Towards attribute reduction in multi-adjoint concept lattices*

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Abstract. In Formal Concept Analysis, attribute reduction is a important step in order to reduce the complexity of the computation of the concept lattice. This reduction is more complex in fuzzy environments. In this paper, we will present a first approximation to reduce the set of attributes in the multi-adjoint concept lattice. The solution found is based on the development of specific results which allow us to reduce the number of attributes in the classical case, by detecting some relatively necessary and absolutely unnecessary attributes and, then, use linguistic labels in order to obtain a method to reduce the number of attributes in a multi-adjoint context, working up to some level of tolerance, and preserving the original lattice structure of the set of concepts.

1 Introduction

Formal concept analysis (FCA) has become an important and appealing research topic both from a theoretical perspective [15] and from the applicative one. Regarding applications, we can find papers ranging from ontology merging [13], to diverse fields of application such as the Semantic Web.

In a nutshell, FCA extracts information from databases containing a set attributes A and a set of objects B together with a relation $R \subseteq A \times B$. This information is classified into concepts and an order among them is defined, the final algebraic structure obtained this way is the so-called concept lattice. Usually, the set of attributes is very large and the complexity to built the concept lattice is very high.

Different fuzzy approaches for generalizing FCA were introduced and, nowadays, there are works which extend the theory with ideas from fuzzy set theory or from rough set theory or even integrated approaches encompassing both approaches.

Rough set theory is an alternative formal tool for modelling and processing information under uncertainty. For both environments, FCA and RST, there

* Partially supported by the Spanish Science Ministry projects TIN2009-14562-C05-01 and TIN2009-14562-C05-03, and by Junta de Andalucía projects P06-FQM-02049 and P09-FQM-5233.

exist mechanisms to reduce the number of attributes [8, 9, 16, 18, 19]. To our knowledge, the existing approaches to attribute reduction in classical FCA, but all of them need to build the concept lattices before obtaining a reduct (in some sense, a minimal set of attributes which maintains the original lattice structure).

In this paper we concentrate on the problem of attribute reduction in a fuzzy extension of formal concept analysis. Specifically, we introduce some results which allow us to reduce the number of attributes in the classical case, by detecting some relatively necessary and absolutely unnecessary attributes. Later, we use linguistic labels in order to obtain a method to reduce the number of attributes in a multi-adjoint context, working up to some level of tolerance. As the method to compute the reduced set of attributes does not require to build the whole concept lattice in advance, the complexity to obtain the multi-adjoint concept lattice, after attribute reduction, decreases notably.

2 Preliminaries

In classical formal concept analysis [17], we consider a set of **a**ttributes A, a set of o**b**jects B and a crisp relation between them $R\colon A \times B \to \{0,1\}$, where, for each $a \in A$ and $b \in B$, we have that $R(a,b) = 1$, if a and b are related, or $R(a,b) = 0$, otherwise. We will also write aRb when $R(a,b) = 1$. The triple (A, B, R) is called *context* and the mappings ${}^{\uparrow}\colon 2^B \to 2^A$, ${}^{\downarrow}\colon 2^A \to 2^B$, are defined, for each $X \subseteq B$ and $Y \subseteq A$, as:

$$X^{\uparrow} = \{a \in A \mid \text{for all } b \in X, aRb\} = \{a \in A \mid \text{if } x \in X, \text{ then } aRb\} \tag{1}$$

$$Y^{\downarrow} = \{b \in B \mid \text{for all } a \in Y, aRb\} = \{b \in B \mid \text{if } a \in Y, \text{ then } aRb\} \tag{2}$$

A *concept* in the context (A, B, R) is defined to be a pair (X, Y), where $X \subseteq B$, $Y \subseteq A$, and which satisfies that $X^{\uparrow} = Y$ and $Y^{\downarrow} = X$. The element X of the concept (X, Y) is the *extent* and Y the *intent*.

The set of concepts in a context (A, B, R) is denoted as $\mathcal{B}(A, B, R)$ and it is a complete lattice [5, 17], with the order: $(X_1, Y_1) \leq (X_2, Y_2)$ if $X_1 \subseteq X_2$ (or, equivalently, $Y_2 \subseteq Y_1$), for all $(X_1, Y_1), (X_2, Y_2) \in \mathcal{B}(A, B, R)$.

An important fact is that the extent and intent mappings form a Galois connection [5]. This concept is defined below:

Definition 1. *Let $(P_1, \leq_1)$ and $(P_2, \leq_2)$ be posets, and ${}^{\downarrow}\colon P_1 \to P_2$, ${}^{\uparrow}\colon P_2 \to P_1$ mappings, the pair $({}^{\uparrow}, {}^{\downarrow})$ forms a* Galois connection *between P_1 and P_2 whenever the following conditions hold:*

1. *${}^{\uparrow}$ and ${}^{\downarrow}$ are order-reversing.*
2. *$x \leq_1 x^{\downarrow\uparrow}$ for all $x \in P_1$.*
3. *$y \leq_2 y^{\uparrow\downarrow}$ for all $y \in P_2$.*

Recently, a fuzzy generalization of the formal concept analysis was presented in [10], which generalizes some other [2, 4, 7, 14]. Now, we will recall some definitions and results from [10].

The first definition introduces the basic building blocks of the multi-adjoint concept lattices, the *adjoint triples*, which are generalisations of the notion of adjoint pair under the hypothesis of having a non-commutative conjunctor.

The lack of commutativity of the conjunctor, directly provides two different ways of generalising the well-known adjoint property between a t-norm and its residuated implication, depending on which argument is fixed in the conjunction.

Definition 2. *Let* $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ *be posets and* $\&\colon P_1 \times P_2 \to P_3$, $\swarrow\colon P_3 \times P_2 \to P_1$, $\nwarrow\colon P_3 \times P_1 \to P_2$ *be mappings,*[3] *then* $(\&, \swarrow, \nwarrow)$ *is an* adjoint triple *with respect to* P_1, P_2, P_3 *if:*

1. $\&$ *is order-preserving in both arguments.*
2. $\swarrow$ *and* $\nwarrow$ *are order-preserving in the consequent and order-reversing in the antecedent.*
3. $x \leq_1 z \swarrow y$ *iff* $x \,\&\, y \leq_3 z$ *iff* $y \leq_2 z \nwarrow x$, *where* $x \in P_1$, $y \in P_2$ *and* $z \in P_3$.

Note that in the domain and codomain of the considered conjunctor we have three (in principle) different sorts, thus providing a more flexible language to a potential user. Furthermore, notice that no boundary condition is required, in difference to the usual definition of multi-adjoint lattice [11] or implication triple [1].

In order to provide more flexibility into our language, we will allow the existence of several adjoint triples for a given triplet of posets. Notice, however, that since these triplets will be used as the underlying structures of our generalization of concept lattice, it is reasonable to require the lattice structure on some of the posets in the definition of adjoint triple.

Definition 3. *A* multi-adjoint frame $\mathcal{L}$ *is a tuple*

$$(L_1, L_2, P, \preceq_1, \preceq_2, \leq, \&_1, \swarrow^1, \nwarrow_1, \dots, \&_n, \swarrow^n, \nwarrow_n)$$

where $(L_1, \preceq_1)$ *and* $(L_2, \preceq_2)$ *are complete lattices,* $(P, \leq)$ *is a poset and, for all* $i \in \{1, \dots, n\}$, $(\&_i, \swarrow^i, \nwarrow_i)$ *is an adjoint triple with respect to* L_1, L_2, P.

For short, a multi-adjoint frame will be denoted as $(L_1, L_2, P, \&_1, \dots, \&_n)$.

Following the usual approach to formal concept analysis, given a frame, a *multi-adjoint context* is a tuple consisting of sets of objects and attributes and a fuzzy relation among them; in addition, the multi-adjoint approach also includes a function which assigns an adjoint triple to each object (or attribute). This feature is important in that it allows for defining subgroups of objects or attributes in terms of different degrees of preference, see [10]. Formally, the definition is the following:

Definition 4. *Given a multi-adjoint frame* $(L_1, L_2, P, \&_1, \dots, \&_n)$, *a* context *is a tuple* (A, B, R, σ) *such that* A *and* B *are non-empty sets (usually interpreted*

[3] Note that the use of $\swarrow$ done in [6] has a different meaning.

as attributes and objects, respectively), R is a P-fuzzy relation $R\colon A \times B \to P$ and $\sigma\colon B \to \{1,\dots,n\}$ is a mapping which associates any element in B with some particular adjoint triple in the frame.[4]

Once we have fixed a multi-adjoint frame and a context for that frame, we can define the following mappings ${}^{\uparrow_\sigma}\colon L_2^B \longrightarrow L_1^A$ and ${}^{\downarrow^\sigma}\colon L_1^A \longrightarrow L_2^B$ which can be seen as generalisations of those given in [3, 7]:

$$g^{\uparrow_\sigma}(a) = \inf\{R(a,b) \swarrow^{\sigma(b)} g(b) \mid b \in B\} \tag{3}$$

$$f^{\downarrow^\sigma}(b) = \inf\{R(a,b) \nwarrow_{\sigma(b)} f(a) \mid a \in A\} \tag{4}$$

It is not difficult to show that these two arrows generate a Galois connection [10].

As usual in the different frameworks of formal concept analysis, a *multi-adjoint concept* is a pair $\langle g, f\rangle$ satisfying that $g \in L_2^B$, $f \in L_1^A$ and that $g^{\uparrow_\sigma} = f$ and $f^{\downarrow^\sigma} = g$; with $({}^{\uparrow_\sigma}, {}^{\downarrow^\sigma})$ being the Galois connection defined above.

Definition 5. *The* multi-adjoint concept lattice *associated to a multi-adjoint frame $(L_1, L_2, P, \&_1, \dots, \&_n)$ and a context (A, B, R, σ) is the set*

$$\mathcal{M} = \{\langle g, f\rangle \mid g \in L_2^B, f \in L_1^A \text{ and } g^{\uparrow_\sigma} = f, f^{\downarrow^\sigma} = g\}$$

in which the ordering is defined by $\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ if and only if $g_1 \preceq_2 g_2$ (equivalently $f_2 \preceq_1 f_1$).

The ordering just defined above actually provides $\mathcal{M}$ with the structure of a complete lattice.

3 Attribute reduction in classical formal concept analysis

Definition 6. *Given two concept lattices $\mathcal{B}(A_1, B, R_1)$ and $\mathcal{B}(A_2, B, R_2)$. If for any $(X, Y) \in \mathcal{B}(A_2, B, R_2)$ there exists $(X', Y') \in \mathcal{B}(A_1, B, R_1)$ such that $X = X'$, then we say that $\mathcal{B}(A_1, B, R_1)$ is* finer *than $\mathcal{B}(A_2, B, R_2)$ and we will write:*

$$\mathcal{B}(A_1, B, R_1) \leq \mathcal{B}(A_2, B, R_2)$$

If $\mathcal{B}(A_1, B, R_1) \leq \mathcal{B}(A_2, B, R_2)$ and $\mathcal{B}(A_2, B, R_2) \leq \mathcal{B}(A_1, B, R_1)$, then these two concept lattices are said to be *isomorphic* to each other, and we will write:

$$\mathcal{B}(A_1, B, R_1) \cong \mathcal{B}(A_2, B, R_2)$$

Given a context (A, B, R), if we consider a subset of attributes, $Y \subseteq A$ and the restriction relation $R_Y = R \cap (Y \times B)$, the triple (Y, B, R_Y) is also a formal context, which can be interpreted as a *subcontext* of the original. Hence, we can apply the mappings ${}^\downarrow$ and ${}^\uparrow$, in this subcontext we will write ${}^{\downarrow^Y}$ and ${}^{\uparrow_Y}$. It is clear that, given $X \subseteq B$ we have that $X^{\uparrow_Y} = X^\uparrow \cap Y$.

[4] A similar theory could be developed by considering a mapping $\tau\colon A \to \{1,\dots,n\}$ which associates any element in A with some particular adjoint triple in the frame.

Theorem 1 ([19]). *Let (A, B, R) be a formal context. For any $Y \subseteq A$, such that $Y \neq \varnothing$, $\mathcal{B}(A, B, R) \leq \mathcal{B}(Y, B, R_Y)$ holds.*

Definition 7. *Given a context (A, B, R), if there exists a set of attributes $Y \subseteq A$ such that $\mathcal{B}(A, B, R) \cong \mathcal{B}(Y, B, R_Y)$, then Y is called a* consistent *set of (A, B, R). Moreover, if $\mathcal{B}(Y \smallsetminus \{y\}, B, R_{Y \smallsetminus \{y\}}) \ncong \mathcal{B}(A, B, R)$, for all $y \in Y$, then Y is called a* reduct *of (A, B, R).*

The intersection of all the reducts of (A, B, R) is called the core *of (A, B, R).*

Theorem 2 ([19]). *Let (A, B, R) be a formal context, $Y \subseteq A$ and $Y \neq \varnothing$. Then,*

$$Y \text{ is a consistent set } \Leftrightarrow \mathcal{B}(Y, B, R_Y) \leq \mathcal{B}(A, B, R)$$

In [19], the authors used the three types of attributes in a formal context originally proposed by Pawlak [12] for rough set theory.

Definition 8. *Let Λ be an index set and (A, B, R) be a formal context, and consider the set $\{Y_i \mid Y_i \text{ is a reduct}, i \in \Lambda\}$ of all reducts of (A, B, R). Then A can be divided into the following three parts:*

1. *Absolutely necessary attributes (core attribute) $A_c = \bigcap_{i \in \Lambda} Y_i$.*
2. *Relatively necessary attributes $A_r = (\bigcup_{i \in \Lambda} Y_i) \smallsetminus (\bigcap_{i \in \Lambda} Y_i)$.*
3. *Absolutely unnecessary attributes $A_u = A \smallsetminus (\bigcup_{i \in \Lambda} Y_i)$.*

It can be checked that $\{a_c\}^{\downarrow} \neq \{a_r\}^{\downarrow}$, $\{a_r\}^{\downarrow} \neq \{a_u\}^{\downarrow}$, $\{a_c\}^{\downarrow} \neq \{a_u\}^{\downarrow}$ for all $a_c \in A_c, a_r \in A_r, a_u \in A_u$.

Now, we will introduce a mechanism in order to obtain a reduct from a given context. The most important feature is that it is not necessary to obtain all the concepts in order to classify the attributes.

Proposition 1. *Let (A, B, R) be a context and consider $a \in A$. If $aR = \bigcap\{a_i R \mid a_i \in \{a\}^{\downarrow\uparrow} \smallsetminus \{a\}\}$, then $A \smallsetminus \{a\}$ is a consistent set.*

Proof. Let us write $\overline{A}$ for $A \smallsetminus \{a\}$. Thus, we need to prove that, given $(X, Y) \in \mathcal{B}(A, B, R)$, there exists $(X', Y') \in \mathcal{B}(\overline{A}, B, R_{\overline{A}})$ such that $X = X'$.

If $a \notin Y$, then $Y \subseteq \overline{A}$ and we consider $(X', Y') = (X, Y)$. Otherwise, we have that $\{a\}^{\downarrow\uparrow} \subseteq Y$ and we consider $Y' = Y \smallsetminus \{a\}$. Hence,

$$\begin{aligned} X = Y^{\downarrow} &= \bigcap\{a_i R \mid a_i \in Y\} \\ &= \bigcap\{a_i R \mid a_i \in Y \smallsetminus \{a\}^{\downarrow\uparrow}\} \cap \bigcap\{a_i R \mid a_i \in \{a\}^{\downarrow\uparrow}\} \\ &\overset{(*)}{=} \bigcap\{a_i R \mid a_i \in Y \smallsetminus \{a\}^{\downarrow\uparrow}\} \cap \bigcap\{a_i R \mid a_i \in \{a\}^{\downarrow\uparrow} \smallsetminus \{a\}\} \\ &= (Y')^{\downarrow^{\overline{A}}} = X' \end{aligned}$$

where $(*)$ follows by hypothesis. □

Lemma 1. *Let (A, B, R) be a context, and $a_1, a_2 \in A$. If $\{a_1\}^{\downarrow\uparrow} = \{a_2\}^{\downarrow\uparrow}$, then $a_1 R = a_2 R$ (which is equivalent to $\{a_1\}^{\downarrow} = \{a_2\}^{\downarrow}$).*

Given a context (A, B, R), with $A = \{a_1, \dots, a_m\}$, we have the following set of intents $I_0 = \{\{a_1\}^{\downarrow\uparrow}, \dots, \{a_m\}^{\downarrow\uparrow}\}$. Set equality defines an equivalence relation in I_0, and we denote by $[Y]$ the equivalence class of Y, for all $Y \subseteq A$.

Proposition 2. *Let (A, B, R) be a context, and $a \in A$. If*

$$[\{a\}^{\downarrow\uparrow}] = \{\{a_1\}^{\downarrow\uparrow}, \dots, \{a_n\}^{\downarrow\uparrow}\}$$

and $|\{a\}^{\downarrow\uparrow}| = n$, with $n \geq 2$, then $a_1, \dots, a_n$ are relatively necessary attributes.

Proof. The hypothesis $|\{a\}^{\downarrow\uparrow}| = n$ implies that $\{a\}^{\downarrow\uparrow} = \{a_1, \dots, a_n\}$, $n \geq 2$. Without loss of generality we can assume that $a = a_n$ and, therefore,

$$[\{a_n\}^{\downarrow\uparrow}] = \{\{a_1\}^{\downarrow\uparrow}, \dots, \{a_n\}^{\downarrow\uparrow}\}$$

now, by Lemma 1, we have that $a_1 R = a_2 R = \cdots = a_n R$. As a result, given the concept $(\{a_n\}^{\downarrow}, \{a_n\}^{\downarrow\uparrow}) \in \mathcal{B}(A, B, R)$, the extent $\{a_n\}^{\downarrow}$ is equal to each of the $a_i R$, with $i \in \{1, \dots, n\}$. In particular,

$$a_1 R = \bigcap \{a_i R \mid a_i \in \{a_n\}^{\downarrow\uparrow} \smallsetminus \{a_1\}\}$$

by Proposition 1, we obtain that $Y = A \smallsetminus \{a_1\}$ is consistent and the pair $(\{a_n\}^{\downarrow^Y}, \{a_n\}^{\downarrow^Y\uparrow_Y})$ is an element of the concept lattice $\mathcal{B}(Y, B, R_Y)$, where $\{a_n\}^{\downarrow^Y} = \{a_n\}^{\downarrow} = a_n R$, and $\{a_n\}^{\downarrow^Y\uparrow_Y} = \{a_2, \dots, a_n\}$.

The procedure above can be successively applied to the attributes $a_2, \dots,$ a_{n-1}, obtaining that $Z = A \smallsetminus \{a_1, \dots, a_{n-1}\}$ is consistent and $(\{a_n\}^{\downarrow^Z}, \{a_n\}^{\downarrow^Z\uparrow_Z})$ is an element of the concept lattice $\mathcal{B}(Z, B, R_Z)$, where $\{a_n\}^{\downarrow^Z} = \{a_n\}^{\downarrow} = a_n R$, and $\{a_n\}^{\downarrow^Z\uparrow_Z} = \{a_n\}$.

Let us see that a_n belongs to at least one reduct. For each reduct $Z' \subseteq Z$, the attribute a_n must belong to Z' since, otherwise, there would not exist an element in the resulting concept lattice which is related to $(\{a_n\}^{\downarrow^Z}, \{a_n\}^{\downarrow^Z\uparrow_Z})$, and this would imply that Z' is not consistent, which is contradictory.

Now, it is easy to check that a_n cannot belong to every reduct, since the initial procedure could have been done with respect to any other attribute among $a_1, \dots, a_{n-1}$, with $n \geq 2$. Therefore, a_n (actually any a_i) is a relatively necessary attribute. □

The previous proposition can be extended to the case in which the cardinality of the intent $\{a\}^{\downarrow\uparrow}$ is greater than n. The obtained result depends on whether the cardinality is equal to $n + 1$ or strictly greater than that value.

Proposition 3. *Let (A, B, R) be a context, and $a \in A$. Assume that*

$$[\{a\}^{\downarrow\uparrow}] = \{\{a_1\}^{\downarrow\uparrow}, \dots, \{a_n\}^{\downarrow\uparrow}\} \text{ and } aR = \bigcap \{a_k R \mid a_k \in \{a\}^{\downarrow\uparrow} \smallsetminus \{a_1, \dots, a_n\}\}$$

then the following statements hold:

- *If $|\{a\}^{\downarrow\uparrow}| = n + 1$, then all the elements in $\{a\}^{\downarrow\uparrow}$ are relatively necessary.*

– *If* $|\{a\}^{\downarrow\uparrow}| \geq n+2$*, then* $a_1, \dots, a_n$ *are absolutely unnecessary.*

Proof. Follows the idea of the previous proposition. □

Under certain circumstances, we can recognize absolute unnecessary and relative necessary attributes, and it is possible to prove that the rest of attributes are absolute necessary.

Therefore, given a context (A, B, R), where $A = \{a_1, \dots, a_m\}$, we have a method that computes the character of all attributes and the reducts of (A, B, R). Firstly, we compute the subsets $\{\{a_1\}^{\downarrow\uparrow}, \dots, \{a_m\}^{\downarrow\uparrow}\}$ and we apply Proposition 1, 2 and 3 in order to obtain consistent sets of attributes and classifying them. Finally, note that when we cannot apply Proposition 1, we have obtained a reduct of (A, B, R).

Notice that it is possible to obtain reducts before building the concept lattice, as in [9,19]. As a result, we can notably reduce the complexity of its computation of the concept lattice.

4 Attribute reduction using linguistic labels

Attribute reduction is an interesting tool in order to reduce the complexity of the computation of concept lattices [9,19] The extension of the methods used in classical formal concept analysis to fuzzy environments is very complex. In this section, starting with the multi-adjoint concept lattice, firstly, we will apply a weak defuzzification, using linguistic labels and a tolerance level given by the user, obtaining a new set of attributes, then we apply the result in the previous section in order to reduce the cardinality of this new set attributes with the goal of reducing the size of the original set of attributes.

From now on we will consider the lattice $(L, \preceq)$ as the unit interval $([0,1], \leq)$. For practical matters, the use of the unit interval is excessively expressive since it is often the case that only several degrees are needed. Thus, to begin with, assume that the user is asked about how many degrees will be required, and we will consider a partition of the unit interval in such a number of subintervals. For instance,we will consider

$$I_{n+1} = \{[x_0, x_1], (x_1, x_2], \dots, (x_n, x_{n+1}]\}$$

such that $x_0 = 0$ and $x_{n+1} = 1$, for all $i \in \{0, 1, \dots, n-1\}$.

Now, a set H of linguistic labels such as low, medium, high, very, more or less, much, essentially, slightly, ..., will be assigned to the previous partition of the unit interval by a mapping $\phi\colon H \to I_{n+1}$.

Note that in the rest of the paper, we will often directly refer to the set of labels as an ordered set $H = \{h_0, h_1, \dots, h_n\}$ to denote that $\phi(h_i) = (x_i, x_{i+1}]$. For example, if we consider $H = \{$Low, Medium-Low, Medium-High, High$\}$ we can assume the following regular partition:

$$I_4 = \{[0, 1/4], (1/4, 2/4], (2/4, 3/4], (3/4, 1]\}$$

where ϕ assigns Low with $[0, 1/4]$, Medium-Low with $(1/4, 2/4]$, Medium-High with $(2/4, 3/4]$, and High with $(3/4, 1]$.

Now, we will consider a fuzzy or multi-adjoint context which set of attributes will be reduced by using the ideas described at the beginning of the section.

Let (A, B, R, σ) be a multi-adjoint context and $H = \{h_0, h_1, \dots, h_n\}$ be a list of labels. The cardinality of H depends on the level of tolerance than the user may assume. Thus, we will be working with a regular partition of the unit interval in $n+1$ pieces, I_{n+1}, and the mapping $\phi\colon H \to I_{n+1}$.

We consider a new *crisp* context (A^H, B, R^ϕ) where the set of objects is equal to the original one, the set of attributes is extended by composing each of the labels with each attribute from the original A, that is, $A^H = \{h_i a \mid i \in \{0, \dots, n\}, a \in A\}$, and finally the relation $R^\phi\colon A^H \times B \to \{0, 1\}$ is defined as

$$R^\phi(h_i a, b) = \begin{cases} 1 & \text{if } R(a, b) \in \phi(h_i) \\ 0 & \text{otherwise} \end{cases}$$

The following example will be used the rest of the paper in order to show the definitions and the procedure we are introducing.

Example 1. Let us consider an example in which a number of journals are considered as objects and several parameters appearing in the ISI Journal Citation Report are the set of attributes.

The sets of attributes and objects are the following:

$$A = \{\text{Impact Factor}, \text{Immediacy Index}, \text{Cited Half-Life}, \text{Best Position}\}$$
$$B = \{\text{AMC}, \text{CAMWA}, \text{FSS}, \text{IEEE-FS}, \text{IJGS}, \text{IJUFKS}, \text{JIFS}\}$$

where the "best position" means the best quartile of the different categories under which the journal is included, and the journals considered are Applied Mathematics and Computation (AMC), Computer and Mathematics with Applications (CAMWA), Fuzzy Sets and Systems (FSS), IEEE transactions on Fuzzy Systems (IEEE-FS), International Journal of General Systems (IJGS), International Journal of Uncertainty Fuzziness and Knowledge-based Systems (IJUFKS), Journal of Intelligent and Fuzzy Systems (JIFS).

The fuzzy relation between them, $R\colon A \times B \to [0, 1]$, is the normalization to the unit interval $[0, 1]$ of the information in the JCR, and can be seen in Table 1.

Before computing the multi-adjoint concept lattice it is certainly advantageous to reduce the number of attributes in order to decrease the complexity of its computation. In order to do this, let as assume a level of tolerance for the defuzzification. In this example we will consider a list of of four labels, $H = \{\text{Low}, \text{Medium-Low}, \text{Medium-High}, \text{High}\}$. Hence, we will be working with the regular partition $I_4 = \{[0, 1/4], (1/4, 2/4], (2/4, 3/4], (3/4, 1]\}$.

Hence, we have a new context (A^H, B, R^ϕ), where A^H and R^ϕ are shown in Table 2, applying the definitions above.

Table 1. Fuzzy relation between the attributes and the objects.

R	AMC	CAMWA	FSS	IEEE-FS	IJGS	IJUFKS	JIFS
Impact Factor	0.34	0.21	0.52	0.85	0.43	0.21	0.09
Immediacy Index	0.13	0.09	0.36	0.17	0.1	0.04	0.06
Cited Half-Life	0.31	0.71	0.92	0.65	0.89	0.47	0.93
Best Position	0.75	0.5	1	1	0.5	0.25	0.25

Table 2. Crisp relation R^ϕ between the new attributes A^H and the objects B.

R^ϕ	AMC	CAMWA	FSS	IEEE-FS	IJGS	IJUFKS	JIFS
Low IF	0	1	0	0	0	1	1
Medium-Low IF	1	0	0	0	1	0	0
Medium-High IF	0	0	1	0	0	0	0
High IF	0	0	0	1	0	0	0
Low II	1	1	0	1	1	1	1
Medium-Low II	0	0	1	0	0	0	0
Medium-High II	0	0	0	0	0	0	0
High II	0	0	0	0	0	0	0
Low CHL	0	0	0	0	0	0	0
Medium-Low CHL	1	0	0	0	0	1	0
Medium-High CHL	0	1	0	1	0	0	0
High CHL	0	0	1	0	1	0	1
Low BP	0	0	0	0	0	1	1
Medium-Low BP	0	1	0	0	1	0	0
Medium-High BP	1	0	0	0	0	0	0
High BP	0	0	1	1	0	0	0

Once we have obtained the crisp context above, we can focus on reducing the number of attributes; any method will do but, in the following, we apply the method based on the results of the previous section. For example, if in an instance of our running example the attributes Low IF, Medium-Low IF and High IF absolutely unnecessary and, moreover, Medium-High IF and Medium-Low II are relatively necessary, then we would delete Medium-High IF instead of deleting Medium-Low II because deleting both Low IF, Medium-Low IF, High IF and Medium-High IF we would reduce the attribute Impact Factor in the original context.

As stated in the previous paragraph, the main idea in this stage is looking for full blocks of superfluous (crisp) attributes in order to reduce original attributes, that is, if we can reduce all the crisp attributes $H_a = \{h_0 a, h_1 a, \dots, h_n a\}$ in the modified context (A^H, B, R^ϕ), then we are able to reduce attribute a in the original context (A, B, R).

Example 2. Continuing with Example 1, we can check that: Medium-High IF, High IF, Medium-Low II, Medium-High II, High II, Low CHL, Medium-High BP are absolutely unnecessary; the final context is shown in Table 3. Therefore, the

Table 3. Crisp relation after reduction.

R^{ϕ}	AMC	CAMWA	FSS	IEEE-FS	IJGS	IJUFKS	JIFS
Low IF	0	1	0	0	0	1	1
Medium-Low IF	1	0	0	0	1	0	0
Low II	1	1	0	1	1	1	1
Medium-Low CHL	1	0	0	0	0	1	0
Medium-High CHL	0	1	0	1	0	0	0
High CHL	0	0	1	0	1	0	1
Low BP	0	0	0	0	0	1	1
Medium-Low BP	0	1	0	0	1	0	0
High BP	0	0	1	1	0	0	0

reduction given in the crisp context has not direct consequences in the original context. As a result, we may affirm that the set of original attributes are not much dependent.

Now, assume the following modification in the original relation, in which we have $R(\text{Best Position}, \text{IJGS}) = {}^{1}\!/_{4}$ and that $R(\text{Best Position}, \text{CAMWA}) = {}^{3}\!/_{4}$. The new context is shown in Table 4.

Table 4. New relation R'^{ϕ} between the new attributes A^{H} and the objects B.

R'^{ϕ}	AMC	CAMWA	FSS	IEEE-FS	IJGS	IJUFKS	JIFS
Low IF	0	1	0	0	0	1	1
Medium-Low IF	1	0	0	0	1	0	0
Medium-High IF	0	0	1	0	0	0	0
High IF	0	0	0	1	0	0	0
Low II	1	1	0	1	1	1	1
Medium-Low II	0	0	1	0	0	0	0
Medium-High II	0	0	0	0	0	0	0
High II	0	0	0	0	0	0	0
Low CHL	0	0	0	0	0	0	0
Medium-Low CHL	1	0	0	0	0	1	0
Medium-High CHL	0	1	0	1	0	0	0
High CHL	0	0	1	0	1	0	1
Low BP	0	1	0	0	0	1	1
Medium-Low BP	0	0	0	0	0	0	0
Medium-High BP	1	0	0	0	1	0	0
High BP	0	0	1	1	0	0	0

The absolutely unnecessary attributes now are Medium-High IF, High IF, Medium-Low II, Medium-High II, High II, Low CHL and Medium-Low BP. But, in addition,

$$[\{\text{Low IF}\}^{\downarrow\uparrow}] = \{\{\text{Low IF}\}^{\downarrow\uparrow}, \{\text{Low BP}\}^{\downarrow\uparrow}\}$$

and

$$[\{\text{Medium-Low IF}\}^{\downarrow\uparrow}] = \{\{\text{Medium-Low IF}\}^{\downarrow\uparrow}, \{\text{Medium-Low BP}\}^{\downarrow\uparrow}\}$$

Hence, we choose to delete both Low IF and Medium-Low IF in order to completely get rid of the full set of labelled versions of Impact Factor. As a result, we obtain the context given in Table 5 and, therefore, Impact Factor can be removed in the original context.

Table 5. Crisp relation after reduction.

R	AMC	CAMWA	FSS	IEEE-FS	IJGS	IJUFKS	JIFS
Low II	1	1	0	1	1	1	1
Medium-Low CHL	1	0	0	0	0	1	0
Medium-High CHL	0	1	0	1	0	0	0
High CHL	0	0	1	0	1	0	1
Low BP	0	1	0	0	0	1	1
Medium-High BP	1	0	0	0	1	0	0
High BP	0	0	1	1	0	0	0

5 Conclusions and future work

We have started to consider the problem of attribute reduction in the multi-adjoint extension of formal concept analysis. However, the solution found is based on the development of specific results which allow us to reduce the number of attributes in the classical case, by detecting some relatively necessary and absolutely unnecessary attributes and, then, use linguistic labels in order to obtain a method to reduce the number of attributes in a multi-adjoint context, working up to some level of tolerance, and preserving the original lattice structure of the set of concepts. Certainly, this idea is applicable to any other fuzzy approach to FCA.

So far we have not conducted any experimental tests but, this is future work to be done in the short term; however, as the method to compute the reduced set of attributes does not require to build the whole concept lattice in advance, the complexity to obtain the multi-adjoint concept lattice, after attribute reduction, is expected to decrease notably.

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