Obtaining the Minimal Polygonal Representation of a Curve by Means of a Fuzzy Clustering

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Abstract. The problem of obtaining of a minimal polygonal representation of a plane digital curve is treated. Means of a fuzzy clustering method are used. The fuzzy clustering is realized by relations of similarity and dissimilarity that are defined on a planar digital curve.

1 Introduction

As a rule the some set of features is extracted in image to analyse and recognize an object in the image. We will distinguish between low- and high-level features in the image. The low-level features are features that may be extracted without information about a special location of certain parts of the image object [14]. In the contrary the information about special location of certain parts of the image object is used to detect high-level features. The edges of image object, the curvature of a curve on the image are the main low-level features in the image. The low-level features are aggregated for receive a compact representation of an image object. As a result we will receive a high-level representation of image object, for example, a curve. The compact curve representation is necessary for image compression, image vectorization etc. In general digital curve Γ depend on a set of parameters so the number of parameters may be equal to the number of points of digital curve. In this case a problem of representation is to find the curve Γ' that depends from a smaller set of parameters and saves a main information about the curve Γ. There are many methods of solving of this problem, which may be divided into two groups – the group of approximate methods and the group of interpolative methods. The methods of first group are based on the replacement of digital curve Γ by a such curve of some fixed class that satisfies to some conditions "nearness". The methods based on the Bezier curves and on the B-spline are the most popular approximate methods of finding the curve representation [15], [13]. The application of those methods requires a prior detection of knots of spline and this task is equal to a general task of a curve representation. The methods of second group assumes the choice of some set of points on Γ and replacement of every part of curve between the two neighbor points by the other curve from some fixed class (for example, class of segments, circles, algebraic curves of some order etc.) in agreement with the defined optimal conditions. The straight-line interpolation is called by polynomial representation of curve. There are two main approaches of solving the task of polynomial representation of curve: heuristic and optimization. The algorithms based on extraction of dominant points, on the using of split-junction procedure for a side of polygon (for example, Douglas-Peucker algorithm), genetic algorithms [6], algorithms of multiple smoothing [24], algorithms on the fuzzy logic [11] etc. are referred to the algorithms of the first approach. These algorithms are rapid but not optimal as a rule. Algorithms of the second approach assume to find such approximate polygonal line which satisfied to a defined optimal condition. The conditions which are considered as an optimal criterion are following: 1) the polygon with fixed number of vertex must have minimal perimeter [23]; 2) the maximal distance between the points of the curve and segments of the polygon must be a minimal [18]; 3) the number of segment of polygon with approximation error must be a minimal [4]; 4) the area of a symmetric difference between the set bounded by a closed curve and set bounded by the polygon must be a minimal [28]; 5) the approximation error of polygon with a fixed length of a segment must be a minimal [21]. Thus the algorithms of second approach solve tasks of nonlinear optimization with boundary conditions. The majority of algorithms mentioned above are suboptimal. Almost all algorithms of finding of compact polygonal representation assumes the preliminary finding of basis set of curve points which must be optimized with a respect to the chosen criteria. The set of points with a high curvature is chosen as a basis set. At this paper we will consider the approach to find polygonal representations of curve with a help of fuzzy clustering methods. The main idea of this approach bases on a fact that the quantitative low-level local features of a curve at the given point may be considered as a degree of membership of this point to polygonal representation. The curve itself is considered as a fuzzy set. Then a problem of finding of a minimal representation of a fuzzy set may be formulated as a solution of a task of a fuzzy clustering.

2 Statement of Problem

We will considered the plane discrete curve $\mathit{\Gamma} = (\mathbf{g}_k)_{k=0}^{n-1}$, $\mathbf{g}_k = x_k\mathbf{i} + y_k\mathbf{j}$. The points $\mathbf{g}_k$, $k = 0, ..., n-1$, belongs to $\mathbb{Z}^2$ and they satisfy to a condition of a connectivity in the initial contour which will be used in an image processing. We will consider the set of points of curve $\mathit{\Gamma}$ as an ordering set. We want to extract some subset $B = \{\mathbf{g}_{i_1}, ..., \mathbf{g}_{i_l}\}$ of points of a curve $\mathit{\Gamma}$ that will be a "good" representation of $\mathit{\Gamma}$.

The minimal polygonal representations of curve must consist of such points $\mathbf{g}$ of curve $\mathit{\Gamma}$ which have a great information capacity relatively to a given set of features $\{\omega_i\}_{i\in I}$. We will consider only local features: low-level features of curve. We may be consider those features as some functions of points of curve: $\omega_i(\mathbf{g})$, $\mathbf{g} \in \mathit{\Gamma}$, $i \in I$. It will be assumed that $\omega_i(\mathbf{g}) \in [0,1]$ for all $\mathbf{g} \in \mathit{\Gamma}$, $i \in I$ and $\omega_i(\mathbf{g}) \leq \omega_i(\mathbf{h})$, if the point $\mathbf{h} \in \mathit{\Gamma}$ is more informative than point $\mathbf{g} \in \mathit{\Gamma}$ relatively of feature ω_i. The normalized estimation of curvature and the normalized variation of contour length after deletion of point $\mathbf{g}$ are by examples of such features functions [2]. The function $\omega_i(\mathbf{g})$ characterizes the degree of membership of point $\mathbf{g}$ to set informative points of curve Γ relatively i-th feature. Therefore the set of informative points of curve Γ relatively i-th feature may be considered as a fuzzy set $\{(\mathbf{g}, \omega_i(\mathbf{g})),\ \mathbf{g} \in \Gamma\}$ with membership function ω_i. If we consider the information capacity of points of curve Γ relatively to the set of features $\{\omega_i\}_{i\in I}$ set Γ can be considered in terms of a fuzzy set with membership function $\omega(\mathbf{g}) = T(\omega_i(\mathbf{g}))$, where $T(\cdot)$ – t-norm on $[0,1]^I$ [9]. For example, $T(\omega_i) = \min\limits_i \omega_i$ or $T(\omega_i) = \prod_{i\in I} \omega_i$. In general case we can use some nonnegative function from feature $\mu_\Gamma(\mathbf{g}) = f(\omega(\mathbf{g}))$ as a membership function. Then we may formulated the task of finding of such minimal fuzzy subset B of set Γ for which the set $\{\omega(\mathbf{g})\}_{\mathbf{g}\in \mathrm{B}}$ will be an optimal representation of $\{\omega(\mathbf{g})\}_{\mathbf{g}\in\Gamma}$. Let's specify a statement of problem. Let's consider α-cut $\mathrm{B}_\alpha = \{\mathbf{g} \in \Gamma : \omega(\mathbf{g}) \geq \alpha\{$ of fuzzy set Γ for some fixed value $\alpha \in [0,1]$. The set B_α is a some representation of a contour Γ. It is necessary to find such value of parameter $\alpha \in [0,1]$ that the representation B_α will be minimal on the one hand and will be optimal on other hand. The finding of minimal representation of a fuzzy set is a task of fuzzy clustering. The main ways to solve a fuzzy clustering task were considered in the works [19],[20], [5], [1] etc. The review of fuzzy clustering methods may be found in [29]. The modern state of problem may is reviewed in [12]. One approach to fuzzy clustering consists to definition of some functionals on the set of all representations, which then are optimized to receive a desired clustering.

3 The Using of Similarity Relation

Let us consider representation B_α of contour Γ, $\alpha \in [0,1]$ with membership function $\mu_\alpha^\omega(\mathbf{g}) = \begin{cases} \omega\ (\mathbf{g})|\mathrm{B}_\alpha|, & \mathbf{g} \in \mathrm{B}_\alpha, \\ 0, & \mathbf{g} \notin \mathrm{B}_\alpha. \end{cases}$ We are introduced into consideration so called fuzzy similarity relation $r(\mathbf{g},\mathbf{h})$ on Γ that is reflexive, symmetric fuzzy relation satisfying to inequality $|r(\mathbf{g},\mathbf{h}) - r(\mathbf{g},\mathbf{e})| \leq 1 - r(\mathbf{h},\mathbf{e})$ for all $\mathbf{e},\mathbf{g},\mathbf{h} \in \Gamma$ for construction of identifying functional. The last inequality is an equivalent to condition for strongly Δ -transitive relation (respect to t-norm $a\Delta b = \max\{a+b-1, 0\}$) [7]. The equivalence of strongly Δ-similarity (that is reflexive, symmetric, strongly Δ-transitive relation) and Δ-similarity was proved in [7]. The coherent nearness relation [3] is weak. By analogy with E.H.Ruspini we called set B_α by fuzzy r-representation of set Γ if the inequality

$$\sum_{\mathbf{h}\in\Gamma} r(\mathbf{g},\mathbf{h})\mu_\alpha^\omega(\mathbf{h}) \geq \mu_\Gamma(\mathbf{g}) \tag{1}$$

is holds for all $\mathbf{g} \in \Gamma$. The efficiency of such clustering depends on a fuzzy similarity relation $r(\mathbf{g},\mathbf{h})$. The choice of this relation is defined by classification features. In particular, the function $r(\mathbf{g},\mathbf{h}) = 1 - n^{-1}\sum_{i=1}^n \rho_i(\omega_i(\mathbf{g}), \omega_i(\mathbf{h}))$ is the similarity relation, where $\omega_i(\mathbf{g})$ is an informativity function of the i-th feature of point $\mathbf{g}$, ρ_i is such metric in R^1 that $\rho_i(a,b) \leq 1$ for all $a, b \in [0,1]$. We will consider the similarity relation $r(\mathbf{g},\mathbf{h}) = 1 - |\omega(\mathbf{g}) - \omega(\mathbf{h})|$ below. Then (1) take the form

$$|\mathrm{B}_\alpha| \sum_{\mathbf{h}\in \mathrm{B}_\alpha} (1 - |\omega(\mathbf{g}) - \omega(\mathbf{h})|)\, \omega(\mathbf{h}) \geq \omega(\mathbf{g})|\Gamma| \tag{2}$$

for all $\mathbf{g} \in \Gamma$. It is obvious to see that $\mathrm{B}_\alpha = \Gamma$ povided (2) is valid. Thus the task consists of a maximal reduction of a cardinality of B_α (with increased α) until (2) remains valid. The set B_α of minimum cardinality for which (2) is valid we will call by minimal r-representation of set Γ and will denote by $\underline{\mathrm{B}}_\alpha$. The following inequality may be get from (2) if we considered that $\omega(\mathbf{g}) < \alpha$ if $\mathbf{g} \in \Gamma\backslash \mathrm{B}_\alpha$:

$$\sum_{\mathbf{h}\in \mathrm{B}_\alpha} (1 - \omega(\mathbf{h}))\omega(\mathbf{h}) \geq \left(\frac{|\Gamma|}{|\mathrm{B}_\alpha|} - \sum_{\mathbf{h}\in \mathrm{B}_\alpha} \omega(\mathbf{h}) \right) \max_{\mathbf{g}\in \Gamma\backslash \mathrm{B}_\alpha} \omega(\mathbf{g}). \tag{3}$$

Thus we proved the validity of the following proposition.

Proposition 1. *If set B_α is a fuzzy r-representation of set Γ then (3) is correct.*

The contrary of this statement may be is not true in general. The algorithm of finding of minimal representation $\underline{\mathrm{B}}_\alpha$ consists of two steps: 1) to find the set $\mathrm{B}_\alpha^{(1)}$ of minimum cardinality for which (3) is valid; 2) to add (if it is necessary) the set $\mathrm{B}_\alpha^{(1)}$ by such points $\mathbf{h} \in \Gamma\backslash \mathrm{B}_\alpha^{(1)}$ that (2) is correct. Let $\widehat{\Gamma} = \{\mathbf{h}_i\}_{i=1}^{|\Gamma|}$ be a set of points of contour Γ ordered by decreasing of weights $\omega(\mathbf{h})$, $\mathbf{h} \in \Gamma$. Calculate the function

$$Q(p) := \sum\nolimits_{i=1}^{p} (1 - \omega(\mathbf{h}_i))\omega(\mathbf{h}_i)$$

and let

$$R(p) := \left(\frac{|\Gamma|}{p} - \sum\nolimits_{i=1}^{p} \omega(\mathbf{h}_i) \right) \max_{p+1\leq j\leq |\Gamma|} \omega(\mathbf{g}_j)$$

for $p = 1, 2, ..., |\Gamma|$. The minimum p for which $Q(p) \geq R(p)$ will be define a boundary of partition of set $\widehat{\Gamma}$ on two classes $\mathrm{B}_\alpha^{(1)} := \left\{ \mathbf{h}_i \in \widehat{\Gamma} :\ i = 1, 2, ..., p \right\}$ and $\Gamma\backslash \mathrm{B}_\alpha^{(1)}$ as a consequence from (3). On the second step are to find such point $\mathbf{h} \in \Gamma\backslash \mathrm{B}_\alpha^{(1)}$ for which $(1 - |\omega(\mathbf{g}) - \omega(\mathbf{h})|)\, \omega(\mathbf{h}) \to \max$ for all $\mathbf{g} \in \Gamma\backslash \left(\mathrm{B}_\alpha^{(1)} \cup \{\mathbf{h}\} \right)$. We will check the validity of condition (2) for set $\mathrm{B}_\alpha^{(2)} = \mathrm{B}_\alpha^{(1)} \cup \{\mathbf{h}\}$. If (2) is not correct then we add the new point from $\Gamma\backslash \mathrm{B}_\alpha^{(2)}$ to the set $\mathrm{B}_\alpha^{(2)}$ etc. We have a question. Will we get a minimal fuzzy r-representation of curve Γ with help of suggested algorithm indeed? The following proposition gives us the condition when we will get the minimal fuzzy r-representation after the first step.

Proposition 2. *If we get after the first step of algorithm such a representation $\mathrm{B} = \mathrm{B}_\alpha^{(1)}$ that*

$$\sum_{\mathbf{h}\in \mathrm{B}} (1 - \omega(\mathbf{h}))^2 \leq 1 + |\mathrm{B}| - \frac{|\Gamma|}{\mathrm{B} + 1} \tag{4}$$

and $|\Gamma| \max_{\mathbf{g}\in\Gamma} \omega(\mathbf{g}) \leq \alpha^2 |\mathrm{B}|^2$ then $\mathrm{B}_\alpha^{(1)}$ will be a minimal fuzzy r-representation of a curve Γ.

Proof. Firstly we show that the representation $\mathrm{B}_\alpha^{(1)}$ formed on the first step of algorithm is a minimal representation for which (3) is satisfied. To show this we consider the set function

$$\phi(\mathrm{B}) = \sum_{\mathbf{h}\in\mathrm{B}} (1-\omega(\mathbf{h}))\omega(\mathbf{h}) \Bigg/ \left(\frac{|\Gamma|}{|\mathrm{B}|} - \sum_{\mathbf{h}\in\mathrm{B}} \omega(\mathbf{h})\right),$$

where $\mathrm{B} \subseteq \Gamma$ such that $\sum_{\mathbf{h}\in\mathrm{B}} \omega(\mathbf{h}) < \frac{|\Gamma|}{|\mathrm{B}|}$. Let $\phi(\emptyset) = 0$. Let us show that ϕ is monotone set function. Let $S_i = \sum_{\mathbf{h}\in\mathrm{B}} \omega^i(\mathbf{h})$, $\delta_i = \frac{|\Gamma|}{|\mathrm{B}|+i-1}$, $i = 1,2$. Then $\phi(\mathrm{B}) = \frac{S_1-S_2}{\delta_1-S_1}$. We have

$$\psi(\omega(\mathbf{g})) = \phi(\mathrm{B}\cup\{\mathbf{g}\}) = \frac{S_1 - S_2 + \omega(\mathbf{g}) - \omega^2(\mathbf{g})}{\delta_2 - S_1 - \omega(\mathbf{g})}$$

for such every $\mathbf{g} \in \Gamma\backslash\mathrm{B}$ that $\omega(\mathbf{g}) + \sum_{\mathbf{h}\in\mathrm{B}} \omega(\mathbf{h}) < \frac{|\Gamma|}{|\mathrm{B}|+1}$. Then $\phi(\mathrm{B}\cup\{\mathbf{g}\}) - \phi(\mathrm{B}) = \frac{\omega(\mathbf{g})-\omega^2(\mathbf{g})}{\delta_2-S_1-\omega(\mathbf{g})} + \frac{\psi(S_1-S_2)(\delta_1-\delta_2+\omega(\mathbf{g}))}{(\delta_2-S_1-\omega(\mathbf{g}))(\delta_2-S_1)} \geq 0$ such as $S_1 \geq S_2$, $\delta_1 \geq \delta_2$. Therefore ϕ is a monotone set function. In addition $\psi(x)$ is a monotone function on $[0,1]$. Indeed, we have $\psi'(x) = \frac{x^2-2x(\delta_2-S_1)+\delta_2-S_2}{(\delta_2-S_1-x)^2}$. Two cases are possible. At the first case the minimal value of numerator of derivative $\psi'(x)$ up to $x = 1$ if $\delta_2 - S_1 > 1$. In this case we have $\psi'(x) \leq 0$ if $\delta_2 \geq 1 + 2S_1 - S_2 \Leftrightarrow (4)$. At the other case $0 \leq \delta_2 - S_1 \leq 1$ and the minimal value of numerator of derivative $\psi'(x)$ up to $x = \delta_2 - S_1$ and $\psi'(x) \geq 0$ if $\delta_2 - S_2 - (\delta_2 - S_1)^2 \geq 0$. The last inequality is true because $\delta_2 - S_2 - (\delta_2 - S_1)^2 \geq \delta_2 - S_1 - (\delta_2 - S_1)^2 \geq 0 = (\delta_2 - S_1)(1 - (\delta_2 - S_1)) \geq 0$. Therefore ϕ is a monotone set function and $\phi(\mathrm{B}\cup\{\mathbf{g}'\}) \geq \phi(\mathrm{B}\cup\{\mathbf{g}''\})$ if $\omega(\mathbf{g}') \geq \omega(\mathbf{g}'')$. We may get two cases when we form the set $\mathrm{B}_\alpha^{(1)}$: 1) $\sum_{\mathbf{h}\in\mathrm{B}_\alpha^{(1)}} \omega(\mathbf{h}) < \frac{|\Gamma|}{|\mathrm{B}_\alpha^{(1)}|}$; 2) there exist such point $\mathbf{g} \in \mathrm{B}_\alpha^{(1)}$ that $\frac{|\Gamma|}{|\mathrm{B}_\alpha^{(1)}|} - \omega(\mathbf{g}) < \sum_{\mathbf{h}\in\mathrm{B}_\alpha^{(1)}\backslash\{\mathbf{g}\}} \omega(\mathbf{h}) < \frac{|\Gamma|}{|\mathrm{B}_\alpha^{(1)}|}$. The set $\mathrm{B}_\alpha^{(1)}$ will be by set with minimal cardinality which satisfied to (3) since the set $\mathrm{B}_\alpha^{(1)}$ in the first case and the set $\mathrm{B}_\alpha^{(1)}\backslash\{\mathbf{g}\}$ in the second case formed from the points $\mathbf{h} \in \Gamma$ with maximal value of feature $\omega(\mathbf{h})$. We will show the validity of inequality (2) for set $\mathrm{B}_\alpha^{(1)}$ if the condition of proposition obeys to completing of proof of proposition. If $\mathbf{g} \in \Gamma\backslash\mathrm{B}_\alpha^{(1)}$, an inequality (2) will lead to inequality (3). Consequently it will be correct. Let $\mathbf{g} \in \mathrm{B}_\alpha^{(1)}$. Then $|\omega(\mathbf{g}) - \omega(\mathbf{h})| \leq 1 - \alpha$ for any $\mathbf{h} \in \mathrm{B}_\alpha^{(1)}$. Then we have

$$\sum_{\mathbf{h}\in\mathrm{B}_\alpha^{(1)}} (1 - |\omega(\mathbf{g}) - \omega(\mathbf{h})|)\,\omega(\mathbf{h}) \geq \alpha \sum_{\mathbf{h}\in\mathrm{B}_\alpha^{(1)}} \omega(\mathbf{h}) \geq$$

$$\alpha^2 \left|\mathrm{B}_\alpha^{(1)}\right| \geq \frac{|\Gamma|}{|\mathrm{B}_\alpha^{(1)}|} \max_{\mathbf{g}\in\Gamma} \omega(\mathbf{g}) \geq \frac{|\Gamma|}{|\mathrm{B}_\alpha^{(1)}|}\omega(\mathbf{g}).$$

The proposition is thus proved.

If we get on the first step of algorithm the set $\mathrm{B}_\alpha^{(1)}$ for which conditions of proposition aren't satisfied we must carry out the second step of algorithm. In this case we may to get the set B_α near to minimal in general.

Remark 1. The function $r_s(\mathbf{g},\mathbf{h}) = 1 - |\omega(\mathbf{g}) - \omega(\mathbf{h})|^s$, $s \in (0,1]$, may be used in the capacity of similarity relation too. This function satisfies to all conditions of similarity relation because the inequality $(a+b)^s \le a^s + b^s$ is correct for $a, b \ge 0$, $0 \le s \le 1$. It is obvious that the inclusion $\underline{\mathrm{B}}_\alpha(s_1) \supseteq \underline{\mathrm{B}}_\alpha(s_2)$ is correct for minimal r_s-representation $\underline{\mathrm{B}}_\alpha(s)$ if $0 < s_1 \le s_2 \le 1$ because $r_{s_1} \le r_{s_2}$ in this case.

4 The Using of Dissimilarity Relation

Other relation may be used in task of fuzzy clustering without similarity relation. For example, the points of minimal polygonal representation must be located far from each other on a curve Γ. We may introduce the fuzzy dissimilarity relation regarding these conditions. This relation must be antireflexive, symmetric fuzzy relation $\tau(\mathbf{g},\mathbf{h})$ and obeying to an inequality $|\tau(\mathbf{g},\mathbf{h}) - \tau(\mathbf{g},\mathbf{e})| \le \tau(\mathbf{h},\mathbf{e})$ for all $\mathbf{e},\mathbf{g},\mathbf{h} \in \Gamma$. Note that definition of fuzzy dissimilarity relation is coordinated with a fuzzy similarity relation that is introduced above. Let $f(\mathbf{g})$ be membership function of point $\mathbf{g} \in \Gamma$ to the set of informative points. Again we will call the set $\mathrm{B}_\beta = \left\{\mathbf{g} \in \Gamma : \mu_\beta^f(\mathbf{g}) \ge \beta\right\}$ with membership function $\mu_\beta^f(\mathbf{g})$ by fuzzy τ-representation of set Γ if the inequality

$$\sum_{\mathbf{h}\in\Gamma} (1 - \tau(\mathbf{g},\mathbf{h}))\left(1 - \mu_\beta^f(\mathbf{h})\right) \ge 1 - f(\mathbf{g}) \tag{5}$$

is correct for all $\mathbf{g} \in \Gamma$. We will considered that condition (5) is valid if $\mathrm{B}_\beta = \emptyset$ and isn't valid if $\mathrm{B}_\beta = \Gamma$. Thus the task is to increase maximally the cardinality of B_β (with decreased β) until (5) remains is valid. The set B_β of maximum cardinality for which (5) is valid we will call a maximal τ-representation of set Γ and will denote by $\bar{\mathrm{B}}_\beta$.

We will use the function $\tau(\mathbf{g},\mathbf{h}) = l(\mathbf{g},\mathbf{h})$ as a dissimilarity relation. Here $l(\mathbf{g},\mathbf{h})$ is a minimal length of arc of the curve Γ located between the points $\mathbf{g},\mathbf{h} \in \Gamma$, that normed by length of all curve Γ. We will use also the functions $f(\mathbf{g}) = \omega(\mathbf{g})|\Gamma|$ and $\mu_\beta^f(\mathbf{g}) = |\Gamma \backslash \mathrm{B}_\beta| \begin{cases} 1 \quad, & \mathbf{g} \in \mathrm{B}_\beta, \\ f(\mathbf{g}), & \mathbf{g} \notin \mathrm{B}_\beta \end{cases}$ as a membership function of curve Γ and set B_β correspondingly. Then inequality (5) can be rewritten:

$$|\Gamma \setminus \mathrm{B}_\beta| \sum_{\mathbf{h}\in\Gamma\backslash\mathrm{B}_\beta} (1 - l(\mathbf{g},\mathbf{h}))(1 - f(\mathbf{h})) \ge |\Gamma|(1 - f(\mathbf{g})) \tag{6}$$

for all $\mathbf{g} \in \Gamma$. Then the new formulation of task about seach of (r,τ)-representation of curve Γ follows from (2) and (6). It is necessary to find such set B for which the system of inequalities

$$\sum_{\mathbf{h}\in\mathrm{B}} (1 - |\omega(\mathbf{g}) - \omega(\mathbf{h})|)\,\omega(\mathbf{h}) \ge \frac{|\Gamma|}{|\mathrm{B}|}\omega(\mathbf{g}),$$

$$\sum_{\mathbf{h}\in\Gamma\backslash\mathrm{B}} (1 - l(\mathbf{g},\mathbf{h}))(1 - \omega(\mathbf{h})) \ge \frac{|\Gamma|}{|\Gamma \setminus \mathrm{B}|}(1 - \omega(\mathbf{g}))$$

are holds for all $\mathbf{g} \in \Gamma$. We have a question: in what case does the algorithm give us the minimal (r,τ)-representation of curve Γ ? The next statement follows from proposition 2.

Proposition 3. *If after the first step of algorithm we get such representation* $\mathrm{B}_\alpha^{(1)}$ *that (4) is true and* $|\Gamma| \max_{\mathbf{g}\in\Gamma} \omega_I(\mathbf{g}) \le \alpha^2 \left|\mathrm{B}_\alpha^{(1)}\right|^2$, $|\Gamma| \min_{\mathbf{g}\in\Gamma} \omega_I(\mathbf{g}) \ge |\Gamma| - 0.5(1-\alpha)\left|\Gamma\backslash\mathrm{B}_\alpha^{(1)}\right|^2$ *then* $\mathrm{B}_\alpha^{(1)}$ *will be a minimal fuzzy* (r,τ)*-representation of closed digital curve* Γ.

Proof. If the conditions of proposition are satisfied, then the set $\mathrm{B}_\alpha^{(1)}$ will be a minimal fuzzy r-representation as was shown in proposition 2. Now we should proof that the set $\mathrm{B}_\alpha^{(1)}$ will be fuzzy τ-representation too. To show this, it's noticed that $\sum_{\mathbf{h}\in A} l(\mathbf{g},\mathbf{h}) \le 0.5|A|$ for closed curve and any point $\mathbf{g} \in \Gamma$, $A \in 2^\Gamma$. Then we have

$$\sum_{\mathbf{h}\in\Gamma\backslash\mathrm{B}_\alpha^{(1)}} (1-l(\mathbf{g},\mathbf{h}))(1-\omega(\mathbf{h})) \ge (1-\alpha) \sum_{\mathbf{h}\in\Gamma\backslash\mathrm{B}_\alpha^{(1)}} (1-l(\mathbf{g},\mathbf{h})) \ge$$

$$0.5(1-\alpha)\left|\Gamma\backslash\mathrm{B}_\alpha^{(1)}\right| \ge \frac{|\Gamma|}{\left|\Gamma\backslash\mathrm{B}_\alpha^{(1)}\right|}\left(1-\min_{\mathbf{g}\in\Gamma}\omega(\mathbf{g})\right).$$

The proposition is thus proved.

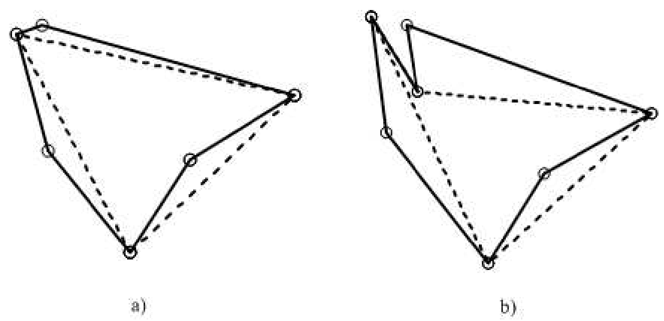

Fig. 1. The initial contour and the minimal polygonal representation of contour found by fuzzy clustering method

The results of the algorithm of a research of minimal polygonal representation of contour are shown in Fig.1. On the Fig.1.a the representation was found by fuzzy clustering method with help of similarity and dissimilarity relations separately. On the Fig.1.b the representation was found by fuzzy clustering method with help of combined using of similarity and dissimilarity relations. We used normalized estimation of curvature in the capacity of feature function $\omega(\mathbf{g})$ (see [10]). Note that the quality of algorithm work may be improved if we will use the fuzzy clustering for the few features.

5 Summary and Conclusion

In this paper we have considered two clusters in a polygonal representation of a curve. The first cluster consists of points that belong to the polygonal representation. The second cluster consists of points that not belong to the polygonal representation. In case of the crisp clustering distance within one cluster is small, whereas clusters are sparse, so two objects from different clusters are distant. The notion of distance at this paper was replaced by similarity and dissimilarity fuzzy relation. We have received the fuzzy clustering method for polygonal representation. The quality of this representation depends on a similarity and dissimilarity fuzzy relation.

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