irreflexive symmetric relation on $D$.
3. ($\vdash\!\dashv$-downward closedness) For any $s, t, u \in D$, $s \vdash\!\dashv t$ and $t \sqsupset u$ imply $s \vdash\!\dashv u$.
4. (Point determinacy) For any $s \in D$ and $x \in p(D)$, $x \sqsubset s$ or $x \vdash\!\dashv s$.
5. (Point minimality) For any $s \in D$ and $x \in p(D)$ such that $s \not\equiv x$, $not(s \sqsubset x)$.

Cf. Lemma 2.4. Observe that the above properties (i), (ii), and (iii) imply that, for any distinct pair of elements of $D$, at most one of the relations $\sqsubset$ and $\vdash\!\dashv$ holds (cf. Proposition 2.3); because if both of them hold, say $s \sqsubset t$ and $s \vdash\!\dashv t$, the property (iii) implies $s \vdash\!\dashv s$, which contradicts the irreflexivity of $\vdash\!\dashv$-relation. [4]

As seen in Section 2.1, given an $\mathsf{EUL}$-diagram $\mathcal{D}$, the set $\mathsf{rel}(\mathcal{D})$ of relations holding on it is uniquely determined by Proposition 2.3. $\mathsf{rel}(\mathcal{D})$ can be regarded as an $\mathsf{EUL}$-structure.

**Proposition 3.2** *Let $\mathcal{D}$ be an* $\mathsf{EUL}$*-diagram. The set of* $\mathsf{EUL}$*-relations* $\mathsf{rel}(\mathcal{D})$ *gives rise to an* $\mathsf{EUL}$*-structure* $(ob(\mathcal{D}), pt(\mathcal{D}), \sqsubset, \vdash\!\dashv)$.

For example, $\mathsf{rel}(\mathcal{D}_1), \mathsf{rel}(\mathcal{D}_5)$ and $\mathsf{rel}(\mathcal{D}_6)$ of Fig. 4 in Example 2.6 are expressed graphically as follows: Here the ordering relations $\sqsubset$ are expressed by $\rightarrow$-edges.



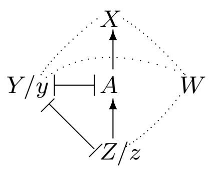

$\mathsf{rel}(\mathcal{D}_1)$ $\mathsf{rel}(\mathcal{D}_5)$ $\mathsf{rel}(\mathcal{D}_6)$

Observe that there is no edge for $\bowtie$-relation.

Now we describe the unification rule of Definition A.1 of Appendix A in terms of a graph-theoretical representation of $\mathsf{EUL}$-diagrams, which may assist with the understanding and motivation of our unification rule.

**Proposition 3.3** *Let $\mathcal{D}$ be an* $\mathsf{EUL}$*-diagram, and $\alpha$ be a minimal diagram. The set of* $\mathsf{EUL}$*-relations* $\mathsf{rel}(\mathcal{D} + \alpha)$*, which is obtained by unifying $\mathcal{D}$ and $\alpha$, gives rise to an* $\mathsf{EUL}$*-structure.*

*Proof.* In order to describe graphically the unification of $\mathsf{EUL}$-diagrams $\mathcal{D}$ and $\alpha$, we focus on the shared object of $\mathcal{D}$ and $\alpha$, say $A$, and express the $\mathsf{EUL}$-structure of $\mathsf{rel}(\mathcal{D})$ as follows:




$\mathsf{rel}(\mathcal{D})$

$\rightarrow$-edge denotes $\sqsubset$-relation

$\vdash\!\dashv$-edge denotes $\vdash\!\dashv$-relation

No edge for $\bowtie$-relation

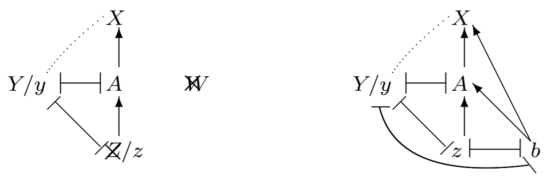
"$\cdots$" denotes one of $\sqsubset, \sqsupset, \vdash\!\dashv, \bowtie$

The variables $X, Y, Z, W$ (resp. $y, z$) are representative circles (resp. points) which are possibly related to $A$. When it makes no difference whether a possibly related object is circle or point, we denote the object as $Y/y$ (instead of simply writing $s$). Each dotted line between objects expresses that there may be one of the relations $\sqsubset, \sqsupset, \vdash\!\dashv, \bowtie$ between the objects. Note that there is no edge for each $\bowtie$-relation, as seen between $A$ and $W$. We omit the trivial transitive edge $Z \rightarrow X$ to avoid notational complexity. In the following description of each unification rule for $\mathcal{D}$ and $\alpha$, we give a graphical

[4]Note that, by the properties (i)–(iii), an $\mathsf{EUL}$-structure $(D, p(D), \sqsubset, \vdash\!\dashv)$ is an *event structure* of Nielsen-Plotkin-Winskel [15].

representation of the $\mathsf{EUL}$-structures of $\mathsf{rel}(\mathcal{D})$ in the left-hand graph, and $\mathsf{rel}(\mathcal{D}+\alpha)$ in the right-hand graph. We begin with $\mathsf{U3}$-rule since $\mathsf{U1}$ and $\mathsf{U2}$ rules are rather untypical cases:

$\mathsf{U3}$ Under the constraint of $\mathsf{U3}$-rule, there is no circle $Z$ such that $Z \sqsubset A$ holds, and no circle $W$ such that $A \bowtie W$ holds, which is expressed by $\times$ in the graph of $\mathsf{rel}(\mathcal{D})$. According to $\mathsf{U3}$-rule of Definition A.1, $\mathsf{rel}(\mathcal{D}+(b \sqsubset A))$ is represented by the graph on the right.

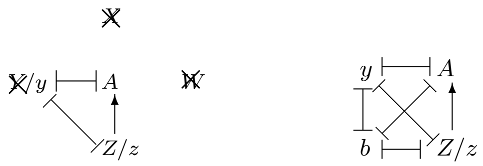



$\mathsf{rel}(\mathcal{D})$ $\mathsf{rel}(\mathcal{D}+(b \sqsubset A))$

It is easily seen that $\mathsf{rel}(\mathcal{D}+(b \sqsubset A))$ is an $\mathsf{EUL}$-structure: I.e., the augmented edges do not violate the properties of $\mathsf{EUL}$-structure.

Note also that, without the constraint, i.e., if there is a circle $Z$ or $W$ as above, in order to preserve Point determinacy, we should fix a relation between $b$ and $Z$ (resp. $W$) to $\sqsubset$ or $\vdash\!\dashv$. However, neither of them is sound with respect to our formal semantics of $\mathsf{EUL}$.

$\mathsf{U4}$ Under the constraint of $\mathsf{U4}$-rule, there is no circle $X$ such that $A \sqsubset X$ holds, no circle $Y$ such that $A \vdash\!\dashv Y$ holds, and no circle $W$ such that $A \bowtie W$ holds, which is expressed by $\times$ in the graph of $\mathsf{rel}(\mathcal{D})$. According to $\mathsf{U4}$-rule of Definition A.1, $\mathsf{rel}(\mathcal{D}+(b \vdash\!\dashv A))$ is represented by the right hand graph below.

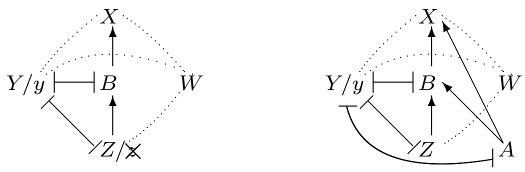



$\mathsf{rel}(\mathcal{D})$ $\mathsf{rel}(\mathcal{D}+(b \vdash\!\dashv A))$

It is easily seen that $\mathsf{rel}(\mathcal{D}+(b \vdash\!\dashv A))$ is an $\mathsf{EUL}$-structure: I.e., the augmented edges do not violate the properties of $\mathsf{EUL}$-structure.

Without the constraint, i.e., if there is a circle $X, Y$ or $W$ as above, in order to preserve Point determinacy, we should fix a relation between $b$ and $X$ (resp. $Y, W$) to $\sqsubset$ or $\vdash\!\dashv$ in $\mathsf{rel}(\mathcal{D}+(b \vdash\!\dashv A))$. However, none of them is sound with respect to our semantics of $\mathsf{EUL}$.

$\mathsf{U5}$ Under the constraint of $\mathsf{U5}$-rule, there is no point $z$ such that $z \sqsubset B$ holds. According to $\mathsf{U5}$-rule of Definition A.1, $\mathsf{rel}(\mathcal{D}+(A \sqsubset B))$ is represented by the right hand graph below.




rel($\mathcal{D}$) rel($\mathcal{D} + (A \sqsubset B)$)

Without the constraint, i.e., if there is a point $z$ as above, in order to preserve Point determinacy, we should fix a relation between $z$ and $A$ to $\sqsubset$ or $\mathrel{\vdash\!\dashv}$. However, none of them is sound with respect to our semantics of EUL.

U6 Under the constraint of U6-rule, there is no point $y$ such that $y \mathrel{\vdash\!\dashv} A$ holds. According to U6-rule of Definition A.1, rel($\mathcal{D} + (A \sqsubset B)$) is represented by the right hand graph below.


rel($\mathcal{D}$) rel($\mathcal{D} + (A \sqsubset B)$)

Without the constraint, i.e., if there is a point $y$ as above, in order to preserve Point determinacy, we should fix a relation between $y$ and $B$ to $\sqsubset$ or $\mathrel{\vdash\!\dashv}$. However, none of them is sound with respect to our semantics of EUL.

U7 Under the constraint of U7-rule, there is no point $y$ such that $y \mathrel{\vdash\!\dashv} A$ holds. According to U7-rule of Definition A.1, rel($\mathcal{D} + (A \mathrel{\vdash\!\dashv} B)$) is represented by the right hand graph below.


rel($\mathcal{D}$) rel($\mathcal{D} + (A \mathrel{\vdash\!\dashv} B)$)

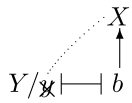
Without the constraint, i.e., if there is a point $y$ as above, in order to preserve Point determinacy, we should fix a relation between $y$ and $B$ to $\sqsubset$ or $\mathrel{\vdash\!\dashv}$. However, none of them is sound with respect to our semantics of EUL.

U8 Under the constraint of U8-rule, there is no point in $\mathcal{D}$. According to U8-rule of Definition A.1, rel($\mathcal{D} + (A \bowtie B)$) is represented by the right hand graph below.

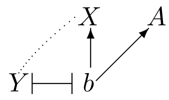

rel($\mathcal{D}$) rel($\mathcal{D} + (A \bowtie B)$)

Without the constraint, i.e., if there is a point $y$ or $z$ as above, in order to preserve Point determinacy, we should fix a relation between $y$ (resp. $z$) and $B$ to $\sqsubset$ or $\vDash$. However, none of them is sound with respect to our semantics of $\mathsf{EUL}$.

$\mathsf{U1}$ Under the constraint of $\mathsf{U1}$-rule, there is no point $y$ in $\mathcal{D}$ other than $b$. According to $\mathsf{U1}$-rule, $\mathsf{rel}(\mathcal{D} + (b \sqsubset A))$ is represented by the right hand graph below.



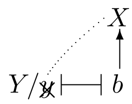

$\mathsf{rel}(\mathcal{D})$



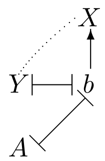

$\mathsf{rel}(\mathcal{D} + (b \sqsubset A))$

Without the constraint, i.e., if there is a point $y$ as above, in order to preserve Point determinacy, we should fix a relation between $y$ and $A$ to $\sqsubset$ or $\vDash$. However, none of them is sound with respect to our semantics of $\mathsf{EUL}$.

$\mathsf{U2}$ Under the constraint of $\mathsf{U2}$-rule, there is no point $y$ in $\mathcal{D}$ other than $b$. According to $\mathsf{U2}$-rule, $\mathsf{rel}(\mathcal{D} + (b \vDash A))$ is represented by the right hand graph below.




$\mathsf{rel}(\mathcal{D})$

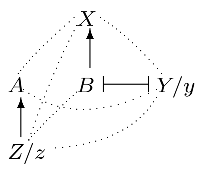



$\mathsf{rel}(\mathcal{D} + (b \vDash A))$

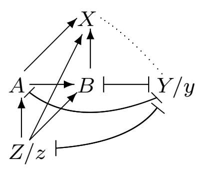
Without the constraint, i.e., if there is a point $y$ as above, in order to preserve Point determinacy, we should fix a relation between $y$ and $A$ to $\sqsubset$ or $\vDash$. However, none of them is sound with respect to our semantics of $\mathsf{EUL}$.

In $\mathsf{U9}$, $\mathsf{U10}$ rules of Definition A.1, the unified diagrams $\mathcal{D}$ and $\alpha$ share two circles, which makes the graphical description of $\mathsf{rel}(\mathcal{D})$ complicated. In order to avoid notational complexity, we omit irrelevant objects and edges, which are retained after the application of $\mathsf{U9}$ and $\mathsf{U10}$ rule, respectively.

$\mathsf{U9}$ Under the constraint of $\mathsf{U9}$-rule, there is no object $s$ such that $s \sqsubset A$ and $s \vDash B$ hold on $\mathcal{D}$, i.e., in the following description of $\mathsf{rel}(\mathcal{D})$, the dotted line between $Y/y$ and $A$ should not be $\rightarrow$, and the dotted line between $Z/z$ and $B$ should not be $\vDash$. According to $\mathsf{U9}$-rule of Definition A.1, $\mathsf{rel}(\mathcal{D} + (A \sqsubset B))$ is represented by the right hand graph below.




$\mathsf{rel}(\mathcal{D})$




$\mathsf{rel}(\mathcal{D} + (A \sqsubset B))$

Observe that, after the unification, some of the dotted lines of $\mathsf{rel}(\mathcal{D})$ are fixed to $\rightarrow$ or $\vdash\!\dashv$ in $\mathsf{rel}(\mathcal{D} + (A \sqsubset B))$ according to Definition A.1. We need to check that $\mathsf{rel}(\mathcal{D} + (A \sqsubset B))$ is an EUL-structure; for example, if the dotted line between $A$ and $X$ in $\mathsf{rel}(\mathcal{D})$ is $A \vdash\!\dashv X$ (or $A \leftarrow X$), after the application of U9-rule, there are two incompatible edges $\vdash\!\dashv$ (resp. $\leftarrow$) and $\rightarrow$ between $A$ and $X$, which violates the irreflexivity of the $\vdash\!\dashv$-relation of EUL-structure. It is shown that, because of our constraint for U9-rule, the dotted line between $A$ and $X$ is $\bowtie$ (i.e., no edge) or $\rightarrow$. Observe that, if we have $A \vdash\!\dashv X$ in $\mathsf{rel}(\mathcal{D})$, by the $\vdash\!\dashv$-downward closedness of $\mathsf{rel}(\mathcal{D})$, we have $Z/z \vdash\!\dashv B$ in $\mathsf{rel}(\mathcal{D})$, which contradicts the constraint. If we have $A \leftarrow X$ in $\mathsf{rel}(\mathcal{D})$, by the transitivity of $\mathsf{rel}(\mathcal{D})$, we have $A \leftarrow B$ in $\mathsf{rel}(\mathcal{D})$, which contradicts the presupposition of U9-rule, i.e., there is no edge between $A$ and $B$ in $\mathsf{rel}(\mathcal{D})$. Thus the dotted line between $A$ and $X$ should be $\bowtie$ (i.e., no edge) or $\rightarrow$, either of which is compatible with the edge $A \rightarrow X$ in $\mathsf{rel}(\mathcal{D} + (A \sqsubset B))$. Similarly, it is shown that the other dotted lines of $\mathsf{rel}(\mathcal{D})$ are compatible with the edges of $\mathsf{rel}(\mathcal{D} + (A \sqsubset B))$. Then it is easily checked that $\mathsf{rel}(\mathcal{D} + (A \sqsubset B))$ satisfies Definition 3.1, and hence it is an EUL-structure.

U10 Under the constraint of U10-rule, there is no object $s$ such that $s \sqsubset A$ and $s \sqsubset B$ hold on $\mathcal{D}$, i.e., in the following graph of $\mathsf{rel}(\mathcal{D})$, the dotted line between $Z'/z'$ and $A$ (and also between $Z/z$ and $B$) should not be $\rightarrow$. According to U10-rule, $\mathsf{rel}(\mathcal{D} + (A \vdash\!\dashv B))$ is represented by by the right hand graph below.

$\mathsf{rel}(\mathcal{D})$ $\qquad$ $\mathsf{rel}(\mathcal{D} + (A \vdash\!\dashv B))$

We show that there are no incompatible edges in $\mathsf{rel}(\mathcal{D} + (A \vdash\!\dashv B))$. For the dotted line between $Z/z$ and $B$, it is not $\rightarrow$ by the constraint for U10-rule. Furthermore, assume to the contrary that we have $Z/z \leftarrow B$ in $\mathsf{rel}(\mathcal{D})$. Then, by the transitivity of $\mathsf{rel}(\mathcal{D})$, we have $A \leftarrow B$ in $\mathsf{rel}(\mathcal{D})$, which contradicts the presupposition of U10-rule, i.e., there is no edge between $A$ and $B$. Hence the dotted line between $Z/z$ and $B$ should be $\bowtie$ (i.e., no edge) or $\vdash\!\dashv$, either of which is compatible with the edge $Z/z \vdash\!\dashv B$ in $\mathsf{rel}(\mathcal{D} + (A \vdash\!\dashv B))$. Similarly, it is shown that the other two dotted lines of $\mathsf{rel}(\mathcal{D})$ are compatible with the edges of $\mathsf{rel}(\mathcal{D} + (A \vdash\!\dashv B))$. Then it is easily checked that $\mathsf{rel}(\mathcal{D} + (A \vdash\!\dashv B))$ satisfies Definition 3.1, and hence it is an EUL-structure. ■

For a given EUL-structure $(D, p(D), \sqsubset, \vdash\!\dashv)$, it can be shown that there is an EUL-diagram $\mathcal{D}$ such that $\mathsf{rel}(\mathcal{D})$ is equivalent to $(D, p(D), \sqsubset, \vdash\!\dashv)$.

# 4 A hierarchy of EUL-diagrams and Venn diagrams

The representation system EUL is extended by introducing new diagrammatic objects, intersection, union, and complement regions, respectively. Extended systems are stratified in terms of their expressive powers.

In what follows, we do not mention named points explicitly, since any named point of EUL can be regarded as a special circle, which does not contain, nor cross, any other objects. If we allow a point (as a special circle) to cross other circles, it amounts to adopting linking between points, although it is slightly restricted compared with usual linking as in Shin [19], among others. [5]

We first extend EUL by considering intersection regions as diagrammatic objects. *Regions* of an EUL-diagram are defined recursively as usual, which are closed under intersection, union, and complement, provided that each is non-empty in a diagram. See, e.g., [10].

**Definition 4.1** A non-empty region $r$ of an EUL-diagram $\mathcal{D}$ is an **intersection region** when, for some $\{A_1, \ldots, A_n\} \subseteq cr(\mathcal{D})$, $r = \bigcap_{1 \le i \le n} in(A_i)$, where $in(A_i)$ is the interior of circle $A_i$. An EUL-**diagrams with intersections** $\mathcal{D}$ is an EUL-diagram where each intersection region $r = \bigcap_{1 \le i \le n} in(A_i)$ has the name $\sqcap_{1 \le i \le n} A_i$, which is sometimes denoted by $\sqcap A_n$ for short. (In particular when $n = 1$, $\sqcap A_1 = A_1$.)

Note that, in an EUL-diagram with intersections, a region may have two names: For example, when $A \sqsubset B$ holds on $\mathcal{D}$, circle $A$ has another name, $A \sqcap B$.

We define an algebraic structure for EUL-diagrams with intersections.

**Definition 4.2** An EUL-**structure with greatest lower bounds (glbs)** $(D, \sqsubset, \vdash\!\dashv, \sqcap)$ is an EUL-structure, where for any subset $\{A_1, \ldots, A_n\} \subseteq D$ such that $\neg\exists_{1 \le j,k \le n}(A_j \vdash\!\dashv A_k$ holds on $D)$, there is the greatest lower bound $\sqcap_{1 \le i \le n} A_i$.

Although we regard named points as special named circles, the operation $\sqcap$ is not applied to them.

**Lemma 4.3** *Let $\mathcal{D}$ be an* EUL*-diagram with intersections. The set of relations* $\mathsf{rel}(\mathcal{D})$ *gives rise to an* EUL*-structure with glbs.*

**Lemma 4.4 (**EUL $\prec$ EUL+$\sqcap$**)** *Let $(D, \sqsubset, \vdash\!\dashv)$ be an* EUL*-structure. It is extended, by introducing glbs, to an* EUL*-structure with glbs $(D^{\sqcap}, \sqsubset, \vdash\!\dashv, \sqcap)$.*

*Proof.* The domain $D^{\sqcap}$ is defined as follows:

$$D^{\sqcap} := D \cup \{\sqcap_{1 \le i \le n} A_i \mid \neg\exists_{1 \le j,k \le n}(A_j \vdash\!\dashv A_k \text{ holds on } D)\}$$

$\sqsubset$ and $\vdash\!\dashv$ relations on $D$ are preserved on $D^{\sqcap}$ and they are extended for any $\sqcap_{1 \le i \le n} A_i \in D^{\sqcap}$ as follows: Let $X, Y \in D^{\sqcap}$.

| | | |
|---|---|---|
| $\sqcap A_n \sqsubset \sqcap A_n$ | | |
| $X \sqsubset \sqcap A_n$ | *iff* | $X \sqsubset A_i$ for all $1 \le i \le n$ |
| $\sqcap A_n \sqsubset X$ | *iff* | $A_i \sqsubset X$ for some $1 \le i \le n$ |
| $X \vdash\!\dashv Y$ | *iff* | $X \sqcap Y \notin D^{\sqcap}$ |

[5] We exclude a crossing relation $c \bowtie d$ between distinct named points, since it amounts to $c = d$ *or* $c \neq d$ (cf. Remark 2.10) but we always assume $c \neq d$ in our framework.

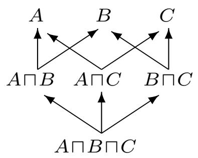
It is immediate that thus constructed $(\mathcal{D}^{\sqcap}, \sqsubset, \vdash\!\dashv, \sqcap)$ is an $\mathsf{EUL}$-structure, which satisfies Definition 3.1, and $\sqcap A_n$ is the glb of Definition 4.2. ■

See also Example 4.17.

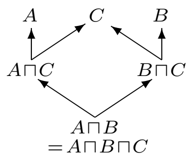
When $\mathcal{D}$ is an $\mathsf{EUL}$-diagram, we denote $\mathcal{D}^{\sqcap}$ an $\mathsf{EUL}$-diagram with intersections whose algebraic structure is constructed from the $\mathsf{EUL}$-structure $\mathsf{rel}(\mathcal{D})$ by Lemma 4.4. We say that the diagram $\mathcal{D}^{\sqcap}$ is **obtained from** $\mathcal{D}$.

By introducing intersection regions as diagrammatic objects, $\mathsf{EUL}$ with intersections are more expressive than the basic $\mathsf{EUL}$ of Section 2.1. Let us see the following example.

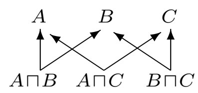
**Example 4.5 ($\mathsf{EUL}$-diagrams with intersections)** The three diagrams $\mathcal{D}_1, \mathcal{D}_2$, and $\mathcal{D}_3$ of Fig. 4 in Example 2.6, which are identified in the original $\mathsf{EUL}$, are distinguished when they are regarded as $\mathsf{EUL}$-diagrams with intersections. The difference among the three diagrams is more clearly seen by drawing their $\mathsf{EUL}$-structures with glbs. (Here, for reasons of simplicity, we omit the point $a$ and abbreviate $\vdash\!\dashv$-relation by stipulating that $X \vdash\!\dashv Y$ holds when $X \sqcap Y \notin \mathsf{rel}(\mathcal{D}^{\sqcap})$.)



$\mathsf{rel}(\mathcal{D}_1^{\sqcap})$



$\mathsf{rel}(\mathcal{D}_2^{\sqcap})$



$\mathsf{rel}(\mathcal{D}_3^{\sqcap})$

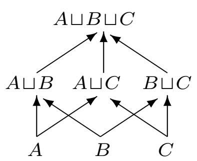
In a similar way as intersections, by considering union regions as diagrammatic objects we have another extension of $\mathsf{EUL}$.

**Definition 4.6** An $\mathsf{EUL}$**-diagrams with unions** $\mathcal{D}$ is an $\mathsf{EUL}$-diagram where each union region $r = \bigcup_{1\leq i\leq n} in(A_i)$ has the name $\sqcup_{1\leq i\leq n} A_i$, provided that it is connected.

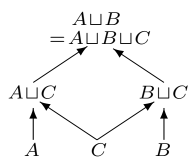
$\mathsf{EUL}$**-structures with least upper bounds (lubs)** for $\mathsf{EUL}$-diagrams with unions are defined in a similar way as $\mathsf{EUL}$-structures with glbs.

$\mathsf{EUL}$ with unions is also more expressive than $\mathsf{EUL}$.

**Example 4.7 ($\mathsf{EUL}$-diagrams with unions)** $\mathcal{D}_1$ and $\mathcal{D}_4$ of Fig. 4 in Example 2.6 are distinguished when they are regarded as $\mathsf{EUL}$-diagrams with unions. The $\mathsf{EUL}$-structures with lubs for these two diagrams are represented by the following different structures.



$\mathsf{rel}(\mathcal{D}_1^{\sqcup})$



$\mathsf{rel}(\mathcal{D}_4^{\sqcup})$

**Definition 4.8** An $\mathsf{EUL}$-**diagram with intersections and unions** $\mathcal{D}$ is an $\mathsf{EUL}$-diagram with intersections where union regions also have names.

Note that we only consider intersection (resp. union) regions of circles, and we exclude other regions such as $(A \cap B) \cup (C \cap D)$.

$\mathsf{EUL}$-structure with glbs and lubs are defined by combining $\mathsf{EUL}$-structure with glbs and $\mathsf{EUL}$-structure with lubs.

By considering the complement region of each circle as a diagrammatic object, we further introduce $\mathsf{EUL}$-diagrams with intersections, unions, and complements.

**Definition 4.9** An $\mathsf{EUL}$-**diagram with intersections, unions, and complements** $\mathcal{D}$ is an $\mathsf{EUL}$-diagram with intersections and unions, where each complement $\overline{A}$ of a circle $A$, i.e., the exterior of $A$, has the name $\overline{A}$.

$\mathsf{EUL}$-structures for $\mathsf{EUL}$-diagrams with $\cap, \cup, {}^{-}$ are defined as follows.

**Definition 4.10** An $\mathsf{EUL}$-**structure with glbs, lubs, and complements** $(D, \sqsubset, \vdash\!\dashv, \sqcap, \sqcup, {}^{-})$ is an $\mathsf{EUL}$-structure with glbs and lubs $(D, \sqsubset, \vdash\!\dashv, \sqcap, \sqcup)$ where, for each $A \in D$ which is not of the form $\sqcap C_j$ nor $\sqcup C_j$ ($j \geq 2$), the complement $\overline{A}$ of $A$ is defined in $D$.

Although we regard named points as special named circles, the operations $\sqcap, \sqcup$, and ${}^{-}$ are not applied to points.

**Lemma 4.11** *Let $\mathcal{D}$ be an $\mathsf{EUL}$-diagram with $\cap, \cup, {}^{-}$. The set of relations* $\mathsf{rel}(\mathcal{D})$ *gives rise to an $\mathsf{EUL}$-structure with glbs, lubs, and complements.*

**Lemma 4.12 ($\mathsf{EUL}{+}\sqcap{+}\sqcup \prec \mathsf{EUL}{+}\sqcap{+}\sqcup{+}{}^{-}$)** *Let $(D^{\Box}, \sqsubset, \vdash\!\dashv, \sqcap, \sqcup)$ be an $\mathsf{EUL}$-structure with glbs and lubs. It is extended, by introducing complements, to an $\mathsf{EUL}$-structure with glbs, lubs, and complements $(D^c, \sqsubset, \vdash\!\dashv, \sqcap, \sqcup, {}^{-})$.*

*Proof.* The domain $D^c$ is defined by adding complement $\overline{A}$ for each $A \in D^{\Box}$ which is not of the form $\sqcap C_j$ nor $\sqcup C_j$ ($j \geq 2$), and by extending glbs (of the form $(\sqcap B_j) \sqcap (\sqcap \overline{A_i})$) and lubs (of the form $(\sqcup B_j) \sqcup (\sqcup \overline{A_i})$) in a similar way as Lemma 4.4.
$\sqsubset$ and $\vdash\!\dashv$ relations on $D^{\Box}$ are preserved on $D^c$ and they are extended as follows:
For any $A, B \in D^c$ not of the form $\sqcap C_j$ nor $\sqcup C_j$ ($j \geq 2$),

$A \vdash\!\dashv \overline{A}$
$\overline{A} \sqsubset \overline{B}$ *iff* $B \sqsubset A$ in $D^{\Box}$
$A \sqsubset \overline{B}$ and $B \sqsubset \overline{A}$ *iff* $A \vdash\!\dashv B$ in $D^{\Box}$

For any $X, Y \in D^c$ of the form $(\sqcap B_j) \sqcap (\sqcap \overline{A_i})$ (resp. $(\sqcup B_j) \sqcup (\sqcup \overline{A_i})$), $\sqsubset$ and $\vdash\!\dashv$ relations are extended to be closed under $\sqcap$ and $\sqcup$ in a similar way as Lemma 4.4. ■

See also Example 4.17.

Euler/Venn diagrams of Swoboda-Allwein [23] are obtained by adding shading of minimal regions and linking of points to $\mathsf{EUL}$-diagrams with $\cap, \cup, {}^{-}$. [6]

[6] There are some differences between our system and Swoboda-Allwein's system: (i) we allow one circle to cross with another circle any number of times; (ii) we allow union regions as diagrammatic objects, which does not increase expressive power as compared to Swoboda-Allwein's system; (iii) we do not allow a circle to be completely shaded given our definition of semantics, where each circle denotes a non-empty set.

EUL-structures for Euler/Venn diagrams, which we call **Venn-structures**, are the directed acyclic graphs DAGs of Swoboda-Allwein [23].

**Lemma 4.13 (EUL$+\sqcap+\sqcup+^{-}$ $\prec$ Venn)** *Let* $(D^c, \sqsubset, \vdash\!\dashv, \sqcap, \sqcup, ^{-})$ *be an* EUL-*structure with glbs, lubs, and complements. It is extended to a Venn-structure* $D^v$ *of Swoboda-Allwein [23] by introducing shading and linking.*

When $\mathcal{D}$ is an EUL-diagram, we denote by $\mathcal{D}^v$ (resp. $\mathcal{D}^{\sqcap}, \mathcal{D}^{\sqcup}, \mathcal{D}^{\square}, D^c$) an Euler/Venn diagram (resp. EUL-diagram with intersections, unions, intersections and unions, intersections and unions and complements) whose algebraic structure is constructed from the EUL-structure $\mathsf{rel}(\mathcal{D})$ by Lemma 4.4, 4.12, and 4.13. We say that the diagram $\mathcal{D}^v$ (resp. $\mathcal{D}^{\sqcap}, \mathcal{D}^{\sqcup}, \mathcal{D}^{\square}, D^c$) is **obtained from** $\mathcal{D}$.

Various extensions of EUL introduced so far can be summarized by the following EUL-**hierarchy**:

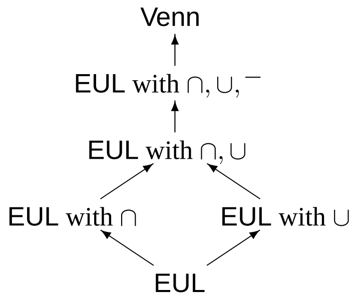



Fig.5 EUL-hierarchy

Note that the semantics of EUL of Section 2.2 is essentially the same as the semantics of Venn diagrams (e.g. [10, 19]), where the interpretation function $I$ of circles is naturally extended to interpret regions: $I(\sqcap X_i) = \bigcap I(X_i)$, $I(\sqcup X_i) = \bigcup I(X_i)$, and $I(\overline{A}) = U \setminus I(A)$. Note that the denotations of intersections, unions, and complements are not assumed to be non-empty, while those of circles and points are non-empty.

Thus when $\mathcal{D}^*$ is a diagram obtained from an EUL-diagram $\mathcal{D}$ for $* \in \{\sqcap, \sqcup, \square, c, v\}$, $\mathcal{D}$ and $\mathcal{D}^*$ are semantically equivalent since any relation of $\mathcal{D}$ is preserved in $\mathcal{D}^*$ by constructions given in Lemmas 4.4, 4.12, and 4.13:

**Lemma 4.14** *Let* $\mathcal{D}$ *be an* EUL-*diagram. For each* $* \in \{\sqcap, \sqcup, \square, c, v\}$*, let* $\mathcal{D}^*$ *be a diagram obtained from* $\mathcal{D}$*. For any model* $M$*,* $M \models \mathcal{D}^*$ *if and only if* $M \models \mathcal{D}$*.*

Based on Lemma 4.14, it is shown that each system of EUL-hierarchy is conservative over any lower system with respect to validity. We denote by $\boldsymbol{\mathcal{D}}$ a sequence of diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$.

**Proposition 4.15 (Semantic conservativity)** *Let* $S'$ *and* $S$ *be any systems of the* EUL-*hierarchy such that* $S'$ *is an extension of* $S$*. Let* $\boldsymbol{\mathcal{D}}, \mathcal{E}$ *be diagrams of* $S$*, and* $\boldsymbol{\mathcal{D}}^*, \mathcal{E}^*$ *be diagrams of* $S'$ *obtained from* $\boldsymbol{\mathcal{D}}, \mathcal{E}$ *for* $* \in \{\sqcap, \sqcup, \square, c, v\}$*. Then* $\boldsymbol{\mathcal{D}}^* \models \mathcal{E}^*$ *iff* $\boldsymbol{\mathcal{D}} \models \mathcal{E}$*.*

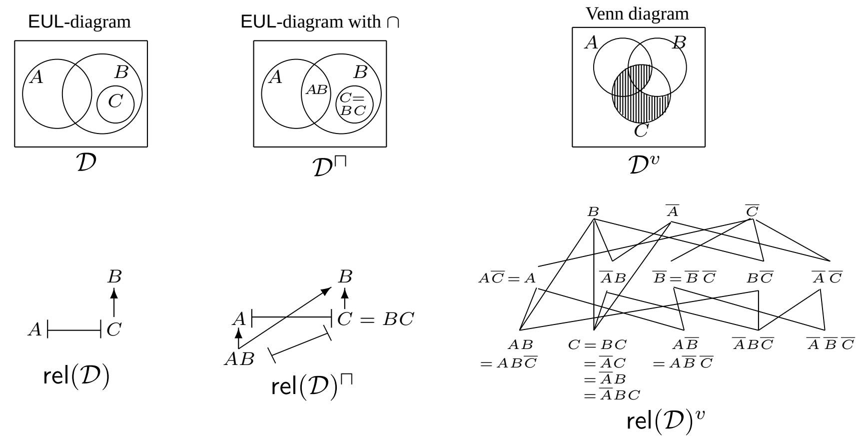
In parallel to the extensions of representation system EUL, we can obtain extended inference systems of GDS of Appendix A. It can be shown that each extended system is a conservative extension of the most elementary GDS with respect to provability. In particular, for Euler/Venn diagrammatic inference system of Swoboda-Allwein [24], we have the following conservativity theorem:

**Theorem 4.16 (Conservativity)** *Let $\boldsymbol{\mathcal{D}}$ and $\mathcal{E}$ be* EUL*-diagrams such that $\boldsymbol{\mathcal{D}}$ has a model. If $\mathcal{E}^v$ is provable from $\boldsymbol{\mathcal{D}}^v$ in Euler/Venn diagrammatic system, then $\mathcal{E}$ is provable from $\boldsymbol{\mathcal{D}}$ in* GDS*.*

*Proof.* Let $\boldsymbol{\mathcal{D}}^v \vdash \mathcal{E}^v$ in Euler/Venn diagrammatic inference system. By soundness (cf. [24]) we have, for any model $M$, $M \models \boldsymbol{\mathcal{D}}^v \Rightarrow M \models \mathcal{E}^v$. Assume $M \models \boldsymbol{\mathcal{D}}$. Then we have $M \models \boldsymbol{\mathcal{D}}^v$ by Lemma 4.14. Thus we have $M \models \mathcal{E}^v$, that is, $M \models \mathcal{E}$. Hence, by the completeness (Theorem A.2) of GDS, we have $\boldsymbol{\mathcal{D}} \vdash \mathcal{E}$ in GDS. ∎

The constructions of extensions of EUL-structures given in Lemma 4.4, 4.12, and 4.13 provide a procedure to transform an EUL-diagram to a Venn diagram. Let us see the following example:

**Example 4.17** Let $\mathcal{D}$ be an EUL-diagram such that $\mathsf{rel}(\mathcal{D}) = \{A \bowtie B, A \vdash\!\dashv C, C \sqsubset B\}$. By transforming the EUL-structure $\mathsf{rel}(\mathcal{D})$ through an EUL-structure with glbs $\mathsf{rel}(\mathcal{D})^{\sqcap}$, we obtain a Venn-structure $\mathsf{rel}(\mathcal{D})^v$. In $\mathsf{rel}(\mathcal{D})^{\sqcap}$ and $\mathsf{rel}(\mathcal{D})^v$ below, we omit $\sqcap$ symbol and write $AB$ for $A \sqcap B$. In $\mathsf{rel}(\mathcal{D})^v$, we further omit lubs and $\vdash\!\dashv$-relation, and represent arrows by lines in a hierarchical structure. By extracting minimal unshaded regions ($AB\overline{C}, \overline{A}BC, A\overline{B}\,\overline{C}, \overline{A}B\overline{C}, \overline{A}\,\overline{B}\,\overline{C}$) from $\mathsf{rel}(\mathcal{D})^v$, we obtain a Venn diagram $\mathcal{D}^v$, which is semantically equivalent to the original EUL-diagram $\mathcal{D}$.




In this paper, we introduced a hierarchy of Euler and Venn diagrammatic reasoning systems in terms of their expressive powers in our topological-relation-based formalization. Because of the space limitation in this paper, we discuss our extensions of EUL mainly at the level of representation and semantics. This is why our conservativity results for these systems (Proposition 4.15) are kept at the level of semantics. We leave our explicit formalization of diagrammatic inference systems for EUL-diagrams with intersections, with unions, with complements, respectively, as future work.

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# A Diagrammatic inference system GDS

In this section, we review Generalized Diagrammatic Syllogistic inference system GDS of [13, 12], which consists of two inference rules: *unification* and *deletion*. In order to motivate our definition of *unification*, let us consider the following question: Given the following diagrams $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$ of Fig. 6, what diagrammatic information on $A, B$ and $c$ can be obtained? (In what follows, in order to avoid notational complexity in a diagram, we express each named point, say $\overset{c}{\bullet}$, simply by its name $c$.) Fig. 6 represents a way of solving the question.

In Fig. 6, at the first step, two diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ are unified to obtain $\mathcal{D}_1 + \mathcal{D}_2$, where point $c$ in $\mathcal{D}_1$ and $\mathcal{D}_2$ are identified, and $B$ is added to $\mathcal{D}_1$ so that $c$ is inside of $B$ and $B$ overlaps with $A$ without any implication of a relationship between $A$ and $B$. We formalize such cases, where two given diagrams share one object, by U1–U8 rules of group (I) of Definition A.1. At the second step, $\mathcal{D}_1 + \mathcal{D}_2$ is combined with another diagram $\mathcal{D}_3$ to obtain $(\mathcal{D}_1 + \mathcal{D}_2) + \mathcal{D}_3$. Note that the diagrams $\mathcal{D}_1 + \mathcal{D}_2$ and $\mathcal{D}_3$ share two circles $A$ and $B$: $A \bowtie B$ holds on $\mathcal{D}_1 + \mathcal{D}_2$ and $A \sqsubset B$ holds on $\mathcal{D}_3$. Since the semantic information of $A \sqsubset B$ on $\mathcal{D}_3$ is more accurate than that of $A \bowtie B$ on $\mathcal{D}_1 + \mathcal{D}_2$, according to our semantics of EUL (recall that $A \bowtie B$ means just "true" in

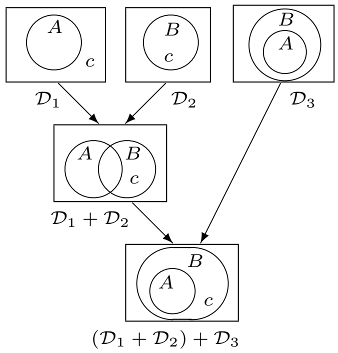



Fig. 6 Unification

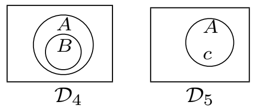



Fig. 7 Indeterminacy

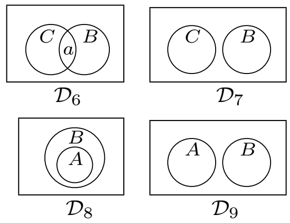



Fig. 8 Inconsistency

our semantics), one keeps the relation $A \sqsubset B$ in the unified diagram $(\mathcal{D}_1 + \mathcal{D}_2) + \mathcal{D}_3$. We formalize such cases, where two given diagrams share two objects, by U9–U10 rules of group (II) of Definition A.1. Observe that the unified diagram $(\mathcal{D}_1 + \mathcal{D}_2) + \mathcal{D}_3$ of Fig. 6 represents the information of these diagrams $\mathcal{D}_1, \mathcal{D}_2$, and $\mathcal{D}_3$, that is, their *conjunction*.

We impose two kinds of constraints on unification. One is the *constraint for determinacy*, which blocks the disjunctive ambiguity with respect to locations of named points. For example, two diagrams $\mathcal{D}_4$ and $\mathcal{D}_5$ in Fig. 7 are not permitted to be unified into one diagram since the location of the point $c$ is not determined (it can be inside $B$ or outside $B$). The other is the *constraint for consistency*, which blocks representing inconsistent information in a single diagram. For example, the diagrams $\mathcal{D}_6$ and $\mathcal{D}_7$ (resp. $\mathcal{D}_8$ and $\mathcal{D}_9$) in Fig. 8 are not permitted to be unified since they contradict each other. Recall that each circle is interpreted by non-empty set in our semantics of Definition 2.8, and hence $\mathcal{D}_8$ and $\mathcal{D}_9$ are also incompatible. [7]

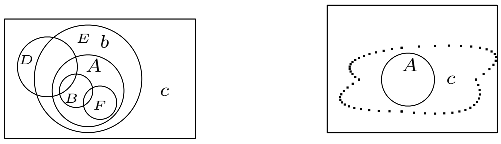
We formalize our unification [8] of two diagrams by restricting one of them to be a *minimal diagram*, except for one rule called the Point Insertion-rule. Our completeness (Theorem A.2) ensures that any diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$ may be unified, under the constraints for determinacy and consistency, into one diagram whose semantic information is equivalent to the conjunction of that of $\mathcal{D}_1, \ldots, \mathcal{D}_n$. We give a formal description of inference rules in terms of EUL-relations: Given a diagram $\mathcal{D}$ and a minimal diagram $\alpha$, the set of relations $\mathsf{rel}(\mathcal{D} + \alpha)$ for the unified diagram $\mathcal{D} + \alpha$ is defined. It is easily checked that the set $\mathsf{rel}(\mathcal{D} + \alpha)$ satisfies the properties of Lemma 2.4 according to our constraints for determinacy and consistency, and hence locations of points are determined in a unified diagram. (See also Section 3, where we give a

[7] In place of our syntactic constraint, it is possible to allow unification of inconsistent diagrams such as $\mathcal{D}_6$ and $\mathcal{D}_7$ (resp. $\mathcal{D}_8$ and $\mathcal{D}_9$) by extending GDS with an inference rule corresponding to the absurdity rule of Gentzen's natural deduction system: We can infer any diagram from a pair of inconsistent diagrams. (For natural deduction systems, see, for example, [3, 17].) Such a rule is introduced in, for example, [10] for spider diagrams; [7] for Venn diagrams; [23, 24] for Euler/Venn diagrams. However, such a rule requires linguistic symbol, say $\perp$, or some arbitrary convention to represent inconsistency, and hence we prefer our syntactic constraint in our framework of a diagrammatic inference system.

[8] The following definition of inference rules of GDS is slightly different from that of [13, 12] since we regard $\sqsubset$-relation as reflexive relation in this paper.

graph-theoretical representation of unification.)

For a better understanding of our unification rule, we also give a schematic diagrammatic representation and a concrete example of each rule. In the schematic representation of diagrams, to indicate the occurrence of some objects in a context on a diagram, we write the indicated objects explicitly and indicate the context by “dots” as in the diagram to the right below. [9] For example, when we need to indicate only $A$ and $c$ on the left hand diagram, we could write it as shown on the right.




**Definition A.1** **Axiom**, **unification**, and **deletion** of GDS are defined as follows.

**Axiom:**

A1: For any circles $A$ and $B$, any minimal diagram where $A \bowtie B$ holds is an axiom.

A2: Any EUL-diagram which consists only of points is an axiom.

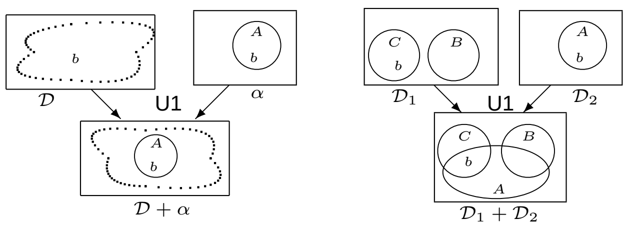
**Unification:** We denote by $\mathcal{D}+\alpha$ the unified diagram of $\mathcal{D}$ with a minimal diagram $\alpha$. $\mathcal{D}+\alpha$ is defined when $\mathcal{D}$ and $\alpha$ share one or two objects. We distinguish the following two cases: (I) When $\mathcal{D}$ and $\alpha$ share one object, they may be unified to $\mathcal{D}+\alpha$ by rules U1–U8 according to the shared object and the relation holding on $\alpha$. Each rule of (I) has a constraint for determinacy. (II) When $\mathcal{D}$ and $\alpha$ share two circles, if the relation which holds on $\alpha$ also holds on $\mathcal{D}$, $\mathcal{D}+\alpha$ is $\mathcal{D}$ itself; otherwise, they may be unified to $\mathcal{D}+\alpha$ by rules U9 or U10 according to the relation holding on $\alpha$. Each rule of (II) has a constraint for consistency. Moreover, there is another unification rule called the Point Insertion-rule (III).

**(I)** The case $\mathcal{D}$ and $\alpha$ share one object:

U1: If $b \sqsubset A$ holds on $\alpha$ and $pt(\mathcal{D}) = \{b\}$, then $\mathcal{D}$ and $\alpha$ may be unified to a diagram $\mathcal{D}+\alpha$ such that the set $\mathsf{rel}(\mathcal{D}+\alpha)$ of relations holding on it is the following:

$$\mathsf{rel}(\mathcal{D}) \cup \{b \sqsubset A\} \cup \{A \bowtie X \mid X \in cr(\mathcal{D})\}$$

U1 is applied as follows:

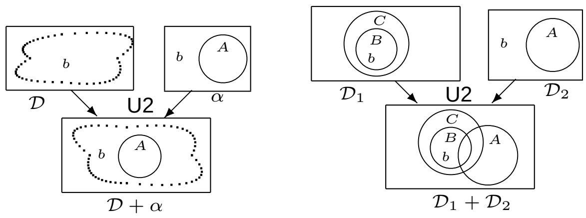



[9] Note that the dots notation is used only for abbreviation of a given diagram. For a formal treatment of such “backgrounds” in a diagram, see, for example, Meyer [11].

U2: If $b \vDash A$ holds on $\alpha$ and $pt(\mathcal{D}) = \{b\}$, then $\mathcal{D}$ and $\alpha$ may be unified to a diagram $\mathcal{D} + \alpha$ such that the set $\mathsf{rel}(\mathcal{D} + \alpha)$ of relations holding on it is the following:

$$\mathsf{rel}(\mathcal{D}) \cup \{b \vDash A\} \cup \{A \bowtie X \mid X \in cr(\mathcal{D})\}$$

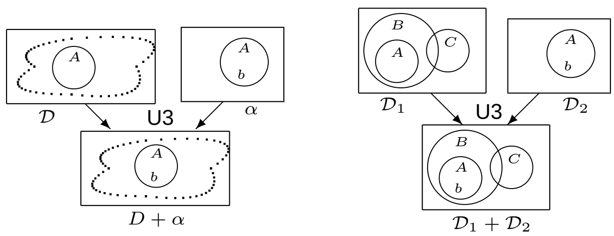
U2 is applied as follows:




U3: If $b \sqsubset A$ holds on $\alpha$ and $A \in cr(\mathcal{D})$, and if $A \sqsubset X$ or $A \vDash X$ holds for all circle $X$ in $\mathcal{D}$, then $\mathcal{D}$ and $\alpha$ may be unified to a diagram $\mathcal{D} + \alpha$ such that the set of relations $\mathsf{rel}(\mathcal{D} + \alpha)$ is the following:

$$\mathsf{rel}(\mathcal{D}) \cup \{b \sqsubset X \mid A \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \cup \{b \vDash X \mid A \vDash X \in \mathsf{rel}(\mathcal{D})\}$$
$$\cup \{b \vDash x \mid x \in pt(\mathcal{D})\}$$

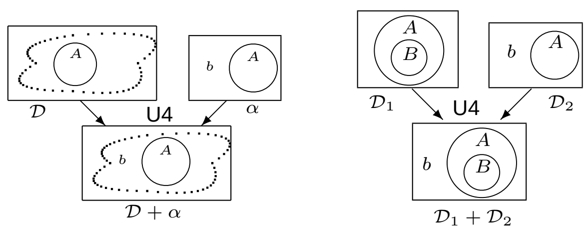
U3 is applied as follows:




U4: If $b \vDash A$ holds on $\alpha$ and $A \in cr(\mathcal{D})$, and if $X \sqsubset A$ holds for all circle $X$ in $\mathcal{D}$, then $\mathcal{D}$ and $\alpha$ may be unified to a diagram $\mathcal{D} + \alpha$ such that the set of relations $\mathsf{rel}(\mathcal{D} + \alpha)$ is the following:

$$\mathsf{rel}(\mathcal{D}) \cup \{b \vDash X \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \cup \{b \vDash x \mid x \in pt(\mathcal{D})\}$$

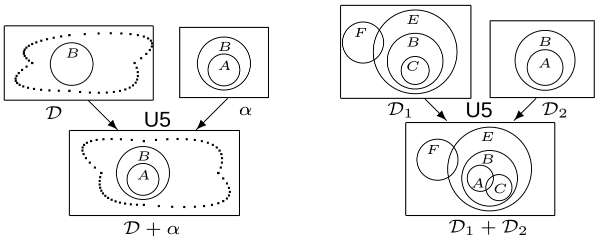
U4 is applied as follows:



U5: If $A \sqsubset B$ holds on $\alpha$ and $B \in cr(\mathcal{D})$, and if $x \mathrel{\vdash\!\dashv} B$ holds for all $x \in pt(\mathcal{D})$, then $\mathcal{D}$ and $\alpha$ may be unified to a diagram $\mathcal{D} + \alpha$ such that the set of relations $\mathsf{rel}(\mathcal{D} + \alpha)$ is the following:

$$\begin{aligned}\mathsf{rel}(\mathcal{D}) &\cup \{A \sqsubset X \mid B \sqsubset X \in \mathsf{rel}(\mathcal{D})\}\\ &\cup \{A \bowtie X \mid X \sqsubset B \text{ or } X \bowtie B \in \mathsf{rel}(\mathcal{D})\}\\ &\cup \{A \mathrel{\vdash\!\dashv} X \mid X \mathrel{\vdash\!\dashv} B \in \mathsf{rel}(\mathcal{D})\} \cup \{x \mathrel{\vdash\!\dashv} A \mid x \in pt(\mathcal{D})\}\end{aligned}$$

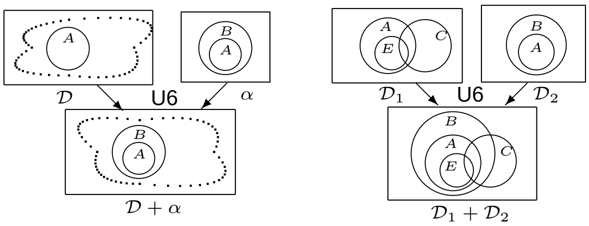
U5 is applied as follows:




U6: If $A \sqsubset B$ holds on $\alpha$ and $A \in cr(\mathcal{D})$, and if $x \sqsubset A$ holds for all $x \in pt(\mathcal{D})$, then $\mathcal{D}$ and $\alpha$ may be unified to a diagram $\mathcal{D} + \alpha$ such that the set of relations $\mathsf{rel}(\mathcal{D} + \alpha)$ is the following:

$$\begin{aligned}\mathsf{rel}(\mathcal{D}) &\cup \{X \sqsubset B \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \cup \{x \sqsubset B \mid x \in pt(\mathcal{D})\}\\ &\cup \{X \bowtie B \mid A \sqsubset X \text{ or } A \mathrel{\vdash\!\dashv} X \text{ or } A \bowtie X \in \mathsf{rel}(\mathcal{D})\}\end{aligned}$$

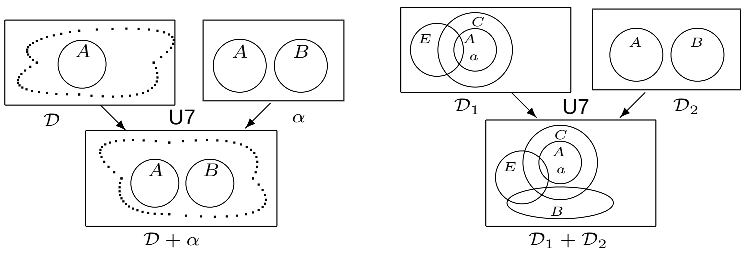
U6 is applied as follows:




U7: If $A \mathrel{\vdash\!\dashv} B$ holds on $\alpha$ and $A \in cr(\mathcal{D})$, and if $x \sqsubset A$ holds for all $x \in pt(\mathcal{D})$, then $\mathcal{D}$ and $\alpha$ may be unified to a diagram $\mathcal{D} + \alpha$ such that the set of relations $\mathsf{rel}(\mathcal{D} + \alpha)$ is the following:

$$\begin{aligned}\mathsf{rel}(\mathcal{D}) &\cup \{X \mathrel{\vdash\!\dashv} B \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \cup \{x \mathrel{\vdash\!\dashv} B \mid x \in pt(\mathcal{D})\}\\ &\cup \{X \bowtie B \mid A \sqsubset X \text{ or } A \mathrel{\vdash\!\dashv} X \text{ or } A \bowtie X \in \mathsf{rel}(\mathcal{D})\}\end{aligned}$$

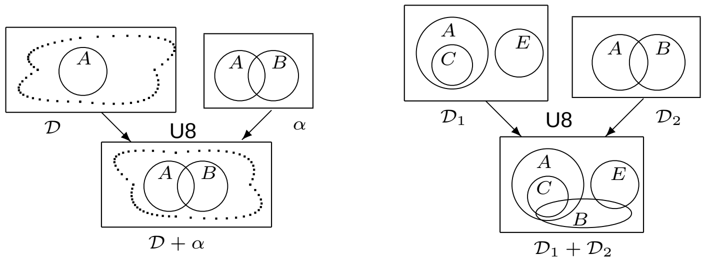
U7 is applied as follows:



U8: If $A \bowtie B$ holds on $\alpha$ and $A \in cr(\mathcal{D})$, and if $pt(\mathcal{D}) = \emptyset$, then $\mathcal{D}$ and $\alpha$ may be unified to a diagram $\mathcal{D} + \alpha$ such that the set of relations $\mathsf{rel}(\mathcal{D} + \alpha)$ is the following:

$$\mathsf{rel}(\mathcal{D}) \cup \{X \bowtie B \mid X \in cr(\mathcal{D})\}$$

U8 is applied as follows:

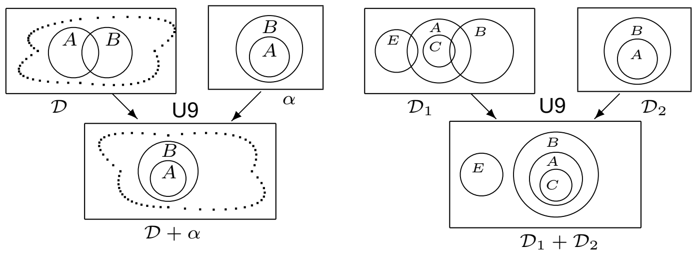



**(II)** When $\mathcal{D}$ and $\alpha$ share two circles, they may be unified to $\mathcal{D} + \alpha$ by the following U9 and U10 rules.

U9: If $A \sqsubset B$ holds on $\alpha$ and $A \bowtie B$ holds on $\mathcal{D}$, and if there is no object $s$ such that $s \sqsubset A$ and $s \mathrel{\vdash\!\dashv} B$ hold on $\mathcal{D}$, then $\mathcal{D}$ and $\alpha$ may be unified to a diagram $\mathcal{D} + \alpha$ such that the set of relations $\mathsf{rel}(\mathcal{D} + \alpha)$ is the following:

$$\big(\mathsf{rel}(\mathcal{D}) \setminus \{Y \bowtie X \mid Y \sqsubset A \text{ and } B \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \setminus \{X \bowtie Y \mid X \sqsubset A \text{ and } Y \mathrel{\vdash\!\dashv} B \in \mathsf{rel}(\mathcal{D})\}\big)$$
$$\cup \{Y \sqsubset X \mid Y \sqsubset A \text{ and } B \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \cup \{X \mathrel{\vdash\!\dashv} Y \mid X \sqsubset A \text{ and } Y \mathrel{\vdash\!\dashv} B \in \mathsf{rel}(\mathcal{D})\}$$

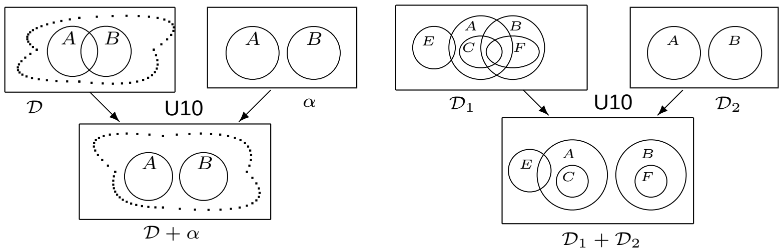
U9 is applied as follows:




U10: If $A \mathrel{\vdash\!\dashv} B$ holds on $\alpha$ and $A \bowtie B$ holds on $\mathcal{D}$, and if there is no object $s$ such that $s \sqsubset A$ and $s \sqsubset B$ hold on $\mathcal{D}$, then $\mathcal{D}$ and $\alpha$ may be unified to a diagram $\mathcal{D} + \alpha$ such that the set of relations $\mathsf{rel}(\mathcal{D} + \alpha)$ is the following:

$$\big(\mathsf{rel}(\mathcal{D}) \setminus \{X \bowtie Y \mid X \sqsubset A \text{ and } Y \sqsubset B \in \mathsf{rel}(\mathcal{D})\}\big) \cup \{X \mathrel{\vdash\!\dashv} Y \mid X \sqsubset A \text{ and } Y \sqsubset B \in \mathsf{rel}(\mathcal{D})\}$$

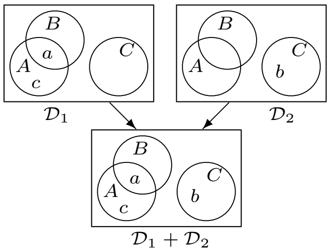
U10 is applied as follows:



**(III)** Point Insertion: If, for any circles $X, Y$ and for any $\square \in \{\sqsubset, \sqsupset, \vdash\!\dashv, \bowtie\}$, $X \square Y \in$ $\mathsf{rel}(\mathcal{D}_1)$ *iff* $X \square Y \in \mathsf{rel}(\mathcal{D}_2)$ holds, and if $pt(\mathcal{D}_2)$ is a singleton $\{b\}$ such that $b \notin pt(\mathcal{D}_1)$, then $\mathcal{D}_1$ and $\mathcal{D}_2$ may be unified to a diagram $\mathcal{D}_1 + \mathcal{D}_2$ such that the set of relations $\mathsf{rel}(\mathcal{D}_1 + \mathcal{D}_2)$ is the following:

$$\mathsf{rel}(\mathcal{D}_1) \cup \mathsf{rel}(\mathcal{D}_2) \cup \{b \vdash\!\dashv x \mid x \in pt(\mathcal{D}_1)\}$$

Point Insertion is applied as follows:




**Deletion:** When $t$ is an object of $\mathcal{D}$, $t$ may be deleted from $\mathcal{D}$ to obtain a diagram $\mathcal{D} - t$ under the constraint that $\mathcal{D} - t$ has at least one objects.

The notion of *diagrammatic proofs (or, d-proofs)* is defined inductively as tree structures consisting of unification and deletion steps. The provability relation between EUL-diagrams is defined as usual. We denote by $\boldsymbol{\mathcal{D}}$ a sequence of diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$.

**Theorem A.2 (Soundness and completeness of GDS [13])** *Let $\boldsymbol{\mathcal{D}}, \mathcal{E}$ be EUL-diagrams, and let $\boldsymbol{\mathcal{D}}$ have a model. $\mathcal{E}$ is a semantically valid consequence of $\boldsymbol{\mathcal{D}}$ ($\boldsymbol{\mathcal{D}} \models \mathcal{E}$), if, and only if, there is a d-proof of $\mathcal{E}$ from $\boldsymbol{\mathcal{D}}$ ($\boldsymbol{\mathcal{D}} \vdash \mathcal{E}$) in GDS.*