Euclid's Diagrammatic Logic and Cognitive Science

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Abstract. For more than two millennia, Euclid's *Elements* set the standard for rigorous mathematical reasoning. The reasoning practice the text embodied is essentially diagrammatic, and this aspect of it has been captured formally in a logical system termed **Eu** [2, 3]. In this paper, we review empirical and theoretical works in mathematical cognition and the psychology of reasoning in the light of **Eu**. We argue that cognitive intuitions of Euclidean geometry might play a role in the interpretation of diagrams, and we show that neither the mental rules nor the mental models approaches to reasoning constitutes by itself a good candidate for investigating geometrical reasoning. We conclude that a cognitive framework for investigating geometrical reasoning empirically will have to account for both the interpretation of diagrams and the reasoning with diagrammatic information. The framework developed by Stenning and van Lambalgen [1] is a good candidate for this purpose.

1 Introduction

A distinctive feature of elementary Euclidean geometry is the natural and intuitive character of its proofs. The prominent place of the subject within the history and education of mathematics attests to this. It was taken to be the foundational starting point for mathematics from the time of its birth in ancient Greece up until the 19th century. And it remains within mathematics education as a subject that serves to initiate students in the method of deductive proof. No other species of mathematical reasoning seem as basic and transparent as that which concerns the properties of figures in Euclidean space.

One may not expect a formal analysis of the reasoning to shed much light on this distinctive feature of it, as the formal and the intuitive are typically thought to oppose one another. Recently, however, a formal analysis, termed **Eu**, has been developed which challenges this opposition [2, 3]. **Eu** is a formal proof system designed to show that a systematic method underlies the use of diagrams in Euclid's *Elements*, the representative text of the classical synthetic tradition of geometry. As diagrams seem to be closely connected with the way we call upon our intuition in the proofs of the tradition, **Eu** holds the promise of contributing to our understanding of what exactly makes the proofs natural.

In this paper, we explore the potential **Eu** has in this respect by confronting it with empirical and theoretical works in the fields of mathematical cognition and the psychology of reasoning. Our investigation is organized around the two following issues:

1. What are the interpretative processes on diagrams involved in the reasoning practice of Euclidean geometry and what are their possible cognitive roots?
2. What would be an appropriate cognitive framework to represent and investigate the constructive and deductive aspects of the reasoning practice of Euclidean geometry?

By providing a formal model of the reasoning practice of Euclidean geometry, **Eu** provides us with a tool to address these two issues. We proceed as follows. To address the first issue, we first state the interpretative capacities that according to the norms fixed by **Eu** are necessary to extract, from diagrams, information for geometrical reasoning. We then present empirical works on the cognitive bases of Euclidean geometry, and suggest that cognitive intuitions might play a role in the interpretative aspects of diagrams in geometrical reasoning. To address the second issue, we compare the construction and inference rules of **Eu** with two major frameworks in the psychology of reasoning—the mental rules and the mental models theories. We argue that both have strengths and weaknesses as a cognitive account of geometrical reasoning as analyzed by **Eu**, but that one will need to go beyond them to provide a framework for investigating geometrical reasoning empirically.

The two main issues are of course intimately related. In a last section, we argue that the framework developed by Stenning and van Lambalgen in [1], which connects interpretative and deductive aspects of reasoning, might provide the right cognitive framework for investigating the relation between them.

2 A Logical Analysis of the Reasoning Practice in Euclid's *Elements*: The Formal System Eu

Eu is based on the seminal paper [4] by Ken Manders. In [4] Manders challenges the received view that the *Elements* is flawed because its proofs sometimes call upon geometric intuition rather than logic. What is left unexplained by the received view is the extraordinary longevity of the *Elements* as a foundational text within mathematics. For over two thousand years there were no serious challenges to its foundational status. Mathematicians through the centuries understood it to display what the basic propositions of geometry are grounded on. The deductive gaps that exist according to modern standards of logic story were simply not seen.

According to Manders, Euclid is not relying on geometric intuition illicitly in his proofs; he is rather employing a systematic method of diagrammatic proof. His analysis reveals that diagrams serve a principled, theoretical role in Euclid's mathematics. Only a restricted range of a diagram's spatial properties are permitted to justify inferences for Euclid, and these self-imposed restrictions can be explained as serving the purpose of mathematical control. **Eu** [2, 3] was designed to build on Manders' insights, and precisely characterize the mathematical significance of Euclid's diagrams in a formal system of geometric proof.

Eu has two symbol types: diagrammatic symbols Δ and sentential symbols A. The sentential symbols A are defined as they are in first-order logic. They express relations between geometric magnitudes in a configuration. The diagrammatic symbols are defined as $n \times n$ arrays for any n. Rules for a well-formed diagram specify how points, lines and circles can be distinguished within such arrays. The points, lines and circles of Euclid's diagrams thus have formal analogues in **Eu** diagrams. The positions the elements of Euclid's diagrams can have to one another are modeled directly by the position their formal analogues can have within a **Eu** diagram.

The content of a diagram within a derivation is fixed via a relation of diagram equivalence. Roughly, two **Eu** diagrams Δ_1 and Δ_2 are equivalent if there is a bijection between its elements which preserves their non-metric positional relations.[3] The equivalence relation is intended to capture what Manders terms the *co-exact* properties of a Euclidean diagram. A close examination of the *Elements* shows that Euclid refrains from making any inferences that depend on a diagram's metric properties. At the same time, Euclid does rely on diagrams for the non-metric positional relations they exhibit—or in Manders' terms, the co-exact relations they exhibit. Diagrams, it turns out, can serve as adequate representations of such relations in proofs.

Eu exhibits this by depicting geometric proof as running on two tracks: a diagrammatic one, and a sentential one. The role of the sentential one is to record metric information about the figure and provide a means for inferring this kind of information. The role of the diagrammatic track is to record non-metric positional information of the figure, and to provide a means for inferring this kind of information about it. Rules for building and transforming diagrams in derivations are sensitive only to properties invariant under diagram equivalence. It is in this way that the relation of diagram equivalence fixes the content of diagrams in derivations.

What is derived in **Eu** are expressions of the form

$$\Delta_1, A_1 \longrightarrow \Delta_2, A_2$$

where Δ_1 and Δ_2 are diagrams, and A_1 and A_2 are sentences. The geometric claim this is stipulated to express is the following:

> $\star$*Given a configuration satisfying the non-metric positional relations depicted in Δ_1 and the metric relations expressed in A_1, then one can obtain a configuration satisfying the positional relations depicted by Δ_2 and metric relations specified by A_2.*

[3] For a more detailed discussion of **Eu** diagrams and diagram equivalence, we refer the reader respectively to sections A and B of the appendix.

3 Interpretative Aspects of Geometrical Reasoning with Diagrams

Diagrams in Euclid's *Elements* are mere pictures on a piece of paper. How can a visual experience triggered by looking at a picture lead to a cognitive representation that can play a role in geometrical reasoning? From a cognitive perspective, taking seriously the use of diagrams in reasoning requires an account of the way diagrams are *interpreted* in order to play a role in geometrical reasoning. The formal system **Eu** provides a theoretical answer to this question. Here we compare this theoretical account with recent works in mathematical cognition probing the existence of cognitive intuitions of Euclidean geometry. We will argue that the **Eu** analysis of geometrical reasoning suggests a possible role for these cognitive intuitions in the interpretation of diagrams.

3.1 Interpretation of Diagrams in Eu

As described in section 2, the key insight behind the **Eu** analysis of the diagrammatic reasoning practice of Euclid's *Elements* is the observation that appeals to diagrams in proofs are highly controlled: proofs in Euclid's *Elements* only make use of the *co-exact* properties of diagrams. From a cognitive perspective, the practice of extracting the co-exact properties from a visual diagram is far from a trivial one. According to **Eu**, the use of diagrams presupposes the two following cognitive abilities:

1. The capacity to categorize elements of the diagram using normative concepts: points, linear elements and circles.
2. The capacity to abstract away irrelevant information from the visual-perceptual experience of the diagram.

The formal syntax of **Eu**, combined with the equivalence relation between **Eu** diagrams, can be interpreted as specifying these capacities precisely. More specifically:

- The first capacity amounts to the ability to see in the diagram the elements p, l, c of a particular **Eu** diagram $\langle n, p, l, c \rangle$, which respectively denote the sets of points, lines and circles.
- The second capacity amounts to the ability to see in the particular diagram positional relations that are invariant under diagram equivalence.

Thus, the interpretation of a visual diagram according to **Eu** results in a formal object which consists in an equivalence class of **Eu** diagrams. It is precisely on these formal objects, the interpreted diagrams, that inference rules operate in the **Eu** formalization of reasoning in elementary geometry.

3.2 Intuitions of Euclidean Geometry in Human Cognition

Recently, several empirical studies in mathematical cognition [5–7] have directly addressed the issue of the cognitive roots of Euclidean geometry. These studies might be classified in two categories: one approach consists in providing empirical evidence for the existence of *abstract geometric concepts* [5, 7], the other approach consists in providing empirical evidence for the perception of *abstract geometric features* [6]. Here we successively report some of the empirical findings provided by these two approaches.

The two studies [5] and [7] have been conducted on an Amazonian Indigene Group called the Munducuru, with no previous education in formal Euclidean geometry. In the first study [5], the experiment consisted in identifying the presence, or the absence, of several topological and geometric concepts in Mundurucu participants. To this end, the experimenter proposed, for each concept under investigation, six slides in which five of the images displayed the given geometric concept (e.g., parallelism), while the last one lacked the considered property (non-parallel lines). The test shows that several basic geometrical concepts are present in Mundurucu conceptual systems, such as the concepts of straight line, parallel line, right angle, parallelogram, equilateral triangle, circle, center of circle. Nevertheless, the study [5] does not address geometric concepts that go beyond perceptual experience. Such concepts are the topic of another study [7]. In [7], empirical evidence is provided for the capacity of Mundurucu to reason about the properties of dots and infinite lines. In particular, most of the Mundurucu participants consider that, given a straight line and a dot, we can always position another straight line on the dot which will never cross the initial line.

The second approach for detecting the presence of intuitions of Euclidean geometry is based on the framework of *transformational geometry* [8]. According to this framework, a geometric theory is identified with respect to the transformations that preserve its theorems. Euclidean geometry is then identified by its four types of transformations—translation, rotation, symmetry and homothecy—leaving then *angle* and *length proportions* as the main defining features of figures in Euclidean geometry. The experimental approach based on this framework consists in investigating the capacity of participants to perceive abstract geometric features in configurations where irrelevant features are varied. The experiments reported in [6] adopt such an approach. The empirical results show that both children and adults are able to use angle and size to classify shapes, but only adults are able to discriminate shapes with respect to sense, i.e., the property distinguishing two figures that are mirror images of each other.

The two approaches reported here to investigate cognitive intuitions of Euclidean geometry bring out cognitive abilities which seem related to the abilities for the interpretation of diagrams postulated by **Eu**. This observation suggests a possible role for the cognitive intuitions of Euclidean geometry in the reasoning practice of elementary geometry, as we will now see.

3.3 Cognitive Intuitions and the Interpretation of Diagrams

How might the cognitive intuitions of Euclidean geometry relate to the deductive aspects of Euclid's theory of geometry? In a correspondence in *Science* [9] on the study reported in [5], Karl Wulff has challenged the claim that Dehaene *et* *al.* address Euclidean geometry by denying a role for these cognitive intuitions in the demonstrative aspects of Euclid's theory of geometry. An opposing position can be found in [10]:

> The axioms of geometry introduced by Euclid [...] define concepts with spatial content, such that any theorem or demonstration of Euclidean geometry can be realized in the construction of a figure. Just as intuitions about numerosity may have inspired the early mathematicians to develop mathematical theories of number and arithmetic, Euclid may have appealed to universal intuitions of space when constructing his theory of geometry. [10, p. 320]

Interestingly, stating precisely the postulated cognitive abilities involved in the interpretation of diagrams according to **Eu** might suggest a role for the cognitive intuitions of Euclidean geometry in the reasoning practice of Euclid's *Elements*.

Firstly, we have noticed that one of the key abilities in the interpretation of diagrams, according to **Eu**, is the capacity to *categorize* or *type* the different figures of the visual diagram using the normative concepts of geometry. This ability seems to be universal, according to the empirical data that we reported in the previous section, as the Mundurucu people, without previous formal education in Euclidean geometry, seem to possess an abstract conceptual systems which contains the key normative concepts of elementary geometry, and are able to use it to categorize elements of visual diagrams.

Secondly, interpretation of diagrams in **Eu** requires an important capacity of *abstraction* which is formally represented by an equivalence relation between diagrams. This equivalence relation aims to capture the idea that some features of the diagram are not relevant for reasoning: for instance, the same diagram rotated, translated or widened, would play exactly the same role in a geometrical proof of the *Elements*. This capacity of abstraction seems to connect with the cognitive ability of perceiving abstract geometric features, such as angle and length proportions, while abstracting away irrelevant information from the point of view of Euclidean geometry.

We now turn to the deductive and constructive aspects of geometrical reasoning. In section 5, we argue that a plausible cognitive framework for an empirical investigation of geometrical reasoning will have to bring together the interpretative, deductive and constructive aspects of geometrical reasoning to be faithful to the mathematical practice of Euclidean geometry.

4 Deductive and Constructive Aspects of Geometrical Reasoning with Diagrams

In this section, we begin by presenting the **Eu** formalization of constructive and deductive steps in Elementary geometry, and then discuss the capacity of the mental rules and the mental models theories to represent these reasoning steps. We conclude that an adequate framework for an empirical investigation of geometrical reasoning will have to go beyond these two theories.

4.1 Eu Construction and Inference Rules

The **Eu** proof rules specify how to derive $\varDelta_1, A_1 \longrightarrow \varDelta_2, A_2$ expressions. They specify, in particular, the operations that can be performed on the pair $\varDelta_1, A_1$ to produce a new pair $\varDelta_2, A_2$. Given the intended interpretation of $\varDelta_1, A_1 \longrightarrow \varDelta_2, A_2$ given by $\star$, the fundamental restriction on these rules is that they be geometrically sound. In other words, if $\varDelta_2, A_2$ is derivable from $\varDelta_, A_1$, then with *any* configuration satisfying the geometric conditions represented by $\varDelta_1 A_1$ one must either have a configuration satisfying the conditions represented by $\varDelta_2, A_2$ (if $\varDelta_1 = \varDelta_2$), or be able to construct a configuration satisfying the conditions represented by $\varDelta_2, A_2$ (if $\varDelta_2$ contains objects not in $\varDelta_1$).

The formal process whereby $\varDelta_2, A_2$ is derived has two stages: a construction and demonstration. Hence **Eu** has two kinds of proof rules: construction rules and inference rules. The construction stage models the application of a proof's construction on a given diagram. The demonstration stage models the inferences made from the assumed metric relations and the particular diagram produced by the construction. The soundness restriction thus applies only to the inference rules. The construction rules together codify a method for producing a representation that can serve as a vehicle of inference. The demonstration rules codify the inferences that can be made from such a vehicle.[4]

4.2 Mental Rules Theory

The mental rules theory [11, 12] represents reasoning as the application of *formal rules*, rules which are akin to those of natural deduction. According to this theory, reasoning is conceived as a syntactic process whereby sentences are transformed. These transformations are made according to specific rules defined precisely in terms of the syntactic structures of sentences. Deduction of one sentence from a set of other sentences (premisses) is seen as the *search* for a *mental proof*, which consists precisely in the production of the conclusion from the premisses by application of the rules a finite number of times. Consequently, the mental rules theory represents reasoning as consisting in syntactic operations on the *logical forms* of sentences.

The formalization of geometrical reasoning provided by **Eu** shares one important feature with the mental rules theory: **Eu** represents geometrical reasoning in terms of syntactic rules of inference. This is made possible by considering diagrams as a kind of *syntax*, and then by stipulating rules that control inferences that are drawn from diagrams. Thus, **Eu** diagrams might be seen as representing something like the *logical form* of concrete visual diagrams. One could perhaps say that **Eu** diagrams represent their *geometric form*, and that **Eu** inference rules operate on these forms. From this point of view, the formalization of geometrical reasoning provided by **Eu** appears to be in direct line with the the mental rules theory of reasoning.

[4] For the formal details, see sections 1.4.1 and 1.4.2 of [2], available online at `www.johnmumma.org`.

Nevertheless, the mental rules theory seems to run into troubles when we consider the *construction* operations on diagrams, which are central to the reasoning practice of Euclidean geometry, and which are formalized by **Eu**'s construction rules. One might still argue that **Eu** construction rules are syntactic operations on **Eu** diagrams, since diagrams are included its syntax, and so **Eu** construction rules fit the framework of mental rules theory. However, this does not seem correct as the mental rules are considered to be *deduction* rules; the soundness restriction applies to them. They are thereby of a very different nature than **Eu**'s construction rules. Consequently, even though the mental rules theory could suitably represent inferential steps in geometrical reasoning, it seems that the theory lacks the necessary resources to account for the construction operations on diagrams, an aspect fundamental to the reasoning practice in Euclidean geometry.

4.3 Mental Models Theory

The mental models theory [13, 14] postulates that reasoning depends on envisaging *possibilities*. When given a set of premisses, an individual constructs mental models which correspond to the possibilities elicited by the premisses. Different reasoning strategies are then available to extract information from the mental models: one may represent several possibilities in a diagram and draw a conclusion from the diagram, one may make an inferential step from a single possibility, or one can use a possibility as a counter-example for falsifying a conclusion. One interesting feature of the theory is that mental models are supposed to be *iconic*: the structure of a mental model is supposed to reflect the structure of the possibility that it represents. This feature is nicely illustrated when the mental models theory is applied to account for reasoning with visual-spatial information [15].

Contrary to the mental rules theory, the mental models theory seems suited to provide an account of the representation of the different construction operations on diagrams: visual diagrams used in geometrical proofs can be conceived as mental models entertained by the reasoner. Mental models associated to diagrams would then be of a visual-spatial nature, reflecting the spatial relations between the different elements of the diagram. This just seems to be a description of **Eu** diagrams, which encode the information that can be legitimately used in geometrical reasoning.

Accordingly, if **Eu** diagrams are conceived as mental models, the construction operations on diagrams would be explained in terms of the ways mental models are constructed. Nevertheless, one might worry about the capacity of the mental models theory to account for the specific use of diagrams in geometrical reasoning. One of the main points of **Eu** is to exhibit precisely that diagrammatic information enters legitimate mathematical inferences in a very controlled way. In order to account for the use of diagrams in geometrical reasoning, the mental models theory would have to be supplemented with a regimentation of the information that can be legitimately extracted from diagrams conceived as mental models. The formal system **Eu** shows that this can be done using syntactic rules that operate on diagrams represented as syntactic objects.

4.4 Beyond the Mental Rules vs Mental Models Debate

Geometrical reasoning, as practiced in Euclid's *Elements*, constitutes an interesting test for current theories in the psychology of reasoning. Our previous comparison predicts that the mental rules and mental models theories present both advantages and disadvantages as an account of the reasoning practice of geometrical reasoning with diagrams. Indeed, it seems that these two theories are actually complementary in their capacity to account for geometrical reasoning as described by **Eu**: the mental rules theory seems adequate to represent *inferences* in geometrical reasoning, while the mental models theory seems adequate to represent the *diagrammatic constructions* that support such inferences. Thus, it seems that to provide a framework for an empirical investigation of geometrical reasoning with diagrams one will need to go *beyond* these two theories. Such a move is also suggested, although for different reasons, in [16, 17]. In this respect, **Eu** constitutes a possible starting point for developing a cognitive framework for geometrical reasoning which would be faithful to both deductive and constructive aspects of the mathematical practice of Euclidean geometry.

5 Interpretation and Reasoning in Elementary Geometry

According to the **Eu** analysis of geometrical reasoning, interpretation and reasoning with diagrams are intimately related. This observation, originating in the study of the reasoning practice in Euclid's *Elements* [4], is directly in line with a recent approach to the psychology of reasoning developed by Stenning and van Lambalgen [1] which attributes a central role to interpretative processes in human reasoning:

> We [. . .] view reasoning as consisting of two stages: first one has to establish the domain about which one reasons and its formal properties (what we will call 'reasoning *to* an interpretation') and only after this initial step has been taken can ones reasoning be guided by formal laws (what we will call 'reasoning *from* an interpretation'). [1, p. 28]

Geometrical reasoning with diagrams, as formalized by **Eu**, precisely fits within this framework: reasoning *to* an interpretation corresponds to the process of interpreting a visual diagram along with an associated metric assertion, resulting in **Eu** into a pair Δ, A; reasoning *from* an interpretation is then represented as the application of the formal rules of **Eu** to prove geometric claims.

This perspective unifies the two main issues addressed in this paper. Our main conclusions can then be restated as follows: (i) intuitions of Euclidean geometry as studied by mathematical cognition are likely to play a role in reasoning *to* an interpretation of a diagrams, (ii) the mental rules and mental models theories of reasoning are inadequate for representing reasoning *from* an interpretation, as none of them is able to account for both deductive and constructive aspects of geometrical reasoning. In the perspective of Stenning and van Lambalgen [1], **Eu** appears as a good candidate for representing the process of reasoning *from* an interpretation in elementary geometry.

6 Conclusion

The empirical investigation of the cognitive bases of Euclidean geometry is a multi-disciplinary enterprise involving both mathematical cognition and the psychology of reasoning, and which shall benefit from works in formal logic and the nature of mathematical practice. In this paper, we used the formal system **Eu** to review existing empirical and theoretical works in cognitive science with respect to this enterprise. Our investigation shows: (i) the necessity of dealing jointly with interpretation and reasoning, (ii) the relevance of works on the cognitive bases of Euclidean geometry for the interpretation of diagrams and (iii) the necessity to go beyond the mental rules vs mental models distinction for accounting for both constructive and deductive aspects of geometrical reasoning.

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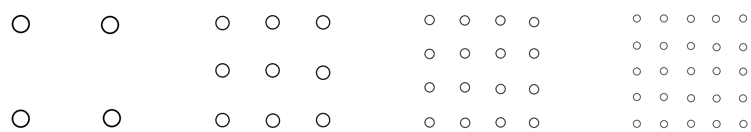
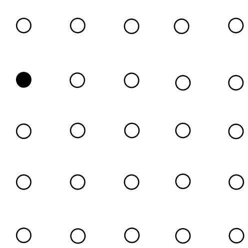
Appendix

A Eu diagrams

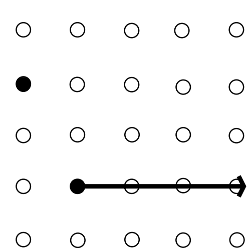
The syntactic structure of diagrams in **Eu** has no natural analogue in standard logic. Their underlying form is a square array of dots of arbitrary finite dimension.



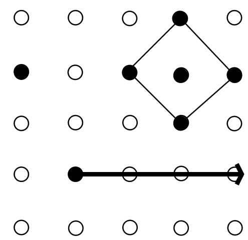
The arrays provide the planar background for an **Eu** diagram. Within them geometric elements—points, linear elements, and circles— are distinguished. A point is simply a single array entry. An example of a diagram with a single point in it is



Linear elements are subsets of array entries defined by linear equations expressed in terms of the array entries. (The equation can be bounded. If it is bounded on one side, the geometric element is a ray. If it is bounded on two sides, the geometric element is a segment.) An example of a diagram with a point and linear element is



Finally, a circle is a convex polygon within the array, along with a point inside it distinguished as its center. An example of a diagram with a point, linear element and a circle is

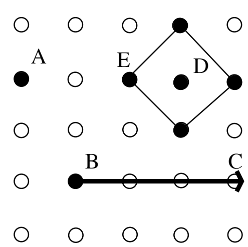


The size of a diagram's underlying array and the geometric elements distinguished within it, comprise a diagram's identity. Accordingly, a diagram in **Eu** is a tuple

$$\langle n, p, l, c \rangle$$

where n, a natural number, is the length of the underlying array's sides, and p, l and c are the sets of points, linear elements and circles of the diagram, respectively.

Like the relation symbols which comprise metric assertions, the diagrams have slots for variables. A diagram in which the slots are filled is termed a *labeled diagram*. The slots a diagram has depends on the geometric elements constituting it. In particular, there is a place for a variable beside a point, beside the end of a linear element (which can be an endpoint or endarrow), and beside a circle. One possible labeling for the above diagram is thus



Having labeled diagrams within **Eu** is essential, for otherwise it would be impossible for diagrams and metric assertions to interact in the course of a proof. We can notate any labeled diagram as

$$\langle n, p, l, c \rangle [\boldsymbol{A}, R]$$

where $\boldsymbol{A}$ denotes a sequence of variables and R a rule matching each variable to each slot in the diagram.

B Diagram equivalence

The definition of diagram equivalence is based on the notions of a diagram's *completion* and that of *co-exact map*. The completion Δ' of a diagram Δ is simply the diagram obtained by adding to Δ intersection points to the intersections present in it. (See page 38-40 of [2]) Two diagrams are then equivalent if there is a co-exact map between their completions.

The key idea behind the definition of diagram equivalence, then, is that of of a co-exact map. A necessary condition for their to be such a map between diagrams Δ_1 and Δ_2 is that both diagrams contain the same number of points, linear elements and circles, and these objects are labeled by the same variables and variable pairs.

Suppose Δ_1 and Δ_2 are two such diagrams, and $\phi_{\boldsymbol{x}}$ is the bijection that maps an element in Δ_1 to the element in Δ_2 with the same label. In virtue of their position with respect to the underlying array of Δ_1, the elements of Δ_1 satisfy various geometric relations. The bijection $\phi_{\boldsymbol{x}}$ is a co-exact map if and only if it preserves a certain sub-set of these relations. Precisely $\phi_{\boldsymbol{x}}$ is a co-exact map if and only if it satisfies the following nine conditions, where P and Q are points of Δ_1, N and M are linear elements of Δ_1 (i.e. segments, rays, or lines) and C_1 and C_2 are circles of Δ_1.

- P lies on M in $\Delta_1 \longleftrightarrow \phi_{\boldsymbol{x}}(P)$ lies on $\phi_{\boldsymbol{x}}(M)$ in Δ_2.
- P and Q lie on a given side M in $\Delta_1 \longleftrightarrow \phi_{\boldsymbol{x}}(P)$ and $\phi_{\boldsymbol{x}}(Q)$ lie on the same side of $\phi_{\boldsymbol{x}}(M)$ in Δ_2. (The same side of $\phi_{\boldsymbol{x}}(M)$ in is determined via the orientation specified by the two variable label of M and $\phi_{\boldsymbol{x}}(M)$.)
- P lies inside/on/outside circle C_1 in $\Delta_1 \longleftrightarrow \phi_{\boldsymbol{x}}(P)$ lies inside/on/outside $\phi_{\boldsymbol{x}}(C_1)$ in Δ_2.
- M intersects N at a point/along a segment in $\Delta_1 \longleftrightarrow \phi_{\boldsymbol{x}}(M)$ intersects $\phi_{\boldsymbol{x}}(N)$ at a point/along a segment in Δ_2.
- M cuts in one point/cuts in two points/is tangent at one point to/is tangent along a segment to C_1 in $\Delta_1 \longleftrightarrow \phi_{\boldsymbol{x}}(M)$ cuts in one point/cuts in two points/is tangent at one point to/is tangent along a segment to $\phi_{\boldsymbol{x}}(C_1)$ in Δ_2.
- C_1 has the same intersection signature with respect to C_2 in Δ_1 as $\phi_{\boldsymbol{x}}(C_1)$ has with respect to $\phi_{\boldsymbol{x}}(C_2)$ in Δ_2. (The intersection signature classifies the way C_1 and C_2 intersect each other as convex polygons. For its precise definition, see appendix A in the thesis.)
- Circle C_1 lies outside/ is contained by circle C_2 in $\Delta_1 \longleftrightarrow \phi_{\boldsymbol{x}}(C_1)$ lies outside/is contained by circle $\phi_{\boldsymbol{x}}(C_2)$ in Δ_2.
- Circle C_1 lies within a given side of M in $\Delta_1 \longleftrightarrow \phi_{\boldsymbol{x}}(C_1)$ lies within the same side of $\phi_{\boldsymbol{x}}(M)$ in Δ_2.
- End-arrow P is on a given side of M in $\Delta_1 \longleftrightarrow \phi_{\boldsymbol{x}}(P)$ is on the same side of $\phi_{\boldsymbol{x}}(M)$ in Δ_2.