An Inference System for Exhaustive Generation of Mixed and Purely Negative Implications from Purely Positive Ones

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Abstract. The objective of this article is to study the problem of generating implications with negation when only a set of purely positive implications related to a formal context $K = (G, M, I)$ is provided. To that end, we define a sound and complete inference system which includes a characterization of implications whose left-hand side is a key in the context $K|\tilde{K}$ representing the apposition of the context K and its complementary $\tilde{K}$.

1 Introduction

In the classical problem of association rule mining, only attributes (items) present in data are recorded and positive rules are extracted. This class of rules is a subclass of the larger and more general set of boolean association rules (i.e., rules with negation, conjunction and disjunction) [11] and can help identify unexpected (surprising) patterns in many real-life applications (e.g., *ostrich is a bird that exceptionally does not fly* or *customers who buy smoked salmon buy also caviar but not orange juice*) [17]. In market basket analysis [1], rules with negation can help identify items that conflict with each other (e.g., *if we buy caviar, then we do not buy canned tuna*) or suggest a classification of customers according to their ability to buy or not a group of products [4]. A negative rule of the form $Y \rightarrow \tilde{Z}$ can also be useful to mine itemset substitutes because the presence of the antecedent Y implies the absence of the positive counterpart of the consequent $\tilde{Z}$, which means that Y may be a substitute for Z [18].

In the formal concept analysis (FCA) framework, a straightforward but not efficient solution to the general problem of extracting association rules with negation from a formal context K consists to conduct the apposition of the initial context K with its complementary context $\tilde{K}$ to get the concept lattice $\underline{\mathfrak{B}}(K|\tilde{K})$ and then extract the rules out of that lattice. However, data collections in many real-life applications tend to be very sparse and hence the corresponding complementary contexts are dense and will generate a very likely huge set of candidate itemsets and a tremendous set of uninteresting rules.

In an initial work [12], we showed that in general cases, it is impossible to compute all mixed implications from the sets of positive and negative implications without the context in hand. We also proposed a set of properties and inference rules to infer a *non exhaustive set* of mixed implications (i.e., implications in which at least a negative attribute and a positive attribute coexist), using either positive, negative or mixed implications, provided the original context K is reduced. Later on, an additional inference rule [15] has been defined to complete the inference system (see the last axiom in Table 4 of the appendix).

In this paper we present an inference system for *the exhaustive generation* of purely negative and mixed implications when *only* a generic basis [13] of *positive implications* is initially provided. Obviously, the whole set of implications (denoted by $\Sigma_{K|\tilde{K}}$) is a superset of purely positive implications (i.e., implications with positive attributes only) and purely negative ones (denoted by $\Sigma_{\tilde{K}}$) since it generally contains mixed implications.

The paper is organized as follows. Section 2 provides a background on formal concept analysis and association rule mining. Section 3 gives a brief overview of related work. Section 4 provides a solution to the problem of generating implications with negation when only a set of purely positive implications is given while the proposed inference system is presented in Section 5. A special case is considered in Section 6 where the set Σ_K of purely positive implications is empty. Finally, a conclusion and further work are given in Section 7.

2 Background

2.1 Formal concept analysis

Formal Concept Analysis [7] has been successfully used for conceptual clustering and rule mining. A formal context is a triple $K := (G, M, I)$ where G, M and I stand for a set of objects, a set of attributes, and a binary relation between G and M respectively. For $A \subseteq G$ and $B \subseteq M$ we define

$$A' := \{a \in M \mid oIa \ \forall o \in A\} \quad \text{and} \quad B' := \{o \in G \mid oIa \ \forall a \in B\},$$

the set of attributes common to objects in A and the set of objects sharing all the attributes in B. The mapping (denoted by $'$) between the powerset of G and the powerset of M defines a Galois connection, and the induced closure operators (on G and M) are denoted by $''$. A formal concept c is a pair (A, B) with $A \subseteq G$, $B \subseteq M$, $A = B'$ and $B = A'$, where A is called the extent of c and B its intent. In the closed *itemset* mining framework [13, 23], G, M, A and B correspond to the transaction database, the set of items (products), the closed *tidset* and the closed *itemset* respectively.

The set $\mathfrak{B}(K)$ of all concepts of the context K, partially ordered by:

$$(X_1, Y_1) \leq (X_2, Y_2) \Leftrightarrow X_1 \subseteq X_2, Y_2 \subseteq Y_1.$$

forms a complete lattice, called a concept lattice and denoted by $\underline{\mathfrak{B}}(K)$. Concept (X_2, Y_2) is called the successor of (X_1, Y_1) and, inversely, (X_1, Y_1) is the predecessor of (X_2, Y_2). Concepts with a single immediate predecessor (successor) are called *join*-irreducible (*meet*-irreducible).

Object (resp. attribute) set reduction of a context $K = (G, M, I)$ consists of discarding from the set G (resp. M) all objects (resp. attributes) that may be obtained through the intersection of some other objects (resp. attributes). The concept lattice of a reduced context is isomorphic to the concept lattice of the initial one.

The apposition $K = K_1|K_2$ of two contexts $K_1 = (G, M_1, I_1)$ and $K_2 = (G, M_2, I_2)$ is the horizontal concatenation of contexts sharing the same set G of objects [7, 19]. It represents the context $K = (G, M_1\dot{\cup}M_2, I_1\dot{\cup}I_2)$ whose lattice is a substructure of the direct product of $\underline{\mathfrak{B}}(K_1)$ and $\underline{\mathfrak{B}}(K_2)$.

In the rest of the paper and unless otherwise indicated, we will use upper-case letters (e.g., B, Y), lower-case letters and letters with tilde to mean sets of attributes (itemsets), atomic attributes and elements with negation respectively. For example, $\tilde{a}$ stands for the negation of attribute a and means that Object o belongs to the extent of $\tilde{a}$ iff it does not belong to the extent of a, and $\tilde{A}$ represents the set $\{\tilde{a} \mid a \notin A\}$.

2.2 Association Rule Mining

Association rule mining [2] is an extensively studied problem in data mining and consists to extract a set of association rules from data (e.g., a set of transactions describing a collection of items bought together). An association rule r is an implication of the form $Y \rightarrow Z$ $[sup, conf]$, where Y and Z are subsets of attributes (called *itemsets*), $Y \cap Z = \emptyset$, and *sup* and *conf* represent the support and the confidence of the rule, respectively. The support of a rule is defined as $Prob(Y \cup Z)$ while the confidence is computed as the conditional probability $Prob(Z/Y)$.

A set of studies in FCA were conducted on the generation of concise representations of rules [9] such as informative rules (i.e., with minimized premise and maximized consequence), Guigues-Duquenne basis [7, 8], generic basis [13], and Luxenburger basis [10]. The notion of *generator* of a closed itemset [13, 14] and *pseudo-intent* [7, 8] play a key role in such studies. A *generic basis* [13] associated with a context K is a concise representation of exact rules (implications) of the form r: $Y \rightarrow Y''\backslash Y$ $[sup, 1]$ such that Y is a generator for Y''. The generator Y [14] of a closed itemset Z is a minimal subset of Z such that $Y'' = Z$. The support of the rule r is $sup = |Y'|/|G|$.

In this work the rule sets Σ_K, $\Sigma_{\tilde{K}}$ and $\Sigma_{K|\tilde{K}}$ are generic bases and any rule in such collections is an implication which will be further represented by $Y \rightarrow Z$ $[sup]$ because the confidence is always equal to 1.

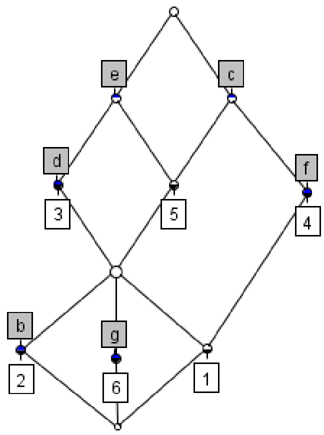

Fig. 1. The concept lattice $\underline{\mathfrak{B}}(K)$ corresponding to Table 1.

2.3 Illustrative Example

To further illustrate notions and properties, let us take the following example in which a context $K = (G, M, I)$ is given, with $G = \{1, 2, 3, 4, 5, 6\}$ and $M = \{b, c, d, e, f, g\}$. The corresponding concept lattice[3] is represented with a reduced labelling as shown by Figure 1. For example, nodes labeled with values 1 and 5 represent the object concepts $(\{1\}, \{c, d, e, f\})$ and $(\{1, 2, 5, 6\}, \{e, c\})$ respectively.

K	b	c	d	e	f	g
1	0	1	1	1	1	0
2	1	1	1	1	0	0
3	0	0	1	1	0	0
4	0	1	0	0	1	0
5	0	1	0	1	0	0
6	0	1	1	1	0	1

$K\|\tilde{K}$	b	c	d	e	f	g	$\tilde{b}$	$\tilde{c}$	$\tilde{d}$	$\tilde{e}$	$\tilde{f}$	$\tilde{g}$
1	0	1	1	1	1	0	1	0	0	0	0	1
2	1	1	1	1	0	0	0	0	0	0	1	1
3	0	0	1	1	0	0	1	1	0	0	1	1
4	0	1	0	0	1	0	1	0	1	1	0	1
5	0	1	0	1	0	0	1	0	1	0	1	1
6	0	1	1	1	0	1	1	0	0	0	1	0

Table 1. A context K, and the apposition of K and its complementary context $\tilde{K}$.

A first glance at Table 2 which provides the complete set of implications generated from $K|\tilde{K}$ shown in Table 1 indicates that the positive implications

[3] The lattice is constructed using the SourceForge project called Concept Explorer [22].

Positive implications Σ_K	Negative implications $\Sigma_{\tilde{K}}$	A subset of implications in $\Sigma_{K\|\tilde{K}}$	
$d \rightarrow e[0.66]$	$\tilde{d} \rightarrow \tilde{b}\tilde{g}[0.33]$	$\tilde{f} \rightarrow e[0.66]$	$b\tilde{b} \rightarrow M\tilde{M}[0]$
$f \rightarrow c[0.66]$	$\tilde{c} \rightarrow \tilde{b}\tilde{f}\tilde{g}[0.16]$	$d\tilde{g} \rightarrow e[0.50]$	$c\tilde{c} \rightarrow M\tilde{M}[0]$
$cd \rightarrow e[0.50]$	$\tilde{d}\tilde{f} \rightarrow \tilde{b}\tilde{g}[0.16]$	$\tilde{b}\tilde{f} \rightarrow e[0.50]$	$d\tilde{d} \rightarrow M\tilde{M}[0]$
$df \rightarrow ce[0.16]$	$\tilde{e} \rightarrow \tilde{b}\tilde{d}\tilde{g}[0.16]$	$cd\tilde{g} \rightarrow e[0.33]$	$e\tilde{e} \rightarrow M\tilde{M}[0]$
$ef \rightarrow cd[0.16]$	$\tilde{c}\tilde{d} \rightarrow \tilde{b}\tilde{e}\tilde{f}\tilde{g}[0]$	$d\tilde{b}\tilde{f} \rightarrow e[0.33]$	$f\tilde{f} \rightarrow M\tilde{M}[0]$
$b \rightarrow cde[0.16]$	$\tilde{c}\tilde{e} \rightarrow \tilde{b}\tilde{d}\tilde{f}\tilde{g}[0]$	$df \rightarrow ce\tilde{b}\tilde{g}[0.16]$	$g\tilde{g} \rightarrow M\tilde{M}[0]$
$g \rightarrow cde[0.16]$	$\tilde{e}\tilde{f} \rightarrow \tilde{b}\tilde{c}\tilde{d}\tilde{g}[0]$	$cd\tilde{b}\tilde{f} \rightarrow eg[0.16]$	$bf \rightarrow M\tilde{M}[0]$
$bf \rightarrow cdeg[0]$		$cd\tilde{b}\tilde{g} \rightarrow ef[0.16]$	$b\tilde{c} \rightarrow M\tilde{M}[0]$
$bg \rightarrow cdef[0]$		$cd\tilde{f}\tilde{g} \rightarrow be[0.16]$	$\tilde{d}ef \rightarrow M\tilde{M}[0]$
$fg \rightarrow bcde[0]$		$d\tilde{b}\tilde{f}\tilde{g} \rightarrow e\tilde{c}[0.16]$	$cd\tilde{b}\tilde{f}\tilde{g} \rightarrow M\tilde{M}[0]$
		$\vdots$	$\vdots$

Table 2. Positive, negative and mixed implications of Table 1.

in Σ_K do not convey interesting information about the absence of some items, and implications with a null support seem useless. However, the set $\Sigma_{K|\tilde{K}}$ brings additional associations about the absence of items (e.g., $\tilde{f} \rightarrow e$ [0.66]), and we will see later that implications with a null support can be exploited to generate mixed implications (see Property 3 and Table 4 in the appendix).

3 Related work

In the area of data mining, the notion of negative associations (relationships) between itemsets was initially discussed by Brin and Motwani [6] who proposed a procedure that exploits the Chi-square test to search for a border between correlated and uncorrelated elements in the itemset lattice. Many studies recognize that mining rules with negation (i.e., rules that contain negative items) is a very challenging problem [5] and there is an urgent need to define pruning strategies, procedures and interestingness measures (other than confidence) to generate negative association rules in an efficient and correct way [6, 20, 21]. For example, Wu *et al.* [21] define a new algorithm for negative association rule generation as well as a new quality measure for an efficient pruning of generated frequent itemsets. In [16], positive frequent itemsets are combined with background knowledge to mine negative association rules, while in [3] a new technique based on Kullback-Leibler divergence is defined. Teng *et al.* [18] exploit negative rules for item substitution (i.e., replacing the purchase of an item with another one) in market basket analysis [1] and propose an approach to identify substitution rules in two steps: the first one identifies concrete itemsets (i.e., frequent itemsets whose elements are statistically dependent among a large number of frequent itemsets) while the second step generates substitution rules.

Two concrete itemsets Y and Z constitute a substitution rule, denoted by $Y \triangleright Z$ to mean that Y is a substitute for Z if and only if Y and Z are negatively correlated and the negative association rule $Y \rightarrow \tilde{Z}$ holds.

The notion of negative rules has different meanings. In [16], it represents rules of the form $Y \nrightarrow Z$ whose actual support deviates at least $MinRI \times MinSup$ from its expected support (based on the support of items in closed itemsets and the taxonomy on attributes). $MinRI$ and $MinSup$ correspond to the minimal value of an interest measure RI and the support, respectively. In [5], the rule $Y \rightarrow \tilde{Z}$ has the standard meaning, *i.e.*, the presence of items in Y implies the absence of all items in Z.

In [12] the problem of computing the generic basis of positive, negative and mixed implications from a given input is analyzed and a set of situations are identified based on the type of available input (e.g., the formal context $K = (G, M, I)$, the set Σ_K of positive implications, the concept lattice $\underline{\mathfrak{B}}(K)$) and the sort of output to produce (e.g., $\underline{\mathfrak{B}}(K)$, the set of negative implications $\Sigma_{\tilde{K}}$, the whole set of implications $\Sigma_{K|\tilde{K}}$). Inference rules to produce a *non exhaustive set* of mixed association rules, using either positive, negative or mixed implications are also defined. A more elaborated inference system is described in [15] and summarized in the appendix.

4 Problem Statement

We have noticed that the implications of the form $A\tilde{C} \rightarrow x$ [sup] such that $|A| > 1$ and $|\tilde{C}| > 1$ and $sup \neq 0$ can never be deduced from the inference axioms initially presented in [12] since the attributes are moved (and negated) only one at a time from one side to another side of a given implication (see the first five rows of Table 4 in the appendix). This leads us to propose an additional inference rule (see the last axiom in Table 4) and later on generalize the idea to characterize implications of the form $A\tilde{B} \rightarrow M\tilde{M}$ [0] such that the former implications could be retrieved. This means that we need to find all keys in $K|\tilde{K}$, including those that contain at least two positive items and two negative ones.

To handle the problem of generating purely negative and mixed implications out of purely positive ones in an exhaustive way, we proceed in two steps: (i) we first find a characterization of keys in $K|\tilde{K}$, and then (ii) show that such a characterization can be combined with Property 3 (previously presented in [12]) to infer the set of negative and mixed implications in a sound and complete manner.

The first step can be stated by *Problem* 1 while the second one can by expressed by *Problem* 2.

Problem 1. : Key Computation (KC)
Instance: A set Σ_K of positive implications of a reduced context.
Question: Compute the set $\mathcal{K}$ of keys in $K|\tilde{K}$.

The reason for imposing a reduced context comes from the fact that an implication base Σ_K may correspond to many contexts, and hence the apposition of each context with its complementary one may generate more than one set $\Sigma_{K|\tilde{K}}$.

Problem 2. : *Exhaustive Implication Computation (EIC)*
Instance: An implication set Σ_K of a reduced context and the set $\mathcal{K}$ of keys.
Question: Compute a cover of $\Sigma_{K|\tilde{K}}$.

First, we need to characterize join-irreducible concepts to further identify keys.

Property 1. Let $F \subseteq M$. F is the intent of a join-irreducible concept in K iff F is closed and $\exists\ b \in M \backslash F$ such that $\forall\ x \in M \backslash F\ Fx \rightarrow b \in \Sigma_K$.

Proof. Suppose that F is the intent of a join-irreducible concept. Then, there exists a unique closed set B that covers F. Let $b \in B \backslash A$. Then, any closed set containing both F and an element of $M \backslash F$, contains also b. Therefore, $\forall$ $x \in M \backslash F\ Fx \rightarrow b \in \Sigma_K$.

Now suppose that there exists an element $b \in M \backslash F$ such that $\forall\ x \in M \backslash F$ $Fx \rightarrow b \in \Sigma_K$. Then F is covered by a unique closed set. So if F is closed, then it is the intent of a join-irreducible concept.

In our illustrative example, there are six join-irreducible concepts. One of them is the concept $(\{1, 2, 5, 6\}, \{c, e\})$ since $\forall\ \mathbf{x} \in \{b, d, f, g\}$ the implication $ce\mathbf{x} \rightarrow d$ holds in Σ_K.

5 Inference System

In the following we establish a corollary which helps generate all the implications whose left-hand side is a key in $M\tilde{M}$. To that end, we first state the property below which is based on the fact that a key cannot be larger than the intent of a join-irreducible concept since its closure is the intent of the lattice infimum.

Property 2. Let $A\tilde{B} \subseteq M\tilde{M}$. Then, $A\tilde{B} \rightarrow M\tilde{M}[0]$ iff $\nexists\ F$ an intent of a join-irreducible concept in K such that $A \subseteq F$ and $B \cap F = \emptyset$.

Proof. A set $A\tilde{B}$ is a key of $K|\tilde{K}$ iff there is no intent of a join-irreducible concept containing $A\tilde{B}$. We only need to note that F is the intent of a join-irreducible concept of K such that $A \subseteq F$ and $B \cap F = \emptyset$ iff $F\tilde{B}$ is the intent of a join-irreducible concept in $K|\tilde{K}$.

For example, the set $A\tilde{B} = \{c, d, \tilde{b}, \tilde{f}\}$ is not a key in $K|\tilde{K}$ since among the join-irreducible concepts that have an intent that includes $A = \{c, d\}$, there is a concept, namely $(\{6\}, \{c, d, e, g\}$, whose intent has an empty intersection with $B = \{b, f\}$. However, the set $A\tilde{B} = \{c, d, \tilde{b}, \tilde{f}, \tilde{g}\}$ is a key in $K|\tilde{K}$.

The corollary below follows directly from the two preceding properties.

Corollary 1. *Let $A\tilde{B} \subseteq M\tilde{M}$. Then, $A\tilde{B} \rightarrow M\tilde{M}[0]$ iff $\nexists$ a closed set $F \subseteq M$ such that $A \subseteq F$, $B \cap F = \emptyset$ and $\forall\ x \in M \backslash F\ Ax \rightarrow b$ with $b \in M \backslash F$.*

Implications of the form: $\{a_i\tilde{a_i}\} \rightarrow M\tilde{M}[0] \ \ \forall i \in \{1 \ldots k\}$ are a special case of implications involving a key, and hence the contradiction axiom (see the third axiom in the appendix) which states that one cannot have both item a_i and its negation $\tilde{a_i}$ becomes redundant in our new inference system.

We recall a property from [12, 15] which allows us to generate new implications (see the fifth axiom in the appendix).

Property 3. Let $Ax \subseteq M\tilde{M}$. Then, $Ax \rightarrow M\tilde{M}\ [0] \Leftrightarrow A \rightarrow \tilde{x}\ [sup]$.

Theorem 1. *Let Σ_K be a non empty set of positive implications in K. Then, Corollary 1 and Property 3 allow the inference of a sound and complete set of implications in $K|\tilde{K}$.*

Proof. Corollary 1 generates all the keys in $K|\tilde{K}$, including $\{a_i\tilde{a_i}\} \rightarrow M\tilde{M}[0]$ $\forall i \in \{1 \ldots k\}$. Each purely negative implication and each mixed implication can be generated from keys using Property 3.

As an illustration, the purely negative implication $\tilde{d} \rightarrow \tilde{b}\tilde{g}$ can be inferred from $b\tilde{d} \rightarrow M\tilde{M}[0]$ and $g\tilde{d} \rightarrow M\tilde{M}[0]$ using Property 3. Moreover, the mixed implication $cd\tilde{b}\tilde{f}\tilde{g} \rightarrow M\tilde{M}[0]$ holds in $\Sigma_{K|\tilde{K}}$ since there does not exist F, an intent of a join-irreducible concept such that $\{cd\} \subseteq F$ and $\{cd\} \cap \{bfg\} = \emptyset$. Using Property 3, the mixed implication involving a key $cd\tilde{b}\tilde{f}\tilde{g} \rightarrow M\tilde{M}[0]$ leads to five other mixed implications by moving one attribute at a time (and taking its negation) from the left-hand side to the right-hand side of the implication (e.g., $cd\tilde{b}\tilde{f} \rightarrow g[0.16]$). It is easy to see that the last axiom presented in Table 4 of the appendix allows the inference of $cd\tilde{b}\tilde{f} \rightarrow g[0.16]$ which leads to $cd\tilde{b}\tilde{f}\tilde{g} \rightarrow M\tilde{M}[0]$ by applying Property 3.

6 Empty Implication Set

There are formal contexts such that their corresponding positive (respectively negative) implication set is empty. A special case of such contexts is the one with the following features: it has the same number of objects and attributes, and each object differs from each one of the other objects by only one attribute (see Table 3). The corresponding concept lattice is a Boolean one. In such a context $K = (G, M, I)$ where $M = \{a_1, ..., a_k\}$ and Σ_K is empty, the set of purely negative implications is $\Sigma_{\tilde{K}} = \{\tilde{a_i}\tilde{a_j}\} \rightarrow M\tilde{M}[0],\ \forall i \in \{1..k\}, \forall j \in \{1..k\}, i \neq j$.

K and $\tilde{K}$	a	b	c	d	$\tilde{a}$	$\tilde{b}$	$\tilde{c}$	$\tilde{d}$
1	0	1	1	1	1	0	0	0
2	1	0	1	1	0	1	0	0
3	1	1	0	1	0	0	1	0
4	1	1	1	0	0	0	0	1

Table 3. A context K with an empty Σ_K and its complementary context $\tilde{K}$.

By applying Property 3 to implications in $\Sigma_{\tilde{K}}$, one can infer the following implications: $\tilde{a_i} \rightarrow a_j, \forall i \in \{1..k\}, \forall j \in \{1..k\}, i \neq j$.

Even though Σ_K is empty, we can observe that the trivial implication $\{a_1...a_k\} \rightarrow M\tilde{M}[0]$ holds and will generate through Property 3 the implications: $M \backslash a_i \rightarrow \tilde{a_i}$, $\forall i \in \{1..k\}$.

The whole set of implications that can be generated when $\Sigma_K = \emptyset$ is then as follows:

$$\begin{aligned}
\Sigma_{K|\tilde{K}} = \quad & \{\tilde{a_i}\tilde{a_j} \rightarrow M\tilde{M}[0], \forall i \in \{1..k\}, \forall j \in \{1..k\}, i \neq j\} \\
& \cup \{\tilde{a_i} \rightarrow a_j, \forall i \in \{1..k\}, \forall j \in \{1..k\}, i \neq j\} \\
& \cup \{a_1...a_k \rightarrow M\tilde{M}[0]\} \\
& \cup \{M \backslash a_i \rightarrow \tilde{a_i}, \forall i \in \{1..k\}\} \\
& \cup \{a_i\tilde{a_i}\} \rightarrow M\tilde{M}[0], \forall i \in \{1..k\}\}
\end{aligned}$$

The second and fourth sets of implications are inferred from the first and third ones respectively by moving (and negating) one attribute from the left side to the right side of the implications (see Property 3). The fifth group of implications reflects contradiction (see the third axiom in the appendix) and holds for any context.

7 Conclusion

This paper is an extension to our first investigation [12] on generating implications with negation. It aims to generate the exhaustive set of rules $\Sigma_{K|\tilde{K}}$ when only the set of positive implications Σ_K is given, provided that the formal context K is reduced.

Our contribution lies in the proposal of a property that identifies all the keys $A\tilde{B}$ in $K|\tilde{K}$ where $A \in M$ and $\tilde{B} \in \tilde{M}$ and the presentation of a new inference rule (see Corollary 1), which together with Property 3, provides a sound and complete inference system for the generation of mixed rules. A special case when Σ_K is empty has been analyzed.

We are currently designing procedures that efficiently compute the set of implications $\Sigma_{K|\tilde{K}}$ and the closure of a set $A\tilde{B}$ when Σ_K is given.

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Appendix: Inference Axioms

The following table summarizes the (complete and sound) inference system presented in [15] where $A \subseteq M$, $\tilde{A} \subseteq \tilde{M}$, $x \in M$, $\tilde{x} \subseteq \tilde{M}$ and the initial context K is reduced. The last axiom completes the set of properties and inference rules initially defined in [12].

ID	**Properties and inference rules**
1	$\Sigma_K \vdash A \rightarrow x \Leftrightarrow \Sigma_{K\|\tilde{K}} \vdash A \rightarrow x$; $\Sigma_{\tilde{K}} \vdash \tilde{A} \rightarrow \tilde{x} \Leftrightarrow \Sigma_{K\|\tilde{K}} \vdash \tilde{A} \rightarrow \tilde{x}$
2	$\Sigma_K \vdash A \rightarrow M[0] \Leftrightarrow \Sigma_{K\|\tilde{K}} \vdash A \rightarrow M\tilde{M}[0]$; $\Sigma_{\tilde{K}} \vdash \tilde{A} \rightarrow \tilde{M}[0] \Leftrightarrow \Sigma_{K\|\tilde{K}} \vdash \tilde{A} \rightarrow M\tilde{M}[0]$
3	$\Sigma_{K\|\tilde{K}} \vdash a\tilde{a} \rightarrow M\tilde{M}[0], \forall a \in M$
4	$\Sigma_K \vdash Ax \rightarrow y \Rightarrow \Sigma_{K\|\tilde{K}} \vdash A\tilde{y} \rightarrow \tilde{x}$; $\Sigma_{\tilde{K}} \vdash \tilde{A}\tilde{x} \Rightarrow \tilde{y} \Rightarrow \tilde{A}y \rightarrow x$
5	$\Sigma_K \vdash Ax \rightarrow M[0] \Leftrightarrow \Sigma_{K\|\tilde{K}} \vdash A \rightarrow \tilde{x}$; $\Sigma_{\tilde{K}} \vdash \tilde{A}\tilde{x} \Leftrightarrow \tilde{M}[0] \Rightarrow \Sigma_{K\|\tilde{K}} \vdash \tilde{A} \rightarrow x$
6	$\Sigma_{K\|\tilde{K}} \vdash B \rightarrow x$, $B \subseteq M\tilde{M}$ with $B = B^+ \cup B^-$, $B^+ \subseteq M$, $B^- \subseteq \tilde{M}$ and $x \in M$ $\Leftrightarrow$ $\forall y \in M \backslash B^+$, $\Sigma_{K\|\tilde{K}} \vdash By \rightarrow x$ and B^+ is not the intent of a join-irreducible concept.

Table 4. Summary of properties and inference axioms to generate $\Sigma_{K|\tilde{K}}$ from Σ_K and $\Sigma_{\tilde{K}}$.