A Linear Algebra Wiki

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Abstract

This paper is a report on an ongoing project of two years. Its purpose is to provide a resource for Linear Algebra at the undergraduate level that can, in particular, be used as a textbook. The experiences, both good and bad are given, and these are used to indicate future directions for the project.

1 Motivation

1.1 Movitation #1: Correctability

The initial motivation for this project came from teaching an introductory course in Linear Algebra for first-year university students after having not done so for many years. I was really dissatisfied with the textbook. For example, the following "proof" was given:

> Theorem 2.2.2 Let A be a square matrix. Then $\det(A) = \det(A^T)$.
>
> Proof By Theorem 2.1.1, the determinant of A found by cofactor expansion along its first row is the same as the determinant of A^T found by cofactor expansion along its first column.

Putting aside the fact that Theorem 2.1.1 is described earlier in the text as a "general theorem, which we state without proof", the purported proof itself is simply irreparably wrong in the absence of some kind of inductive argument.

Here is another sample from the same book:

> Theorem 5.2.1 If W is a set of one or more vectors from a vector space V, then W is a subspace of V if and only if the following conditions hold.
>
> (a) If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u}+\mathbf{v}$ is in W.
>
> (b) If k is any scalar and $\mathbf{u}$ is any vector in W, then $k\mathbf{u}$ is in W.

This Theorem as stated is completely correct. But a student who uses these two conditions as the defining properties of a subspace will eventually come upon a potential subspace that is empty: it will satisfy (a) and (b) without being a subspace. The phrase "one or more vectors" hidden in the hypothesis makes the result correct, but hides a property that is really essential.

Presumably errors are not included by design. Being able to make corrections between editions of a book is impossible, and new editions must be infrequent because of financial constraints; it is trivial to make corrections if the source is online.

1.2 Movitation #2: Breaking the Linear Structure of a Book

Looking at other textbooks is really no help. All seem to have problems such as those given above to a greater or lesser extent. At least part of the problem lies not with the authors, but rather with the nature of the textbook. In particular a textbook must

- assume a certain background and level of innate ability for the student, and give argument in detail consistent with that assumption,
- have a manageable finite length (although some linear algebra textbooks are now over 700 pages long!),
- be essentially linear (each chapter uses results proven in earlier chapters).

1.3 Movitation #3: It's Easier to be an Editor than an Author/Editor

Writing a book is a huge task. Revising it for later editions is almost as bad. However, writing a small section of a book is pretty easy, and it's usually not too difficult to get volunteers to undertake such a task. Similarly, the quality and the levels of difficulty of the exercises in a book is an important (and often underestimated) aspect of the presentation. Getting voluteers to contribute a modest number of exercises is not too difficult. Of course, as with any open project, there must be an editor (dictator) in charge to ensure that all contributions to the project are consistent.

2 Why a wiki

The motivations given in the last section drive the choice of software towards a wiki[3], and many of the problems mentioned in the previous section are solved easily by this choice. The greatest asset is that the data does not have a linear structure as a book does but, rather, is organized as a tree. There are several impications of this structure:

- The same material may be examined from different viewpoints. The most straightforward one may be given in a page with links to deeper or more subtle arguments.
- Forward references are natural; many times it is both expedient and useful to defer a proof of a mathematical concept until a better understanding of the underlying structures has been acquired. By custom, a book will give the proof immediately after its statement (or not at all!).
- Tangential but interesting topics can be explored via a link without interrupting the general flow of the main text.
- More repetition is possible: a theorem may have several similar parts with subtly different proofs. There is simply not enough room in a book to repeat similar proofs (a standard trick is to move some proofs to the exercises), but it is not a problem (and even encouraged) with separate links for comparable material.

There are other advantages with a wiki:

- The use of colour is quite expensive for a book; it costs nothing in a wiki.
- Photographs are quite expensive for a book; it costs nothing in a wiki.
- Putting a copy of a textbook as a file on a student's computer as, say, a PDF file is (copyright) problematic; it is straightforward with a simple wiki extension.
- The inclusion of accurate graphics (e.g., two dimensional or three dimensional plots) is also straigt-forward.
- The use of animations is easy.

Students taking a first-year university course are usually still in their teens. This means, for the current crop of students, that the home computer has always been commonplace, and that any software more than ten years old has been around forever. In particular, they have never known a time when computers didn't come with a browser that could access Wikipedia. Wikis for them are like radios for us: we know that there was a time when they didn't exist, but we can't really imagine it. Just as we can't really appreciate the awe that must have been felt by an adult who, for the first time, heard sound coming out of thin air, our first-year students feel that it is no big deal to have an enormous database like Wikipedia available from their bedrooms on a 24/7 basis.

This is, in fact, a wonderful opportunity. The student comes with a skill set that can be very useful. I have, by intention, made my wiki look much like Wikipedia. By doing this, I don't have to tell students where to look for help, or where the print button is: they just do the right thing.

As another example, a theorem may have several parts with each dependent on the proofs of the previous ones or with all of them logically equivalent. In either case, the part may be stated with a link labelled "Proof" next to it. No further explanation is necessary; the student knows exactly what to do.

3 The first steps

To test the efficacy of the wiki in the classroom, I volunteered (!) to teach a large-section, one-term linear algebra course over several terms. The students in this course have varied levels of ability and ambition. Some just want to be through with their math requirements while others know that they want an honours degree in mathematics. The content of the course is not atypical: Solving systems of linear equations, Gauss-Jordan reduction, some two and three dimensional geometry, and an introduction to vector spaces.

The initial wiki pages were actually expanded lecture notes. Students were encouraged to read them and print them out in advance of the lectures. They were also encouraged to find errors (and indeed they did!). The basic structure for the course and the problems assigned came from the textbook as usual. In a certain sense, this was an anti-wiki: the material was edited by one person and the advantage of the cooperative nature of a wiki was just thrown away. Argueably, the first step in building a quality implementation is to run a simpler prototype and find out what actually happens (goes wrong, in particular), and so that is the approach that was used. Feedback from the students could then be used as an aid for deciding the best way to proceed. The reactions of students were gauged in two ways:

- The access count of each page was monitored. In this way it was possible to see which pages were actually used.
- Comments on the wiki were invited as part of our standard student evaluation questionnaire given at the end of the course.

A second initiative concerning the wiki implementation involved the method of presentation of the mathematics on the web pages. Three approaches were used:

- The "default" approach: using software on the server to generate and cache `png` files of the mathematics used on the page.
- `MathJax`, which uses JavaScript within the browser to create the mathematics for the page on-the-fly.
- Translate a math snippet into `MathML`, which is then displayed using the abilities of the browser.

All three of these have been used as part of the initial tests of the wiki with varying results which are described in more detail in the Subsection 5.1.

4 Initial results

The response from students to the questionnaires has been overwhelming positive. Over 90% are strongly supportive of the idea of having a wiki for the presentation of course materials. They had some additional interesting ideas within the latest questionaire (May 5, 2011):

> Very fond of the homemade Wikipedia. Made finding information very simple.
>
> The wiki was actually very awesome—the only thing missing was some kind of interactive quiz that tested knowledge.
>
> It would be great if the wiki showed where we were in the course at any given time.
>
> Wonderful substitute for textbook and great was to save students a few bucks, and take advantage of new and different technologies.

There were a few really negative responses, but unfortunately they didn't go into detail. It might have been really helpful to know the reasons for their feelings.

There were also some qualified endorsements.

> The wiki is helpful but there are a few math errors that need some editing.
>
> The wiki pages took a long time in downloading and sometimes didn't work. The note printouts were missing the symbol θ sometimes so there would be a blank space left.

Many responses indicated that the student prefered a dynamic wiki to a standard textbook. How could anyone be surprised by this? Nonetheless, some questions to ask are:

- Will the wiki actually increase the level of competence of the students? For example, giving solutions to (at least some of the) exercises is certainly desirable. If these solutions are only a click away, will the temptation to look before actually working on the exercises be too great to resist?
- Will the tail wag the dog? The goal is to help the student learn the mathematics, and not to entertain them. A sense of proportion is necessary.
- Mathematics must be seen as more than a collection of algorithms. Does the wiki format help or hinder this process?

5 Implications of Display Methods

5.1 Inserting Mathematics within the Text on the Page

Several methods were used to display mathematics on the wiki pages. Each came with assets and liabilities:

- The default (`texvc`) method [2]: the material designated as mathematics is sent through LaTeX to create a `dvi` file. This file is sent through `dvipng` to create a `png` file. The `png` file is then inserted into the page as a graphic image. This method works quite well: it is very fast and produces accurate results. However, the results are bitmapped, and, as such, can not be changed when the font sizes in the display are changed (by the user or otherwise). There are usually no problems with pixellation, but the potential for problems is there for complicated constructions (the rendering of subsubscript i versus subsubscript iota, for example).

- The `MathJax` method: `MathJax` is a javascript application [1] which may be used to render mathematics on wiki pages. It produces beautiful results that may be enlarged when the font size of the surrounding text is changed. It can also be used with some of the special LaTeX packages (for example, `amsmath`) available for mathematicians. However, each time a new page is accessed, it is run through `MathJax` and the time required to render a complicated page may be lengthy. In addition, when jumping to a specific location on a new page, the position may be lost as the page is rendered.
- The `MathML` method: `MathJax` can translate the material designated as mathematics into `MathML` code. An appropriate browser is necessary, of course. When such a browser is used, the mathematics is rendered much more quickly than with `MathJax` using javascript. There still is some delay on complicated pages. The problem of the loss of position when the page is rendered is also a problem with this method.

In short, the rendering of the mathematics on wiki pages is still a work in progress. The solutions currently being used are still pretty new. I expect them to improve a lot in the next few years.

It is interesting to note the students' solution to the display problems. If they couldn't read the screen, then just pushed the print button and looked at the resulting pdf file. It looked fine most of the time.

5.2 Multimedia displays

The use of multimedia software is, of course, one the most exciting prospects within a wiki site. The dynamic content at this point consists of animated gifs. These have been produced using different approaches:

- 1. Use TeX to make a `dvi` file of each image.
  2. Make a cropped PostScript file of each image using `dvips`.
  3. Convert the PostScript file to a gif.
  4. Stitch the gifs together to make an animated gif.
- Use a symbolic manipulator (Maple in this case) to make an animation and save it as an animated gif.
- Make a shell script to generate PostScript files, and stitch them together in an animated gif.

6 Next steps

The work done so far is just one step in a larger project. The next goal is a wiki covering undergraduate linear algebra. It will be structured is such a way to allow the material to be used in different ways for different courses.

Any list of topics is necessarily idiosyncratic. Initially the list of topics covered will include:

- Vector spaces over $\mathbb{R}$, $\mathbb{C}$, finite fields and general fields.
- Subspaces. Linear independence, generating (spanning) sets, bases. Dimension. Infinite dimensional vector spaces. Direct sums. Every vector space has a basis (application of Zorn's Lemma or some other form of AC).
- Matrices. Matrix algebra. Inverses of matrices. Singular matrices. Determinants. Rank of matrices. Equivalent conditions for nonsingularity. Row space, column space, null space, rank of matrices and their interrelations.

- The theory of systems of linear equations. Elementary matrices. Gaussian Elimination. The reduced row echelon form. Solution space of a homogeneous system, particular and general solutions of an arbitrary system.
- Linear transformations. Representation by matrices in the finite dimensional case. Fundamental subspaces associated with a transformation (kernel, range). Fundamental subspaces associated with a matrix and their connection to linear transformations. Dimension theorem.
- Linear operators on finite and infinite dimensional spaces.
- Linear functionals. Dual spaces. Duals of finite dimensional spaces; duals of infinite dimensional spaces.
- Eigenvalues and eigenvectors of a linear operator; of a matrix. Characteristic equation of a matrix. The minimal polynomial. Diagonalization of matrices (operators) and applications. Invariant subspaces. Cayley-Hamilton theorem.
- Inner product spaces. Cauchy-Schwarz theorem. Orthogonal complements. The Gram-Schmidt algorithm. Adjoint of a linear operator. Unitary, self-adjoint, normal and orthogonal operators and their matrices. Diagonalization and self-adjoint operators.
- Jordan canonical form.
- Bilinear forms and their representations. Quadratic forms. Positive definite and positive semidefinite bilinear forms.

Each topic will have a page with a table of contents (automatically generated) that includes the list of theorems, and will include separate links to proofs when appropriate, and many exercises of varying difficulty. Solutions to all problems will be available.

It will also become a collaborative project. A working group has been established and is looking at finishing the next stage in a 12–18 month time period.

The beauty of the wiki is that the material may be expanded or changed easily and on a collaborative basis. It also keeps track of the dependencies between pages, an invaluable tool when choosing which topics to cover. Alternative approaches are easily implemented.

The approach so far has been incremental—this matches the philosophy of the wiki perfectly. Further expansion after completing the next stage is anticipated.

References

[1] `MathJax`: `http://www.mathjax.org/`

[2] `MediaWiki` with `texvc`: `http://www.mediawiki.org/wiki/Texvc`

[3] The experimental wiki can be accessed at `http://wikitest.cs.umanitoba.ca/mathwiki/index.php/Math1300:MainPage`