

A decision procedure for a two-sorted extension of Multi-Level Syllogistic with the Cartesian product and some map constructs

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Abstract. We present a nondeterministic polynomial-time decision procedure for an extension of multi-level syllogistic with the singleton operator, the Cartesian product, various map constructs, and special predicates asserting respectively that a map term is single valued, injective, and bijective, but with no map image or domain operators. We also prove that the extension of multi-level syllogistic with a map image operator has an EXPTIME-hard decision problem, via a reduction of the decision problem for the description logic \mathcal{ALC} .

Keywords: Decision procedures, multi-level syllogistics, Cartesian product, map constructs, NP-completeness, EXPTIME-hardness.

1 Introduction

The decision problem in set theory has been intensively investigated in the last decades. The initial motivations can be traced back in 1978, when Jack Schwartz¹ envisaged an interactive proof checker supposed to accept sequences of logical formulae, such that any formula in the sequence follows logically from earlier formulae (cf. [13]). To keep at a reasonable level the mass of details that a user must key in, he argued that such a system should include among other fundamental components also an *inferential core*, comprised of an extensive (and extendible) collection of efficient (semi-)decision procedures to handle elementary inferential steps. Additionally, the language of set theory, which pervades most mathematical reasoning, seemed to be the most appealing one for such an enterprise.

Multi-level syllogistic (in short, MLS) has been the first unquantified sublanguage of set theory to be studied in this context. We recall that MLS involves besides set variables and the ‘null set’ constant \emptyset , also the common Boolean set operators \cup , \cap , \setminus , the set predicates \in , \subseteq , and $=$, and the propositional connectives.

Over the years, decision procedures or proofs of undecidability have been provided for several extensions of MLS and also for various quantified fragments

¹ January 9, 1930 – March 2, 2009.

of set theory, contributing to the rise of the field of *Computable Set Theory* (see [2, 5] for a thorough account of the state-of-the-art until 2001). It is to be noticed, however, that several decision procedures found so far are not practical at all and their interest is limited to the foundational purpose of identifying the boundary between the decidable and the undecidable in set theory.

The quite recent implementation of the proof-verification system *ÆtnaNova/Referee* described in [11, 6, 10] has given a new impulse to the investigation of effective² decision procedures with the goal of enhancing its inferential capabilities. For instance, [3] presents some commonly occurring decidable extensions of MLS.

In this paper we present an effective procedure (namely, having a nondeterministic polynomial-time complexity) for an extension of MLS with the singleton operator, the Cartesian product, and various map constructs, which we denote $\text{MLSS}_{2,m}^\times$. More specifically, $\text{MLSS}_{2,m}^\times$ is a two-sorted language with set and map variables. Set variables range over sets of the von Neumann cumulative hierarchy of sets and map variables range over collections of pairs of sets (this acceptance of the term ‘map’ agrees with the map data structure in the SETL language; see [14]). Since map variables and set variables have different sorts, we do not need to be specific on the internal representation of pairs, as long as the basic property of pairs holds:

$$[x, y] = [x', y'] \iff x = x' \wedge y = y'.$$

Map terms can be formed by means of the Cartesian product, various map restriction operators, a map inverse operator, and the Boolean set operators on maps. Additionally, the language allows assertions meaning that a map term is single-valued, or injective, or bijective. However, for efficiency reasons, domain and range operators are not allowed, since, as we will show in Section 4, their presence triggers the EXPTIME-hardness of the decision problem.

A similar two-sorted language, extended with domain and range operators, but with no Cartesian product and no Boolean set operators among map terms, has been considered in [9], but the decision procedure proposed there had a double exponential time complexity. The language in [9] has been subsequently extended with various topological constructs, without disrupting decidability.

We mention also a one-sorted version of the language considered in [9], further extended with map evaluation (cf. [7]). In this case, since there is no distinction between set and map variables, maps (i.e., sets) can be combined with the Boolean set operators as well. However, the language in [7] does not allow a predicate $\text{ISMAP}(x)$, asserting that x is a collection of pairs. Therefore, for instance, a predicate of type $\text{INVERSE}(f, g)$ expresses only that g is an inverse of f , up to non-pair elements, so that $\text{INVERSE}(f, g)$ and $\text{INVERSE}(f, g')$ do not imply that $g = g'$, but only that g and g' contain the same pairs (namely the inverse of

² Needless to say, any decision procedure for a language at least as expressive as propositional logic has at least a nondeterministic polynomial-time complexity. This is the case for all extensions of MLS. Thus, our meaning of ‘effective’ is just nondeterministic polynomial-time.

the pairs contained in f). Despite the peculiarity of such semantics, the decision procedures given in [7] has a nondeterministic exponential time complexity.

The paper is organized as follows. Section 2 gives the precise syntax and semantics of the language $\text{MLSS}_{2,m}^\times$ of our interest. A decision procedure for $\text{MLSS}_{2,m}^\times$ and its complexity analysis are then reported in Section 3. In Section 4 we prove that the extension of MLS with the image operator has an EXPTIME -hard decision problem. Finally, we briefly give some hints to future work in Section 5.

2 The language $\text{MLSS}_{2,m}^\times$

2.1 Syntax

$\text{MLSS}_{2,m}^\times$ (Multi-Level Syllogistic with singleton, Cartesian product, and various map constructs) is a two-sorted language which contains:

- (i) a countably infinite collection of *set variables* $\text{Vars}_s = \{x, y, z, \dots\}$;
- (ii) a countably infinite collection of *map variables* $\text{Vars}_m = \{f, g, h, \dots\}$;
- (iii) the predicate symbols $\in, =, \subseteq, \text{injective}(\cdot), \text{single_valued}(\cdot), \text{bijective}(\cdot)$;
- (iv) the operator symbols $\cap, \cup, \setminus, \times$ (Cartesian product), $\{\cdot\}$ (singleton), $(\cdot)_\perp, (\cdot)_\uparrow, (\cdot)_\downarrow$ (map restriction operators), $(\cdot)^{-1}$ (map inverse);
- (v) the constant \emptyset (empty set);
- (vi) parentheses (to construct compound terms);
- (vii) the logical connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ (to construct compound formulae).

$\text{MLSS}_{2,m}^\times$ -*set terms* are recursively defined as follows:

1. each set variable and the constant \emptyset are set terms;
2. if X, Y are set terms, then so are $X \cup Y, X \cap Y, X \setminus Y, \{X\}$.

$\text{MLSS}_{2,m}^\times$ -*map terms* are recursively defined as follows:

1. each map variable is a map term;
2. if X, Y are set terms and F, G are map terms then
 - $X \times Y$,
 - $F_{X|}, F_{|Y}, F_{X|Y}$ (left, right, and left-right restriction, respectively),
 - $F^{-1}, F \cup G, F \cap G, F \setminus G$
are map terms.

$\text{MLSS}_{2,m}^\times$ -*atomic formulae* are defined from set terms and map terms in the following way. Let X, Y be set terms and F, G be map terms, then

1. $X \in Y, X \subseteq Y, X = Y$,
2. $[X, Y] \in F, F \subseteq G, F = G$,
3. $\text{injective}(F), \text{single_valued}(F), \text{bijective}(F)$

are atomic formulae. Observe that a formula of type $[X, Y] \in F$ is to be considered just as a ternary relationship among X , Y , and F , since we did not introduce any set term of type $[X, Y]$.

Then the set of $\text{MLSS}_{2,m}^\times$ -formulae is the smallest set containing all the $\text{MLSS}_{2,m}^\times$ -atomic formulae and that is closed with respect to the logical connectives \neg , \wedge , \vee , \rightarrow , \leftrightarrow . Usually, to denote $\text{MLSS}_{2,m}^\times$ -formulae we will use the metavariables φ and ψ .

For an $\text{MLSS}_{2,m}^\times$ -formula φ , we denote by $\text{Vars}_s(\varphi)$ and $\text{Vars}_m(\varphi)$ the collections of set variables and of map variables occurring in φ , respectively, and we put $\text{Vars}(\varphi) =_{\text{def}} \text{Vars}_s(\varphi) \cup \text{Vars}_m(\varphi)$.

2.2 Semantics

The semantics of $\text{MLSS}_{2,m}^\times$ is based upon the von Neumann standard cumulative hierarchy \mathcal{V} of sets defined by:

$$\begin{aligned} \mathcal{V}_0 &= \emptyset \\ \mathcal{V}_{\gamma+1} &= \mathcal{P}(\mathcal{V}_\gamma), \quad \text{for each ordinal } \gamma \\ \mathcal{V}_\lambda &= \bigcup_{\mu < \lambda} \mathcal{V}_\mu, \quad \text{for each limit ordinal } \lambda \\ \mathcal{V} &= \bigcup_{\gamma \in \text{On}} \mathcal{V}_\gamma, \end{aligned}$$

where, given a set S , $\mathcal{P}(S)$ is the power set of S , and On indicates the class of all ordinals. It can be proved that the membership relation is well-founded in \mathcal{V} and, therefore, no membership cycle can occur in \mathcal{V} .

We denote with $\text{rank}(u)$ the least ordinal γ such that $u \subseteq \mathcal{V}_\gamma$ (i.e., $u \in \mathcal{V}_{\gamma+1}$), for every set u in \mathcal{V} .

An *assignment* is a total function $M : \text{Vars}_s \cup \text{Vars}_m \rightarrow \mathcal{V}$ that maps each set and map variable into a set of the von Neumann hierarchy \mathcal{V} , such that Mf is a set of ordered pairs, for any $f \in \text{Vars}_m$. It is not relevant which representation is chosen for ordered pairs, as long as the basic ordered pair property holds. One possibility could be to adopt Kuratowski's definition and put $[u, v] =_{\text{def}} \{\{u\}, \{u, v\}\}$, for all sets u and v .

An assignment M is said to be *injective with respect to a set of variables* $S \subseteq \text{Vars}_s$, if $Mx \neq My$, for all distinct variables x, y in S .

Assignments extend naturally to set terms and map terms. For instance, we have:

$$\begin{aligned} M(X \times Y) &= \{[u, v] : u \in MX \wedge v \in MY\}, \\ MF_{X|} &= \{[u, v] : [u, v] \in MF \wedge u \in MX\}, \\ MF|_Y &= \{[u, v] : [u, v] \in MF \wedge v \in MY\}, \\ MF_{X|Y} &= \{[u, v] : [u, v] \in MF \wedge u \in MX \wedge v \in MY\}, \end{aligned}$$

where X, Y are set terms and F is a map term.³

³ For simplicity, in the following we will use the same operator and predicate symbols also at the semantic level. So, for instance, for any map m and set s , $m_s|$ will denote the map $\{[u, v] : [u, v] \in m \wedge u \in s\}$. This will generate no confusion.

Assignments evaluate atomic formulae of type $X \in Y$, $X \subseteq Y$, $X = Y$, $F \subseteq G$, and $F = G$, to a truth value **true** or **false** in the standard way. Atomic formulae of the other types are evaluated as follows:

$$\begin{aligned} M([X, Y] \in F) \text{ is } \mathbf{true} &\iff [MX, MY] \in MF, \\ M(\text{single_valued}(F)) \text{ is } \mathbf{true} &\iff (\forall u, v, v')(\{[u, v], [u, v']\} \subseteq MF \rightarrow v = v'), \\ M(\text{injective}(F)) \text{ is } \mathbf{true} &\iff (\forall u, u', v)(\{[u, v], [u', v]\} \subseteq MF \rightarrow u = u'), \\ M(\text{bijective}(F)) \text{ is } \mathbf{true} &\iff (\forall u, v, u', v')(\{[u, v], [u', v']\} \subseteq MF \\ &\rightarrow (u = u' \leftrightarrow v = v')). \end{aligned}$$

Finally, evaluation of compound formulae by an assignment M follows the standard rules of propositional logic.

Let M be an assignment and φ a formula of $\text{MLSS}_{2,m}^\times$. We say that M *satisfies* φ (and write $M \models \varphi$) if M evaluates φ to **true**. In this case M is said to be a *model* for φ . A formula φ is *satisfiable* if there is an assignment that satisfies it. Two formulae φ and ψ are *equisatisfiable*, when φ is satisfiable if and only if ψ is also satisfiable. A formula φ is *injectively satisfiable with respect to a set of variables* S if there is an injective assignment with respect to the set of variables S that satisfies it.

The *satisfiability problem* for $\text{MLSS}_{2,m}^\times$ is then the problem of establishing for any given formula of $\text{MLSS}_{2,m}^\times$ whether it is satisfiable or not.

Since the evaluation of an $\text{MLSS}_{2,m}^\times$ -formula by an assignment depends solely on the values which it assumes over the variables occurring in the formula, in most cases we will deal with partial assignments, though we will refer to them just as ‘assignments’.

By way of a simple normalization process of the type described in [2, §3.8], it is easy to show that the satisfiability problem for $\text{MLSS}_{2,m}^\times$ -formulae reduces to the satisfiability problem for *normalized* $\text{MLSS}_{2,m}^\times$ -conjunctions, namely conjunctions of $\text{MLSS}_{2,m}^\times$ -literals of the types reported in Table 1.⁴ For instance, to see that literals of type $\text{single_valued}(f)$ and $\neg\text{single_valued}(f)$ can be dismissed, it is enough to observe that $\neg\text{single_valued}(f)$ is equisatisfiable with

$$[x, y] \in f \wedge [x, y'] \in f \wedge y \in z \wedge y' \notin z,$$

where x, y, y', z are newly introduced set variables, and that $\text{single_valued}(f)$ is equivalent to $\text{injective}(f^{-1})$, which in turn is equisatisfiable with $g = f^{-1} \wedge \text{injective}(g)$, where g is a newly introduced map variable.

3 A decision procedure

In this section we describe a nondeterministic polynomial-time decision procedure for the satisfiability problem for normalized $\text{MLSS}_{2,m}^\times$ -conjunctions. The key idea behind such procedure is that a normalized $\text{MLSS}_{2,m}^\times$ -conjunction is

⁴ Plainly, the expressions $x \notin y$ and $[x, y] \notin f$ in Table 1 are shorthands for the literals $\neg(x \in y)$ and $\neg([x, y] \in f)$, respectively.

$x \in y$	$x \notin y$	$[x, y] \in f$	$[x, y] \notin f$	$y = \{x\}$
$x = y \cup z$	$x = y \setminus z$	$f = g \cup h$	$f = g \setminus h$	$f = x \times y$
$g = f_x $	$g = f^{-1}$	$\text{injective}(f)$		

Table 1. Literals in normalized $\text{MLSS}_{2,m}^\times$ -conjunctions.

satisfiable if and only if there exists a finite structure, to be introduced later, which witnesses this fact, in a sense that will be clarified below. In addition, the total size of such structure has a bound depending solely on the size of the conjunction, This decision procedure is a generalization of the one presented in [1] for the extension MLSS of MLS with a singleton operator.

To begin with, it is convenient to introduce some notations and definitions.

Given a relation ϱ over a set N and a nonempty subset $V \subseteq N$, we say that ϱ is V -extensional if $\langle - \varrho a \rangle \neq \langle - \varrho a' \rangle$, for all distinct $a, a' \in V$, where

$$\langle - \varrho a \rangle =_{\text{def}} \{b \in N : b \varrho a\}.$$

Next, let $\widehat{\epsilon}$ be an acyclic relation over a finite set N and let $V \subseteq N$. We introduce a notion of height relative to V , for $x \in V$, with the following recursion:

$$V\text{-height}(x) = \begin{cases} 0 & \text{if } y \not\widehat{\epsilon} x, \text{ with } y \in V, \\ \max \{V\text{-height}(y) : y \in V \wedge y \widehat{\epsilon} x\} + 1 & \text{otherwise.} \end{cases}$$

We will use $\text{MLSS}_{2,m}^\times$ -relation systems to witness the satisfiability of normalized $\text{MLSS}_{2,m}^\times$ -conjunctions. These are defined as follows.

Definition 1 (MLSS $_{2,m}^\times$ -relation system). An $\text{MLSS}_{2,m}^\times$ -relation system is a tuple $\mathcal{G} = (V, T, F, \widehat{\epsilon}, \{\widehat{f} : f \in F\})$, where

- V and T are finite disjoint collections of set variables such that $V \neq \emptyset$,
- F is a finite collection of map variables,
- $\widehat{\epsilon}$ is an acyclic relation over $V \cup T$,
- $\{\widehat{f} : f \in F\}$ is a collection of relations over $V \cup T$. □

$\text{MLSS}_{2,m}^\times$ -relation systems allow to define special assignments, which are called *realizations*.

Definition 2 (Realization of an $\text{MLSS}_{2,m}^\times$ -relation system).

Let $\mathcal{G} = (V, T, F, \widehat{\epsilon}, \{\widehat{f} : f \in F\})$ be an $\text{MLSS}_{2,m}^\times$ -relation system and let $\{u_t : t \in T\}$ be a collection of sets. Then the realization of \mathcal{G} relative to $\{u_t : t \in T\}$ is the assignment \mathcal{R} over $(V \cup T) \cup F$ defined recursively by:

$$\begin{aligned} \mathcal{R}x &= \{\mathcal{R}z : z \widehat{\in} x\}, & \text{for } x \in V, \\ \mathcal{R}t &= \{\mathcal{R}z : z \widehat{\in} t\} \cup \{u_t\}, & \text{for } t \in T, \\ \mathcal{R}f &= \{[\mathcal{R}x, \mathcal{R}y] : x \widehat{f} y, \text{ for } x, y \in V \cup T\}, & \text{for } f \in F. \end{aligned} \quad \square$$

The following lemma states conditions on $\text{MLSS}_{2,m}^\times$ -relation systems which guarantee that it has realizations satisfying specific $\text{MLSS}_{2,m}^\times$ -literals. It will be used later to prove the correctness of the decision procedure outlined in Theorem 1 below.

Lemma 1. *Let $\mathcal{G} = (V, T, F, \widehat{\in}, \{\widehat{f} : f \in F\})$ be an $\text{MLSS}_{2,m}^\times$ -relation system and let $\{u_t : t \in T\}$ be a collection of sets such that*

- (a) $\widehat{\in}$ is V -extensional;
- (b) $u_t \neq u_{t'}$, for all distinct $t, t' \in T$;
- (c) $u_t \neq \mathcal{R}x$, for all $t \in T, x \in V \cup T$.

Then

- (i) \mathcal{R} is injective w.r.t. $V \cup T$ (i.e., $\mathcal{R}x \neq \mathcal{R}x'$, for all distinct $x, x' \in V \cup T$);
- (ii) $\mathcal{R}x \in \mathcal{R}x' \iff x \widehat{\in} x'$, for all $x, x' \in V \cup T$;
- (iii) $\mathcal{R}x = \mathcal{R}y \cup \mathcal{R}z \iff \langle _ \widehat{\in} x \rangle = \langle _ \widehat{\in} y \rangle \cup \langle _ \widehat{\in} z \rangle$, for all $x, y, z \in V$;
- (iv) $\mathcal{R}x = \mathcal{R}y \setminus \mathcal{R}z \iff \langle _ \widehat{\in} x \rangle = \langle _ \widehat{\in} y \rangle \setminus \langle _ \widehat{\in} z \rangle$, for all $x, y, z \in V$;
- (v) $\mathcal{R}x = \{\mathcal{R}y\} \iff \langle _ \widehat{\in} x \rangle = \{y\}$, for all $x \in V$ and $y \in V \cup T$;
- (vi) $[\mathcal{R}x, \mathcal{R}y] \in \mathcal{R}f \iff x \widehat{f} y$, for all $x, y \in V \cup T$ and $f \in F$;
- (vii) $\mathcal{R}f = \mathcal{R}x \times \mathcal{R}y \iff \widehat{f} = \langle _ \widehat{\in} x \rangle \times \langle _ \widehat{\in} y \rangle$, for all $x, y \in V$ and $f \in F$;
- (viii) $\mathcal{R}f = (\mathcal{R}g)_{\mathcal{R}x} \iff \widehat{f} = (\widehat{g})_{\langle _ \widehat{\in} x \rangle}$, for all $x \in V$ and $f, g \in F$;
- (ix) $\mathcal{R}f = \mathcal{R}g \cup \mathcal{R}h \iff \widehat{f} = \widehat{g} \cup \widehat{h}$, for all $f, g, h \in F$;
- (x) $\mathcal{R}f = \mathcal{R}g \setminus \mathcal{R}h \iff \widehat{f} = \widehat{g} \setminus \widehat{h}$, for all $f, g, h \in F$;
- (xi) $\mathcal{R}f = (\mathcal{R}g)^{-1} \iff \widehat{f} = (\widehat{g})^{-1}$, for all $f, g \in F$;
- (xii) $\text{injective}(\mathcal{R}f) \iff \text{injective}(\widehat{f})$, for every $f \in F$.

Proof. (i) To prove that \mathcal{R} is injective with respect to $V \cup T$ we reason as follows. Let x, x' be two distinct set variables in $V \cup T$. We have to prove that $\mathcal{R}x \neq \mathcal{R}x'$. To begin with, let us assume that $\{x, x'\} \cap T \neq \emptyset$, and let, without loss of generality, $x \in T$. If $\mathcal{R}x = \mathcal{R}x'$, then $u_x \in \mathcal{R}x'$, which by (c) implies $u_x = u_{x'}$, contradicting (b). Thus $\mathcal{R}x \neq \mathcal{R}x'$. Instead, if $x, x' \in V$, we proceed by induction on $\mu = \max(V\text{-height}(x), V\text{-height}(x'))$. In the base case $\mu = 0$, by V -extensionality, there must exist a $t \in T$ such that $t \widehat{\in} x$ if and only if $t \widehat{\in} x'$. Suppose, without loss of generality, that $t \widehat{\in} x$ and $t \not\widehat{\in} x'$. Thus $\mathcal{R}t \in \mathcal{R}x \setminus \mathcal{R}x'$. Indeed, if $\mathcal{R}t \in \mathcal{R}x'$, then there exists $y \in V \cup T$ such that $y \widehat{\in} x'$ and $\mathcal{R}t = \mathcal{R}y$. But then, as in the previous case, we would have that t and y have to be identical and therefore $t \widehat{\in} x'$, which is a contradiction. Thus $\mathcal{R}x \neq \mathcal{R}x'$. For the inductive step, suppose by contradiction that $\mathcal{R}x = \mathcal{R}x'$, whereas (i) is true for every distinct $y, y' \in V$ such that

$$\max(V\text{-height}(y), V\text{-height}(y')) < \max(V\text{-height}(x), V\text{-height}(x')). \quad (1)$$

By V -extensionality, since x and x' are distinct, there must be a $y \in V \cup T$ such that $y \hat{\in} x$ if and only if $y \notin x'$. Suppose, without loss of generality, that $y \hat{\in} x$ and $y \notin x'$. Since $\mathcal{R}x = \mathcal{R}x'$, there must be a $y' \in V \cup T$ such that $y' \hat{\in} x'$ and $\mathcal{R}y = \mathcal{R}y'$. But then, from the base case if $\{y, y'\} \cap T \neq \emptyset$, or from the inductive hypothesis if $y, y' \in V$, y and y' must be identical. This implies that $w \hat{\in} v'$, a contradiction. Thus we have $\mathcal{R}x \neq \mathcal{R}x'$ even if $x, x' \in V$, whenever x and x' are distinct.

Next we prove (ii). Plainly, if $x \hat{\in} x'$ then $\mathcal{R}x \in \mathcal{R}x'$. On the other hand, if $\mathcal{R}x \in \mathcal{R}x'$, then by (c) there must exist a $y \in V \cup T$ such that $y \hat{\in} x'$ and $\mathcal{R}x = \mathcal{R}y$. But then, by (i), x and y must coincide and therefore $x \hat{\in} x'$.

In order to prove (iii), let $x, y, z \in V$ and assume first that $\mathcal{R}x = \mathcal{R}y \cup \mathcal{R}z$. Then for every $x' \in V \cup T$ we have: $x' \in \langle _ \hat{\in} x \rangle \iff \mathcal{R}x' \in \mathcal{R}x \iff \mathcal{R}x' \in \mathcal{R}y \cup \mathcal{R}z \iff \mathcal{R}x' \in \mathcal{R}y$ or $\mathcal{R}x' \in \mathcal{R}z \iff x' \in \langle _ \hat{\in} y \rangle$ or $x' \in \langle _ \hat{\in} z \rangle \iff x' \in \langle _ \hat{\in} y \rangle \cup \langle _ \hat{\in} z \rangle$, so that $\langle _ \hat{\in} x \rangle = \langle _ \hat{\in} y \rangle \cup \langle _ \hat{\in} z \rangle$. Analogously one can prove that $\langle _ \hat{\in} x \rangle = \langle _ \hat{\in} y \rangle \cup \langle _ \hat{\in} z \rangle$ implies $\mathcal{R}x = \mathcal{R}y \cup \mathcal{R}z$.

Properties (iv) and (v) can be proved much as (iii).

(vi) follows immediately the very definition of $\mathcal{R}f$ and from the injectivity of the realization \mathcal{R} w.r.t. $V \cup T$, already proved in (i).

To prove (vii), let $f \in F$, $x, y \in V$ and suppose initially that $\mathcal{R}f = \mathcal{R}x \times \mathcal{R}y$. Let x', y' be such that $x' \hat{f} y'$. Then, by Definition 2, $[\mathcal{R}x', \mathcal{R}y'] \in \mathcal{R}f$, so that $\mathcal{R}x' \in \mathcal{R}x$ and $\mathcal{R}y' \in \mathcal{R}y$. Thus, (ii) yields $x' \hat{\in} x$ and $y' \hat{\in} y$, and therefore $\hat{f} \subseteq \langle _ \hat{\in} x \rangle \times \langle _ \hat{\in} y \rangle$. Similarly, it can be shown that $\langle _ \hat{\in} x \rangle \times \langle _ \hat{\in} y \rangle \subseteq \hat{f}$, which together with the previous inclusion implies $\hat{f} = \langle _ \hat{\in} x \rangle \times \langle _ \hat{\in} y \rangle$. Conversely, assume that $\hat{f} = \langle _ \hat{\in} x \rangle \times \langle _ \hat{\in} y \rangle$. By (i), (vi), and Definition 2, $[u, v] \in \mathcal{R}f$ holds if and only if there are $x', y' \in V \cup T$ such that $\mathcal{R}x' = u$, $\mathcal{R}y' = v$, and $x' \hat{f} y'$. From our assumption, it follows that $x' \hat{\in} x$ and $y' \hat{\in} y$. Thus, by (ii), we have that $u \in \mathcal{R}x$ and $v \in \mathcal{R}y$, so that $\mathcal{R}f \subseteq \mathcal{R}x \times \mathcal{R}y$. To show that $\mathcal{R}x \times \mathcal{R}y \subseteq \mathcal{R}f$, let $[u, v] \in \mathcal{R}x \times \mathcal{R}y$. Since $x, y \in V$, there exist x', y' in $V \cup T$ such that $\mathcal{R}x' = u$, $\mathcal{R}y' = v$, $x' \hat{\in} x$, and $y' \hat{\in} y$. Hence, by our assumption, $x' \hat{f} y'$, and therefore $[u, v] \in \mathcal{R}f$, proving that $\mathcal{R}x \times \mathcal{R}y \subseteq \mathcal{R}f$.

To prove (viii), assume first that $\mathcal{R}f = (\mathcal{R}g)_{\mathcal{R}x}$. Then $\hat{f} = (\hat{g})_{\langle _ \hat{\in} x \rangle}$ follows plainly from (ii) and (vi). Next suppose that $\hat{f} = (\hat{g})_{\langle _ \hat{\in} x \rangle}$ and let u, v be any two sets such that $[u, v] \in \mathcal{R}f$. Then there exist x', y' in $V \cup T$ such that $\mathcal{R}x' = u$, $\mathcal{R}y' = v$, and $x' \hat{f} y'$. But then, since $\hat{f} = (\hat{g})_{\langle _ \hat{\in} x \rangle}$, we have $x' \hat{g} v'$, and $x' \hat{\in} x$, so that $[u, v] \in \mathcal{R}g$ and $u \in \mathcal{R}x$, which in turn implies $[u, v] \in (\mathcal{R}g)_{\mathcal{R}x}$, and therefore $\mathcal{R}f \subseteq (\mathcal{R}g)_{\mathcal{R}x}$. The converse inclusion can be proved analogously, completing the proof of (viii).

Properties (ix), (x), and (xi) are direct consequences of (vi).

Finally, let us prove (xii). First suppose that $\mathcal{R}f$ is injective and let $x, y, y' \in V \cup T$ such that $x \hat{f} y$ and $x \hat{f} y'$. By (vi), $[\mathcal{R}x, \mathcal{R}y]$ and $[\mathcal{R}x, \mathcal{R}y']$ are in $\mathcal{R}f$. Thus, by the assumed injectivity of $\mathcal{R}f$ and by (i), it follows that $y = y'$, proving that the map \hat{f} is injective. Conversely, let us assume that \hat{f} is injective. If $[u, v]$ and $[u, v']$ are in $\mathcal{R}f$, then there must exist x, y, y' in $V \cup T$ such that $\mathcal{R}x = u$,

$\mathcal{R}y = v$, $\mathcal{R}y' = v'$, $x \hat{f} y$, and $x \hat{f} y'$. The injectivity of \hat{f} implies $y = y'$, so that $v = \mathcal{R}y = \mathcal{R}y' = v'$, proving that $\mathcal{R}f$ is injective too. \square

To any given assignment M and given collections of set and map variables, it is possible to associate a particular $\text{MLSS}_{2,m}^\times$ -relation system, called *canonical*, which represents all membership relations among pairs of sets (Mx, My) and pairs of type $([Mx, My], Mf)$, where x, y are set variables and f is a map variable belonging to the given collections of variables.

Definition 3 (Canonical $\text{MLSS}_{2,m}^\times$ -relation system). *Given*

- two disjoint finite collections V and T of set variables, such that $V \neq \emptyset$,
- a finite collection F of map variables,
- an assignment M over $(V \cup T) \cup F$, injective with respect to the variables in $V \cup T$,

then the canonical $\text{MLSS}_{2,m}^\times$ -relation system \mathcal{G}^M of M relative to (V, T, F) is the $\text{MLSS}_{2,m}^\times$ -relation system (cf. Definition 1)

$$\mathcal{G}^M =_{\text{Def}} \left(V, T, F, \hat{\epsilon}^M, \{\hat{f}^M: f \in F\} \right),$$

where

- $x \hat{\epsilon}^M y \iff Mx \in My$, for every $x, y \in V \cup T$,
- $x \hat{f}^M y \iff [Mx, My] \in Mf$, for every $x, y \in V \cup T$ and $f \in F$. \square

The following lemma states some properties of canonical $\text{MLSS}_{2,m}^\times$ -relation systems which will be used to prove the completeness of the decision procedure given in Theorem 1.

Lemma 2. *Let V, T, F, M , and $\mathcal{G}^M = \left(V, T, F, \hat{\epsilon}^M, \{\hat{f}^M: f \in F\} \right)$ be as in Definition 3. Then, the following properties hold:*

- (i) $\hat{\epsilon}^M$ is acyclic;
- (ii) if $Mx \in My$ then $x \hat{\epsilon}^M y$, for $x, y \in V \cup T$;
- (iii) if $Mx \notin My$ then $\neg(x \hat{\epsilon}^M y)$, for $x, y \in V \cup T$;
- (iv) if $Mx = My \cup Mz$ then $\langle _ \hat{\epsilon}^M x \rangle = \langle _ \hat{\epsilon}^M y \rangle \cup \langle _ \hat{\epsilon}^M z \rangle$, for $x, y, z \in V \cup T$;
- (v) if $Mx = My \setminus Mz$ then $\langle _ \hat{\epsilon}^M x \rangle = \langle _ \hat{\epsilon}^M y \rangle \setminus \langle _ \hat{\epsilon}^M z \rangle$, for $x, y, z \in V$;
- (vi) if $Mx = \{My\}$ then $\langle _ \hat{\epsilon}^M x \rangle = \{y\}$, for $x, y \in V \cup T$;
- (vii) if $[Mx, My] \in Mf$ then $x \hat{f}^M y$, for $x, y \in V \cup T$ and $f \in F$;
- (viii) if $[Mx, My] \notin Mf$ then $\neg(x \hat{f}^M y)$, for $x, y \in V \cup T$ and $f \in F$;
- (ix) if $Mf = Mx \times My$ then $\hat{f}^M = \langle _ \hat{\epsilon}^M x \rangle \times \langle _ \hat{\epsilon}^M y \rangle$, for $x, y \in V \cup T$ and $f \in F$;
- (x) if $Mf = (Mg)_{Mx}$ then $\hat{f}^M = (\hat{g}^M)_{\langle _ \hat{\epsilon}^M x \rangle}$, for $f \in F$ and $x \in V$;
- (xi) if $Mf = Mg \cup Mh$ then $\hat{f}^M = \hat{g}^M \cup \hat{h}^M$, for $f, g, h \in F$;
- (xii) if $Mf = Mg \setminus Mh$ then $\hat{f}^M = \hat{g}^M \setminus \hat{h}^M$, for $f, g, h \in F$;
- (xiii) if $Mf = (Mg)^{-1}$ then $\hat{f}^M = (\hat{g}^M)^{-1}$, for $f, g \in F$;

(xiv) if injective(Mf) then injective(\widehat{f}^M), for $f \in F$.

Proof. (i) plainly follows from the definition of $\widehat{\in}^M$ and the acyclicity of the membership relation on sets. The remaining statements are immediate consequences of Definition 3. \square

The decidability of the satisfiability problem for normalized $\text{MLSS}_{2,m}^\times$ -conjunctions is proved in the following theorem.

Theorem 1. *Let φ be a normalized $\text{MLSS}_{2,m}^\times$ -conjunction, $V = \text{Vars}_s(\varphi)$, and $F = \text{Vars}_m(\varphi)$. Then φ is injectively satisfiable w.r.t. V if and only if there exist:*

- a finite collection $T = \{t_1, \dots, t_m\}$ of set variables, disjoint from V , with $m = |T| \leq |V| - 1$,
- an $\text{MLSS}_{2,m}^\times$ -relation system $\mathcal{G} = (V, T, F, \widehat{\in}, \{\widehat{f} : f \in F\})$

such that

- (a) $\widehat{\in}$ is V -extensional;
- (b) if $x \in y$ occurs in φ then $x \widehat{\in} y$, for $x, y \in V$;
- (c) if $x \notin y$ occurs in φ then $x \not\widehat{\in} y$, for $x, y \in V$;
- (d) if $[x, y] \in f$ occurs in φ then $x \widehat{f} y$, for $x, y \in V, f \in F$;
- (e) if $[x, y] \notin f$ occurs in φ then $[x, y] \notin \widehat{f}$, for $x, y \in V, f \in F$;
- (f) if $x = y \cup z$ occurs in φ then $\langle _ \widehat{\in} x \rangle = \langle _ \widehat{\in} y \rangle \cup \langle _ \widehat{\in} z \rangle$, for $x, y, z \in V$;
- (g) if $x = y \setminus z$ occurs in φ then $\langle _ \widehat{\in} x \rangle = \langle _ \widehat{\in} y \rangle \setminus \langle _ \widehat{\in} z \rangle$, for $x, y, z \in V$;
- (h) if $f = g \cup h$ occurs in φ then $\widehat{f} = \widehat{g} \cup \widehat{h}$, for $f, g, h \in F$;
- (i) if $f = g \setminus h$ occurs in φ then $\widehat{f} = \widehat{g} \setminus \widehat{h}$, for every $f, g, h \in F$;
- (j) if $f = x \times y$ occurs in φ then $\widehat{f} = \langle _ \widehat{\in} x \rangle \times \langle _ \widehat{\in} y \rangle$, for $x, y \in V, f \in F$;
- (k) if $f = g_x$ occurs in φ then $\widehat{f} = \widehat{f}_{\langle _ \widehat{\in} x \rangle}$, for $x \in V, f \in F$;
- (l) if injective(f) occurs in φ then \widehat{f} is injective, for every $x \in V, f \in F$;

Proof. We show first that the conditions of the theorem are sufficient for φ to be injectively satisfiable w.r.t. V . Thus, let $T = \{t_1, \dots, t_m\}$ and $\mathcal{G} = (V, T, F, \widehat{\in}, \{\widehat{f} : f \in F\})$ be as in the hypotheses and such that (a)–(l) hold.

We use Lemma 1 to prove that some realization of the $\text{MLSS}_{2,m}^\times$ -relation system \mathcal{G} is an injective model for φ w.r.t. V . To this end, it is enough to exhibit a collection of sets $\{u_{t_i} : t_i \in T\}$ such that

- (A) $u_{t_i} \neq u_{t_j}$, for all distinct $t_i, t_j \in T$, and
- (B) $u_{t_i} \neq \mathcal{R}x$, for all $t_i \in T$ and $v \in V \cup T$,

where \mathcal{R} is the realization of \mathcal{G} relative to $\{u_{t_i} : t_i \in T\}$ and (V, T) . Let us put $u_{t_i} =_{\text{Def}} \{n, i\}$, for $t_i \in T$, where $n = |V|$.⁵ The sets u_{t_i} are obviously pairwise

⁵ We are assuming the von Neumann representation of natural numbers $0 =_{\text{Def}} \emptyset$, $1 =_{\text{Def}} \{0\}$, $2 =_{\text{Def}} \{0, 1\}$, and so on.

distinct. In addition, we have $\text{rank}(u_{t_i}) = n + 1$, for $u_{t_i} \in T$. Hence, to prove (B), we can just show that $\text{rank}(\mathcal{R}x) \neq n + 1$, for each $x \in V \cup T$. Thus, let $x \in V \cup T$. If $u_{t_i} \in^+ \mathcal{R}x$ (where \in^+ denotes the transitive closure of the membership relation) then $\text{rank}(\mathcal{R}v) > n + 1$. On the other hand, if $u_{t_i} \notin^+ \mathcal{R}x$ for all $u_{t_i} \in T$, by induction on $V\text{-height}(x)$ it follows that $\text{rank}(\mathcal{R}x) < n$.

Properties (b)–(l) and Lemma 1 guarantee that \mathcal{R} satisfies all literals in φ . For instance, if the literal $x \in y$ occurs in φ , then by (b) we have $x \hat{\in} y$, so that Lemma 1(ii) yields $\mathcal{R}x \in \mathcal{R}y$. Reasoning in a similar way, one can show that \mathcal{R} satisfies all the conjuncts in φ . Additionally, by Lemma 1(i), \mathcal{R} is injective w.r.t. $V \cup T$, completing the proof of the sufficiency part of the theorem.

Conversely, let M be a model for φ which is injective w.r.t. the collection of variables $V = \text{Vars}_s(\varphi)$. We show next how to construct a collection $T = \{t_1, \dots, t_m\}$ of set variables with $m = |T| \leq |V| - 1$, and an $\text{MLSS}_{2,m}^\times$ -relation system $\mathcal{G} = (V, T, F, \hat{\in}, \{\hat{f} : f \in F\})$ such that conditions (a)–(l) of Lemma 1 are satisfied.

It is convenient to introduce the following notion (cf. [1]). We say that a set Σ *distinguishes* a set S if $s \cap \Sigma \neq s' \cap \Sigma$ holds for any two distinct $s, s' \in S$.

*Claim.*⁶ Any finite set S admits a set Σ which distinguishes it and such that $|\Sigma| \leq |S| - 1$.

Proof. If $|S| \leq 1$, our claim is vacuously true. Otherwise, let $|S| > 1$ and, inductively, let us assume that our claim holds for any set S' such that $|S'| < |S|$. Then, pick $s \in S$. By our inductive hypothesis the set $S \setminus \{s\}$ admits a set Σ' which distinguishes it and such that $|\Sigma'| \leq |S| - 2$. If Σ' distinguishes S , we are done. Otherwise, there is an $s' \in S \setminus \{s\}$ such that $s \cap \Sigma' = s' \cap \Sigma'$. Let $d \in (s \setminus s') \cup (s' \setminus s)$ and consider $\Sigma =_{\text{def}} \Sigma' \cup \{d\}$. We assert that Σ distinguishes S . Indeed, if this were not the case there would exist an $s'' \in S \setminus \{s, s'\}$ such that $s \cap \Sigma = s'' \cap \Sigma$, so that $s \cap \Sigma' = s'' \cap \Sigma'$ and, therefore, $s' \cap \Sigma' = s'' \cap \Sigma'$, contradicting our assumption that Σ' distinguishes $S \setminus \{s\}$. It only remains to observe that $|\Sigma| \leq |S| - 1$. \square

In view of the above claim, let Σ^M be a set which distinguishes $\{Mx : x \in V\}$ and is such that $|\Sigma^M| \leq |V| - 1$. Also, let $m = |\Sigma^M \setminus \{Mx : x \in V\}| \leq |V| - 1$ and let $T = \{t_1, \dots, t_m\}$ be a collection of m distinct set variables not belonging to V . Let us extend/redefine M over the variables in T in such a way that $\{Mt_i : t_i \in T\} = \Sigma^M \setminus \{Mx : x \in V\}$ and let $\mathcal{G}^M = (V, T, F, \hat{\in}^M, \{\hat{f}^M : f \in F\})$ be the canonical $\text{MLSS}_{2,m}^\times$ -relation system of M relative to (V, T) and F . From the well-foundedness of the membership relation in \mathcal{V} , we have immediately that $\hat{\in}^M$ is acyclic. Additionally, by construction, if $x, x' \in V$ are any two distinct variables, then there exists $y \in V \cup T$ such that $My \in ((Mx \setminus Mx') \cup (Mx' \setminus Mx))$. Hence $y \hat{\in}^M x \iff y \hat{\notin}^M x'$, proving the V -extensionality of $\hat{\in}^M$ (condition (a)). Conditions (b)–(l) follows from Lemma 2 and from the fact that M satisfies all the literals of φ . For instance, if $x \in y$ occurs in φ , then we have plainly

⁶ See [1]. [12] also provides an extension to infinite sets.

$Mx \in My$. Thus, by Lemma 2(ii), we have $x \hat{\in}^M y$. The remaining conditions can be proved much in the same way, concluding the proof that the conditions of the theorem are also necessary for the injective satisfiability of φ w.r.t. V . \square

Since an $\text{MLSS}_{2,m}^\times$ -conjunction ψ of literals of the types listed in Table 1 is satisfiable if and only if there exists an equivalence relation \sim on $\text{Vars}_s(\psi)$ such that $\tilde{\psi}$ is injectively satisfiable (w.r.t. $\text{Vars}_s(\tilde{\psi})$), where $\tilde{\psi}$ is the $\text{MLSS}_{2,m}^\times$ -conjunction obtained from ψ by identifying \sim -equivalent set variables, namely by replacing them by a common representative, we have the following result.

Corollary 1. *The ordinary satisfiability problem for the class of $\text{MLSS}_{2,m}^\times$ -formulae is solvable.* \square

3.1 Complexity issues

Theorem 1 leads to a nondeterministic polynomial-time decision test for the injective satisfiability problem for normalized $\text{MLSS}_{2,m}^\times$ -conjunctions (w.r.t. the collection of its set variables).

Lemma 3. *The injective satisfiability problem for normalized $\text{MLSS}_{2,m}^\times$ -conjunctions w.r.t. set variables is NP-complete.*

Proof. To begin with, the NP-hardness of the class of formulae of our interest follows immediately from the NP-hardness of MLSS (cf. [4]). Concerning its membership to NP, we reason as follows. Let φ be a normalized $\text{MLSS}_{2,m}^\times$ -conjunction which admits an injective model, relative to its set variables. This fact can be witnessed by the existence of an $\text{MLSS}_{2,m}^\times$ -relation system

$$\mathcal{G} = \left(\text{Vars}_s(\varphi), T, \text{Vars}_m(\varphi), \hat{\in}, \{\hat{f} : f \in \text{Vars}_m(\varphi)\} \right),$$

with $|T| < |\text{Vars}_s(\varphi)|$, such that conditions (a)–(l) of Theorem 1 are satisfied. Since the size of \mathcal{G} is at most quadratic in the size of φ and since conditions (a)–(l) can be verified in polynomial-time, it follows that the injective satisfiability problem for normalized $\text{MLSS}_{2,m}^\times$ -conjunctions is in NP and, therefore, is NP-complete. \square

In view of Corollary 1 and the remarks just before its statement, we have also the NP-completeness of the ordinary satisfiability problem for normalized $\text{MLSS}_{2,m}^\times$ -conjunctions. To this end, it is enough to observe that given an equivalence relation \sim on the set variables of a normalized $\text{MLSS}_{2,m}^\times$ -conjunction φ , then $\tilde{\varphi}$ can be computed in linear time (obviously, relative to the size of φ). Thus we have:

Corollary 2. *The ordinary satisfiability problem for normalized $\text{MLSS}_{2,m}^\times$ -conjunctions is NP-complete.*

3.2 Remarks on the domain and image operators

The same approach of Theorem 1 can not readily be applied to deal also with the extension of $\text{MLSS}_{2,m}^\times$ with literals of the type $x = \text{dom}(f)$, where the semantics of the $\text{dom}(\cdot)$ operator is the obvious one, namely

$$M(\text{dom}(f)) =_{\text{def}} \{s : [s, u] \in Mf, \text{ for some set } u\},$$

for any assignment M .

Consider for instance a formula φ containing the literals $x = \text{dom}(f_0)$ and $y \in x$ and let M be a model for φ . Let $\mathcal{G}^M = (V, T, F, \widehat{\in}^M, \{\widehat{f}^M : f \in F\})$ be the canonical $\text{MLSS}_{2,m}^\times$ -relation system of M , relative to a certain set T of auxiliary variables and let us assume that $Mx = \{My\}$ and $Mf_0 = \{[My, v]\}$, where $v \neq Mz$ for every set variable $z \in V \cup T$. Then $\widehat{f}_0^M = \emptyset$ and

$$\emptyset = \text{dom}(\widehat{f}_0^M) \neq \langle _ \widehat{\in} x \rangle = \{y\}.$$

Most likely, any extension of the decision test contained in Theorem 1 to deal also with literals of type $x = \text{dom}(f)$ will involve the introduction in T of exponentially many auxiliary set variables in the size of the input formula. This is supported by the fact that any decidable extension of MLS with the literals of type $y = f[x]$ (map image operator) is EXPTIME -hard (see next section) and the fact that the literal $y = f[x]$ can be expressed in any extension of $\text{MLSS}_{2,m}^\times$ with literals of type $x' = \text{dom}(f')$ by the literal

$$y = \text{dom}((f_{x|})^{-1}).$$

On the other hand, we observe that the language $\text{MLSS}_{2,m}^\times$ allows one to express literals of the types $\text{dom}(f) \subseteq x$ and $f[x] \subseteq y$, in view of the equivalences

$$\begin{aligned} \text{dom}(f) \subseteq x &\iff f = f_{x|} \\ f[x] \subseteq y &\iff f = f_{x|y}. \end{aligned}$$

4 ExpTime-hardness of MLS with the image operator

In this section we provide a complexity lower bound for the satisfiability problem of the decidable extension MLS_{Im} of MLS with literals of the type $y = f[x]$.⁷ The semantics of the image operator $f[x]$ is the obvious one, namely

$$M(f[x]) =_{\text{def}} \{v : [u, v] \in Mf, \text{ for some } u \in Mx\},$$

for any assignment M .

Our result is achieved by reducing the decision problem for the description logic \mathcal{ALC} to the satisfiability problem for MLS_{Im} .

\mathcal{ALC} is a two-sorted language which contains:

⁷ MLS_{Im} is a subfragment of an extension of MLSS with some map constructs, whose decidability has been proved in [9].

- a countably infinite collection of *concept names* $\mathcal{N}^c = \{A, B, \dots\}$,
- a countably infinite collection of *role names* $\mathcal{N}^r = \{P, Q, \dots\}$,
- the symbols \top, \perp , representing the *universal concept* and the *bottom concept*,
- the *concept constructors* \neg (complement), \sqcap (conjunction), \sqcup (disjunction), \forall (universal restriction), \exists (existential restriction) to form *complex concepts*.

\mathcal{ALC} -concepts are defined recursively as follows:

- concept names are concepts;
- \top, \perp are concepts;
- if C, D are concepts, then $\neg C$, $C \sqcap D$, and $C \sqcup D$ are concepts;
- if C is a concept and R is a role name, then $\forall R.C$ and $\exists R.C$ are concepts.

\mathcal{ALC} -axioms have the following forms:

- $C \sqsubseteq D$ (*inclusion axiom*),
- $C \equiv D$ (*equivalence axiom*),

where C, D are \mathcal{ALC} -concepts.

The semantics of \mathcal{ALC} is given in terms of interpretations.⁸ An *interpretation* \mathcal{I} consists of a nonempty set $\Delta^{\mathcal{I}}$, also called the *domain* of the interpretation, and of an *interpretation function* assigning to each concept name $A \in \mathcal{N}^c$ a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, and to every role name $R \in \mathcal{N}^r$, a binary relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$.

An interpretation \mathcal{I} extends recursively to concepts as follows:

$$\begin{aligned}
\top^{\mathcal{I}} &=_{\text{Def}} \Delta^{\mathcal{I}}, \\
\perp^{\mathcal{I}} &=_{\text{Def}} \emptyset, \\
(\neg C)^{\mathcal{I}} &=_{\text{Def}} \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}, \\
(C \sqcup D)^{\mathcal{I}} &=_{\text{Def}} C^{\mathcal{I}} \cup D^{\mathcal{I}}, \\
(C \sqcap D)^{\mathcal{I}} &=_{\text{Def}} C^{\mathcal{I}} \cap D^{\mathcal{I}}, \\
(\forall R.C)^{\mathcal{I}} &=_{\text{Def}} \{u \in \Delta^{\mathcal{I}} : (\forall v \in \Delta^{\mathcal{I}})([u, v] \in R^{\mathcal{I}} \longrightarrow v \in C^{\mathcal{I}})\}, \\
(\exists R.C)^{\mathcal{I}} &=_{\text{Def}} \{u \in \Delta^{\mathcal{I}} : (\exists v \in \Delta^{\mathcal{I}})([u, v] \in R^{\mathcal{I}} \wedge v \in C^{\mathcal{I}})\}.
\end{aligned}$$

Let \mathcal{I} be an interpretation and let C, D be two concepts. Then \mathcal{I} *satisfies* $C \sqsubseteq D$ (resp., $C \equiv D$) if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ (resp., $C^{\mathcal{I}} = D^{\mathcal{I}}$). In addition, let \mathcal{T} be a finite collection of axioms. Then \mathcal{I} *satisfies* \mathcal{T} if and only if it satisfies each axiom in \mathcal{T} ; also, \mathcal{I} satisfies the concept C with respect to \mathcal{T} if it satisfies \mathcal{T} and $C^{\mathcal{I}} \neq \emptyset$. An \mathcal{ALC} -concept C is said to be *satisfiable* with respect to a finite collection of axioms \mathcal{T} if there exists an interpretation \mathcal{I} that satisfies C with respect to \mathcal{T} , otherwise it is *unsatisfiable*. If \mathcal{I} satisfies a concept C (resp., a finite collection of axioms \mathcal{T}), then \mathcal{I} is said to be a *model* for C (resp., \mathcal{T}).

In [8, Theorem 3.27, page 132] the EXPTIME-hardness of the problem of deciding if a given concept C is unsatisfiable with respect to a given finite collection \mathcal{T} of inclusion axioms is proved for the sublogic \mathcal{AL} of \mathcal{ALC} . Hence, we have:

⁸ Here we recall just the descriptive semantic. There are several other semantics that are out of the scope of this paper.

Theorem 2. *The problem of deciding whether a given \mathcal{ALC} -concept C is unsatisfiable with respect to a given finite collection \mathcal{T} of \mathcal{ALC} -inclusion axioms is EXPTIME-hard. \square*

Next we show how the satisfiability problem for \mathcal{ALC} -concepts with respect to a finite set of axioms can be reduced to the satisfiability of MLS_{Im} -formulae. Since $\exists R.C \equiv \neg(\forall R.(\neg C))$, without loss of generality we can consider only concepts that do not contain any occurrence of universal restriction.

Theorem 3. *The satisfiability problem for MLS_{Im} is EXPTIME-hard.*

Proof. In view of Theorem 2, it is enough to exhibit a reduction from \mathcal{ALC} to MLS_{Im} . Thus, given a finite collection \mathcal{T} of \mathcal{ALC} -inclusion axioms and an \mathcal{ALC} -concept C , we show how to construct an MLS_{Im} -formula which is satisfiable if and only if the concept C is satisfiable w.r.t. \mathcal{T} .

Let $\text{Cpts} \subseteq \mathcal{N}^c$ and $\text{Rls} \subseteq \mathcal{N}^r$ be the collections of the concept names and of the role names, respectively, occurring in C and in \mathcal{T} . Additionally, let π be a function that injectively associates every concept name in Cpts to a set variable of the language MLS_{Im} and every role name in Rls to a map variable of MLS_{Im} . The function π extends naturally to concepts and axioms in the following recursive way:

$$\begin{aligned} \pi(\top) &=_{\text{Def}} \mathbb{U} \\ \pi(\perp) &=_{\text{Def}} \emptyset \\ \pi(\neg C) &=_{\text{Def}} \mathbb{U} \setminus \pi(C) \\ \pi(C \sqcap D) &=_{\text{Def}} \pi(C) \cap \pi(D) \\ \pi(C \sqcup D) &=_{\text{Def}} \pi(C) \cup \pi(D) \\ \pi(\exists R.C) &=_{\text{Def}} (\pi(R))[\pi(C)] \\ \pi(C \sqsubseteq D) &=_{\text{Def}} \pi(C) \subseteq \pi(D) \\ \pi(C \equiv D) &=_{\text{Def}} \pi(C) = \pi(D), \end{aligned}$$

where \mathbb{U} is a set variable of MLS_{Im} not in $\pi[\text{Cpts}]$.

Let $\varphi =_{\text{Def}} \psi_1 \wedge \psi_2 \wedge \psi_3$ be the MLS_{Im} -formula in which:

$$\begin{aligned} \psi_1 &=_{\text{Def}} \mathbb{U} \neq \emptyset \wedge \bigwedge_{A \in \text{Cpts}} \pi(A) \subseteq \mathbb{U} \wedge \bigwedge_{R \in \text{Rls}} (\pi(R))[\mathbb{U}] \subseteq \mathbb{U} \\ \psi_2 &=_{\text{Def}} \bigwedge_{\Gamma \in \mathcal{T}} \pi(\Gamma) \\ \psi_3 &=_{\text{Def}} \pi(C) \neq \emptyset. \end{aligned}$$

We observe that the size of the $\text{MLSS}_{2,m}^\times$ -formula φ is linear in the total size of \mathcal{T} and C .

Next we show that φ is satisfiable (relative to the semantics of MLS_{Im}) if and only if C is satisfiable w.r.t. \mathcal{T} (relative to the semantics of \mathcal{ALC}).

To begin with, let us assume that φ is satisfiable, and let M be a model for φ . We construct an interpretation \mathcal{I} , induced by M , with domain $\Delta^{\mathcal{I}} =_{\text{Def}} M\mathbb{U}$, by putting

$$\begin{aligned} A^{\mathcal{I}} &=_{\text{Def}} M(\pi(A)) \\ R^{\mathcal{I}} &=_{\text{Def}} ((M(\pi(R)))_{M\mathbb{U}})^{-1}, \end{aligned}$$

for every concept name A and role name R occurring in \mathcal{T} or in C . Otherwise, the action of the interpretation \mathcal{I} over the remaining concept and role names can be defined arbitrarily, as long as the constraints $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ and $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ hold, for each concept name A and role name R not occurring in \mathcal{T} or in C .

Notice that since M models correctly all the literals in ψ_1 , then we actually have $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ and $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ for all concept names A and role names R , respectively, showing that \mathcal{I} is a valid interpretation. Moreover, for every concept D involving only variables that occur in \mathcal{T} and in C we have

$$D^{\mathcal{I}} = M(\pi(D)). \quad (2)$$

We prove (2) by structural induction on the concept D . If D is of type \top , \perp , $\neg D'$, $D' \sqcup D''$, or $D' \sqcap D''$, then (2) follows directly from the definition of \mathcal{I} . Thus, the only interesting case occurs when the concept D is of type $(\exists R.D_0)$, with R a role name in RIs and D_0 a concept structurally simpler than D .

Let us show that $(\exists R.D_0)^{\mathcal{I}} = M(\pi(\exists R.D_0))$.

Let $v \in (\exists R.D_0)^{\mathcal{I}}$. Then there is a $u \in D_0^{\mathcal{I}} = M(\pi(D_0))$ such that $[v, u] \in R^{\mathcal{I}} = ((M(\pi(R)))_{M\mathbb{U}})^{-1}$. The latter implies $[u, v] \in (M(\pi(R)))_{M\mathbb{U}}$, so that $[u, v] \in M(\pi(R))$ (since $u \in M\mathbb{U}$), and therefore $v \in (M(\pi(R)))[M(\pi(D_0))] = M(\pi(\exists R.D_0))$. Hence, $(\exists R.D_0)^{\mathcal{I}} \subseteq M(\pi(\exists R.D_0))$.

To show the converse inclusion, let now $v \in M(\pi(\exists R.D_0))$. Then $v \in (M(\pi(R)))[M(\pi(D_0))]$, so that $[u, v] \in M(\pi(R))$ for some $u \in M(\pi(D_0))$. Therefore $[u, v] \in (M(\pi(R)))_{M(\mathbb{U})}$ (since by inductive hypothesis $M(\pi(D_0)) = D_0^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} = M(\mathbb{U})$ and therefore $u \in M(\mathbb{U})$). Hence, $[v, u] \in ((M(\pi(R)))_{M(\mathbb{U})})^{-1} = R^{\mathcal{I}}$. And since $u \in D_0^{\mathcal{I}}$, then $v \in (\exists R.D_0)^{\mathcal{I}}$. Therefore $M(\pi(\exists R.D_0)) \subseteq (\exists R.D_0)^{\mathcal{I}}$ which together with the previous inclusion yields $(\exists R.D_0)^{\mathcal{I}} = M(\pi(\exists R.D_0))$.

From (2) and the fact that M models correctly all the conjuncts of ψ_2 , it follows that \mathcal{I} is a model for \mathcal{T} . Additionally, since M satisfies ψ_3 , it also follows that \mathcal{I} satisfies C , so that the interpretation \mathcal{I} induced by the model M satisfies C w.r.t. \mathcal{T} . This completes the first half of the proof.

Conversely, let \mathcal{I} be a model for C w.r.t. \mathcal{T} . Without loss of generality, we may assume that $\Delta^{\mathcal{I}}$ is a set belonging to the von Neumann hierarchy \mathcal{V} (otherwise, we embed $\Delta^{\mathcal{I}}$ in \mathcal{V}).

We construct an assignment $M_{\mathcal{I}}$ induced by \mathcal{I} as follows:

$$\begin{aligned} M_{\mathcal{I}}(\mathbb{U}) &=_{\text{Def}} \Delta^{\mathcal{I}}; \\ M_{\mathcal{I}}(\pi(A)) &=_{\text{Def}} A^{\mathcal{I}}; \\ M_{\mathcal{I}}(\pi(R)) &=_{\text{Def}} (R^{\mathcal{I}})^{-1} \end{aligned}$$

for all concept names A and role names R occurring in \mathcal{T} and in C (as usual, we do not need to be specific on the remaining variables of MLS_{Im}), and show that $M_{\mathcal{I}}$ is a model for φ .

Much as was done before, we prove by structural induction that

$$M_{\mathcal{I}}(\pi(D)) = D^{\mathcal{I}}, \quad (3)$$

for every concept D involving only concept and role names occurring in \mathcal{T} and C . As before, the only relevant case to be considered is when D is of type

$(\exists R.D_0)$. To prove (3) for a concept D of type $(\exists R.D_0)$, it is enough to show that $M_{\mathcal{I}}(\pi(R))[M_{\mathcal{I}}(\pi(D_0))] = (\exists R.D_0)^{\mathcal{I}}$.

Let $u \in (\exists R.D_0)^{\mathcal{I}}$. Then there is a $v \in D_0^{\mathcal{I}}$ such that $[u, v] \in R^{\mathcal{I}}$. Therefore $[v, u] \in M_{\mathcal{I}}(\pi(R))$ and since by inductive hypothesis $v \in M_{\mathcal{I}}(\pi(D_0))$, it follows that $u \in M_{\mathcal{I}}(\pi(R))[M_{\mathcal{I}}(\pi(D_0))]$. Hence we have $(\exists R.D_0)^{\mathcal{I}} \subseteq M_{\mathcal{I}}(\pi(R))[M_{\mathcal{I}}(\pi(D_0))]$.

To prove the converse inclusion, let $u \in M_{\mathcal{I}}(\pi(R))[M_{\mathcal{I}}(\pi(D_0))]$. Hence, there exists $v \in M_{\mathcal{I}}(\pi(D_0))$ such that $[v, u] \in M_{\mathcal{I}}(\pi(R)) = (R^{\mathcal{I}})^{-1}$. But then $[u, v] \in R^{\mathcal{I}}$ and since by inductive hypothesis $M_{\mathcal{I}}(\pi(D_0)) = D_0^{\mathcal{I}}$, we have $v \in D_0^{\mathcal{I}}$, and thus $u \in (\exists R.D_0)^{\mathcal{I}}$. Therefore we have $M_{\mathcal{I}}(\pi(R))[M_{\mathcal{I}}(\pi(D_0))] \subseteq (\exists R.D_0)^{\mathcal{I}}$, which together with the previously established inclusion yields $M_{\mathcal{I}}(\pi(R))[M_{\mathcal{I}}(\pi(D_0))] = (\exists R.D_0)^{\mathcal{I}}$.

Having established (3), it is immediate to check that the assignment $M_{\mathcal{I}}$ satisfies the $\text{MLSS}_{2,m}^{\times}$ -formula φ , completing the proof of the theorem. \square

5 Conclusions and Future Work

We have introduced the unquantified fragment $\text{MLSS}_{2,m}^{\times}$ of set theory, involving besides the basic set constructors, also the Cartesian product operator and some map constructs. We have shown that the satisfiability problem for $\text{MLSS}_{2,m}^{\times}$ -formulae is NP-complete. We have also proved that any decidable extension of the basic fragment MLS extended with map literals of the form $y = f[x]$ has an EXPTIME-hard decision problem. Such lower bound has been obtained by exhibiting a reduction from the description logic \mathcal{ALC} .

We plan to further investigate the relationship between description logics and other MLS extensions in order to find new lower bounds. In particular, we conjecture that the presence of the map union and map difference operators together with the image operator leads to NEXPTIME-hardness.

We also intend to extend the fragment $\text{MLSS}_{2,m}^{\times}$ with some further map constructs such as reflexive closure, transitive closure, symmetric closure, and restricted forms of map composition.

Furthermore, we plan to continue our investigations of decision procedure for a one-sorted variant of the language $\text{MLSS}_{2,m}^{\times}$ in which maps (and the Cartesian product) are regarded just as primitive sets.

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