Existential definability of modal frame classes

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Abstract. A class of Kripke frames is called modally definable if there is a set of modal formulas such that the class consists exactly of frames on which every formula from that set is valid, i. e. globally true under any valuation. Here, existential definability of Kripke frame classes is defined analogously, by demanding that each formula from a defining set is satisfiable under any valuation. This is equivalent to the definability by the existential fragment of modal language enriched with the universal modality. A model theoretic characterization of this type of definability is given.

Keywords: modal logic, model theory, modal definability

1 Introduction

Some questions about the power of modal logic to express properties of relational structures are addressed in this paper. For the sake of notational simplicity, only the *basic propositional modal language* (BML) is considered in this paper. Let Φ be a set of *propositional variables*. The syntax of BML is given by

$$\varphi ::= p \mid \bot \mid \varphi_1 \vee \varphi_2 \mid \neg\varphi \mid \Diamond\varphi,$$

where $p \in \Phi$. We define other connectives and $\Box$ as usual. Namely, $\Box\varphi := \neg\Diamond\neg\varphi$.

Only the Kripke semantics is considered in this paper. The basic notions and results are only briefly recalled here (see [1] for details if needed). A *Kripke frame* for the basic modal language is a relational structure $\mathfrak{F} = (W, R)$, where $W \neq \emptyset$ and $R \subseteq W \times W$. A *Kripke model* based on a frame $\mathfrak{F}$ is $\mathfrak{M} = (W, R, V)$, where $V : \Phi \to 2^W$ is a mapping called *valuation*. For $w \in W$, we call $(\mathfrak{M}, w)$ a *pointed model*.

The *truth* of a formula is defined locally and inductively as usual, and denoted $\mathfrak{M}, w \Vdash \varphi$. Namely, a formula of a form $\Diamond\varphi$ is *true* at $w \in W$ if $\mathfrak{M}, u \Vdash \varphi$ for some u such that Rwu. A valuation is naturally extended to all modal formulas by putting $V(\varphi) = \{w \in W : \mathfrak{M}, w \Vdash \varphi\}$.

We say that a formula is *globally true* on $\mathfrak{M}$ if it is true at every $w \in W$, and we denote this by $\mathfrak{M} \Vdash \varphi$. On the other hand, a formula is called *satisfiable* in $\mathfrak{M}$ if it is true at some $w \in W$.

If a formula φ is true at w under any valuation on a frame $\mathfrak{F}$, we write $\mathfrak{F}, w \Vdash \varphi$. We say that a formula is *valid* on a frame $\mathfrak{F}$ if we have $\mathfrak{M} \Vdash \varphi$ for any model $\mathfrak{M}$ based on $\mathfrak{F}$. This is denoted $\mathfrak{F} \Vdash \varphi$. A class $\mathcal{K}$ of Kripke frames is *modally definable* if there is a set Σ of formulas such that $\mathcal{K}$ consists exactly of frames on which every formula from Σ is valid, i. e. $\mathcal{K} = \{\mathfrak{F} : \mathfrak{F} \Vdash \Sigma\}$. If this is the case, we say that $\mathcal{K}$ is *defined* by Σ and denote $\mathcal{K} = \mathrm{Fr}(\Sigma)$.

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Model theoretic closure conditions that are necessary and sufficient for an elementary class of frames (i. e. first-order definable property of relational structures) to be modally definable are given by the famous Goldblatt-Thomason Theorem.

Theorem (Goldblatt-Thomason [3]). *An elementary class $\mathcal{K}$ of frames is definable by a set of modal formulas if and only if $\mathcal{K}$ is closed under surjective bounded morphisms, disjoint unions and generated subframes, and reflects ultrafilter extensions.*

All of the frame constructions used in the theorem – bounded morphisms, disjoint unions, generated subframes and ultrafilter extensions – are presented in detail in [1] (the same notation is used in this paper). Just to be clear, we say that a class $\mathcal{K}$ *reflects* a construction if its complement $\mathcal{K}^c$, that is the class of all Kripke frames not in $\mathcal{K}$, is closed under that construction.

Now, an alternative notion of definability is proposed here as follows.

Definition. A class $\mathcal{K}$ of Kripke frames is called *modally $\exists$-definable* if there is a set Σ of modal formulas such that for any Kripke frame $\mathfrak{F}$ we have: $\mathfrak{F} \in \mathcal{K}$ if and only if each $\varphi \in \Sigma$ is satisfiable in $\mathfrak{M}$, for any model $\mathfrak{M}$ based on $\mathfrak{F}$. If this is the case, we denote $\mathcal{K} = \mathrm{Fr}_\exists(\Sigma)$.

The definition does not require that all formulas of Σ are satisfied at the same point – it suffices that each formula of Σ is satisfied at some point.

In the sequel, a notation $\mathrm{Mod}(F)$ is used for a class of structures defined by a first-order formula F. Similarly, if $\Sigma = \{\varphi\}$ is a singleton set of modal formulas, we write $\mathrm{Fr}_\exists(\varphi)$ instead of $\mathrm{Fr}_\exists(\{\varphi\})$.

Example 1. It is well-known that the formula $p \to \Diamond p$ defines reflexivity, i. e. $\mathrm{Fr}(p \to \Diamond p) = \mathrm{Mod}(\forall x Rxx)$. Now, it is easy to see that $\mathrm{Fr}_\exists(p \to \Diamond p)$ is the class of all frames such that $R \neq \emptyset$, that is $\mathrm{Fr}_\exists(p \to \Diamond p) = \mathrm{Mod}(\exists x \exists y Rxy)$. This class is not modally definable in the usual sense, since it is clearly not closed under generated subframes. Note that the condition $R \neq \emptyset$ is $\exists$-definable also by a simpler formula $\Diamond\top$.

The main result of this paper is the following characterization.

Theorem 1. *Let $\mathcal{K}$ be an elementary class of Kripke frames. Then $\mathcal{K}$ is modally $\exists$-definable if and only if it is closed under surjective bounded morphisms and reflects generated subframes and ultrafilter extensions.*

This is an analogue of a similar characterization of existentially definable Kripke model classes, given in [6].

2 First and second-order standard translations

The starting point of correspondence between first-order and modal logic is the *standard translation*, a mapping that translates each modal formula φ to the first-order formula $ST_x(\varphi)$, as follows:

$ST_x(p) = Px$, for each $p \in \Phi$,
$ST_x(\bot) = \bot$,
$ST_x(\neg\varphi) = \neg ST_x(\varphi)$,
$ST_x(\varphi \vee \psi) = ST_x(\varphi) \vee ST_x(\psi)$,
$ST_x(\Diamond\varphi) = \exists y(Rxy \wedge ST_y(\varphi))$.

Clearly, we have $\mathfrak{M}, w \Vdash \varphi$ if and only if $\mathfrak{M} \models ST_x(\varphi)[w]$, and $\mathfrak{M} \Vdash \varphi$ if and only if $\mathfrak{M} \models \forall x ST_x(\varphi)$. But, validity of a formula on a frame generally is not first-order expressible, since we need to quantify over valuations. We have a second-order standard translation, that is, $\mathfrak{F} \Vdash \varphi$ if and only if $\mathfrak{F} \models \forall P_1 \ldots \forall P_n \forall x ST_x(\varphi)$, where $P_1, \ldots, P_n$ are monadic second-order variables, one for each propositional variable occurring in φ. So, the notion of modal definability is equivalent to the definability by a set of second-order formulas of the form $\forall P_1 \ldots \forall P_n \forall x ST_x(\varphi)$. However, in many cases a formula of this type is equivalent to a first-order formula. Namely, this holds for any *Sahlqvist formula* (the definition is omitted here – see [7] or [1]), for which a first-order frame correspondent is effectively computable. On the other hand, the Goldblatt-Thomason Theorem characterizes those first-order properties that are modally definable.

Now, $\exists$-definability is clearly also equivalent to the definability by a type of second-order formulas – those of the form $\forall P_1 \ldots \forall P_n \exists x ST_x(\varphi)$. Consider another example of a modally $\exists$-definable class.

Example 2. The condition $F = \exists x \forall y (Rxy \to \exists z Ryz)$ is not modally definable, since it is not closed under generated subframes, but it is modally $\exists$-definable by the formula $\varphi = p \to \Box\Diamond p$.

To prove this, we need to show $\mathrm{Fr}_\exists(\varphi) = \mathrm{Mod}(F)$. But $\mathfrak{F} = (W, R) \in \mathrm{Fr}_\exists(\varphi)$ if and only if $\mathfrak{F} \models \forall P \exists x (Px \to \forall y (Rxy \to \exists z (Ryz \wedge Pz)))$. So in particular, under the assignment which assigns the entire W to the second-order variable P, we get $\mathfrak{F} \models \exists x \forall y (Rxy \to \exists z Ryz)$, thus $\mathfrak{F} \in \mathrm{Mod}(F)$. The reverse inclusion is proved similarly.

Other changes of quantifiers or the order of first and second-order quantifiers would result in other types of definability, perhaps also worthy of exploring. In fact, this has already been done by Venema [9] and Hollenberg [5], who consider *negative definability*, which corresponds to second-order formulas of the form $\forall x \exists P_1 \ldots \exists P_n ST_x(\neg\varphi)$. A class of frames negatively defined by Σ is denoted $\mathrm{Fr}^-(\Sigma)$. A general characterization of negative definability has not been obtained, and neither has a characterization of elementary classes which are negatively definable – it even remains unknown if all negatively definable classes are in fact elementary. But, to digress a little from the main point of this paper, we easily get the following fairly broad result.

Proposition 1. *Let φ be a modal formula which has a first-order local correspondent, i. e. there is a first-order formula $F(x)$ such that for any frame $\mathfrak{F} = (W, R)$ and any $w \in W$ we have $\mathfrak{F}, w \Vdash \varphi$ if and only if $\mathfrak{F} \models F(x)[w]$. (In particular, this holds for any Sahlqvist formula.)*

Then we have $\mathrm{Fr}^-(\varphi) = \mathrm{Mod}(\forall x \neg F(x))$.

Proof. We have $\mathfrak{F} \in \mathrm{Fr}^-(\varphi)$ if and only if $\mathfrak{F} \models \forall x \exists P_1 \ldots \exists P_n ST_x(\neg\varphi)$ if and only if $\mathfrak{F} \not\models \exists x \forall P_1 \ldots \forall P_n ST_x(\varphi)$. But this means that there is no $w \in W$ such that $\mathfrak{F} \models \forall P_1 \ldots \forall P_n ST_x(\varphi)[w]$. The latter holds if and only if $\mathfrak{F}, w \Vdash \varphi$, which is by assumption equivalent to $\mathfrak{F} \models F(x)[w]$. The fact that such w does not exist, is equivalent to $\mathfrak{F} \in \mathrm{Mod}(\forall x \neg F(x))$. $\Box$

So for example, since $p \to \Diamond p$ locally corresponds to Rxx, we have that $p \to \Diamond p$ negatively defines irreflexivity, which is not modally definable property, since it is not preserved under surjective bounded morphisms.

3 Model-theoretic constructions

This section can be used, if needed, for a quick reference of the basic facts about constructions used in the main theorem. Otherwise it can be omitted.

A *bisimulation* between Kripke models $\mathfrak{M} = (W, R, V)$ and $\mathfrak{M}' = (W', R', V')$ is a relation $Z \subseteq W \times W'$ such that:

(at) if wZw' then we have: $w \in V(p)$ if and only if $w' \in V'(p)$, for all $p \in \Phi$,

(forth) if wZw' and Rwv, then there is v' such that vZv' and $R'w'v'$,

(back) if wZw' and $R'w'v'$, then there is v such that vZv' and Rwv.

The basic property of bisimulations is that (at) extends to all formulas: if wZw' then $\mathfrak{M}, w \Vdash \varphi$ if and only if $\mathfrak{M}', w' \Vdash \varphi$, i. e. $(\mathfrak{M}, w)$ and $(\mathfrak{M}', w')$ are modally equivalent. We get the definition of bisimulation between frames by omitting the condition (at).

A *bounded morphism* from a frame $\mathfrak{F} = (W, R)$ to $\mathfrak{F}' = (W', R')$ is a function $f : W \to W'$ such that:

(forth) Rwv implies $R'f(w)f(v)$,

(back) if $R'f(w)v'$, then there is v such that $v' = f(v)$ and Rwv.

Clearly, a bounded morphism is a bisimulation that is also a function.

A *generated subframe* of $\mathfrak{F} = (W, R)$ is a frame $\mathfrak{F}' = (W', R')$ where $W' \subseteq W$ such that $w \in W'$ and Rwv implies $v \in W'$, and $R' = R \cap (W' \times W')$. A *generated submodel* of $\mathfrak{M} = (W, R, V)$ is a model based on a generated subframe, with the valuation $V'(p) = V(p) \cap W'$, for all $p \in \Phi$. It is easy to see that the global truth of a modal formula is preserved on a generated submodel.

To define the ultraproducts and ultrafilter extensions, we need the notion of ultrafilters. An *ultrafilter* over a set $I \neq \emptyset$ is a family $U \subseteq \mathcal{P}(I)$ such that:

(1) $I \in U$,

(2) if $A, B \in U$, then $A \cap B \in U$,

(3) if $A \in U$ and $A \subseteq B \subseteq I$, then $B \in U$,

(4) for all $A \subseteq I$ we have: $A \in U$ if and only if $I \setminus A \notin U$.

The existence of ultrafilters is provided by a fact that any family of subsets which has the finite intersection property (that is, each finite intersection is non-empty) can be extended to an ultrafilter (see e. g. [2]).

Let $\{\mathfrak{M}_i = (W_i, R_i, V_i) : i \in I\}$ be a family of Kripke models and let U be an ultrafilter over I. The *ultraproduct* of this family over U is the model $\prod_U \mathfrak{M}_i = (W, R, V)$ such that:

(1) W is the set of equivalence classes f^U of the following relation defined on the Cartesian product of the family: $f \sim g$ if and only if $\{i \in I : f(i) = g(i)\} \in U$,

(2) $f^U R g^U$ if and only if $\{i \in I : f(i) R_i g(i)\} \in U$,

(3) $f^U \in V(p)$ if and only if $\{i \in I : f(i) \in V_i(p)\} \in U$, for all p.

The basic property of ultraproducts is that (3) extends to all formulas.

Proposition 2. *Let $\{\mathfrak{M}_i : i \in I\}$ be a family of Kripke models and let U be an ultrafilter over I.*

Then we have $\prod_U \mathfrak{M}_i, f^U \Vdash \varphi$ if and only if $\{i \in I : \mathfrak{M}_i, f(i) \Vdash \varphi\} \in U$, for any f^U. Furthermore, we have $\prod_U \mathfrak{M}_i \Vdash \varphi$ if and only if $\{i \in I : \mathfrak{M}_i \Vdash \varphi\} \in U$.

This is an analogue of the Łoś Fundamental Theorem on ultraproducts from the first-order model theory (see [2] for this, and [1] for the proof of the modal analogue). The Łoś Theorem also implies that an elementary class of models is closed under ultraproducts.

An ultraproduct such that $\mathfrak{M}_i = \mathfrak{M}$ for all $i \in I$ is called an *ultrapower* of $\mathfrak{M}$ and denoted $\prod_U \mathfrak{M}$. From the Łoś Theorem it follows that any ultrapower of a model is elementarily equivalent to the model, that is, the same first-order sentences are true on $\mathfrak{M}$ and $\prod_U \mathfrak{M}$. Definition of an ultraproduct of a family of frames is obtained by omitting the clause regarding valuation.

Another notion needed in the proof of the main theorem is *modal saturation*, the modal analogue of *ω-saturation* from the classical model theory. The definition of saturation is omitted here, since we only need some facts which it implies. Most importantly, saturation implies a converse of the basic property of bisimulations, which generally does not hold. In fact, modal equivalence between points of modally saturated models is a bisimulation. Also, we use the fact that any ω-saturated Kripke model is also modally saturated (see [1] for proofs of these facts).

Finally, the *ultrafilter extension* of a model $\mathfrak{M} = (W, R, V)$ is the model $\mathfrak{ue}\mathfrak{M} = (\mathrm{Uf}(W), R^{\mathfrak{ue}}, V^{\mathfrak{ue}})$, where $\mathrm{Uf}(W)$ is the set of all ultrafilters over W, $R^{\mathfrak{ue}}uv$ holds if and only if $A \in v$ implies $m_\Diamond(A) \in u$, where $m_\Diamond(A)$ denotes the set of all $w \in W$ such that Rwa for some $a \in A$, and $u \in V^{\mathfrak{ue}}(p)$ if and only if $V(p) \in u$. The basic property is that this extends to any modal formula, i. e. we have $u \in V^{\mathfrak{ue}}(\varphi)$ if and only if $V(\varphi) \in u$ (see [1]). From this it easily follows that the global truth of a modal formula is preserved on the ultrafilter extension. Another important fact is that the ultrafilter extension of a model is modally saturated (see [1]).

The ultrafilter extension of a frame $\mathfrak{F} = (W, R)$ is $\mathfrak{ue}\mathfrak{F} = (\mathrm{Uf}(W), R^{\mathfrak{ue}})$.

4 Proof of the main theorem

In this section Theorem 1 is proved in detail. Arguments and techniques used in the proof are similar to the ones used in the proof of Goldblatt-Thomason theorem as presented in [1], so the reader might find it interesting to compare these proofs to note analogies and differences.

Proof (of Theorem 1). Let $\mathcal{K} = \mathrm{Fr}_\exists(\Sigma)$. Let $\mathfrak{F} = (W, R) \in \mathcal{K}$ and let f be a surjective bounded morphism from $\mathfrak{F}$ to some $\mathfrak{F}' = (W', R')$. Take any $\varphi \in \Sigma$ and any model $\mathfrak{M}' = (W', R', V')$ based on $\mathfrak{F}'$. Put $V(p) = \{w \in W : f(w) \in V'(p)\}$. Then V is a well defined valuation on $\mathfrak{F}$. Put $\mathfrak{M} = (W, R, V)$. Since $\mathfrak{F} \in \mathcal{K}$, there exists $w \in W$ such that $\mathfrak{M}, w \Vdash \varphi$. But then $\mathfrak{M}', f(w) \Vdash \varphi$. This proves that $\mathcal{K}$ is closed under surjective bounded morphisms.

To prove that $\mathcal{K}$ reflects generated subframes and ultrafilter extensions, let $\mathfrak{F} = (W, R) \notin \mathcal{K}$. This means that there is $\varphi \in \Sigma$ and a model $\mathfrak{M} = (W, R, V)$ based on $\mathfrak{F}$ such that $\mathfrak{M} \Vdash \neg\varphi$. Let $\mathfrak{F}' = (W', R')$ be a generated subframe of $\mathfrak{F}$. Define $V'(p) = V(p) \cap W'$, for all p. Then we have $\mathfrak{M}' \Vdash \neg\varphi$, which proves $\mathfrak{F}' \notin \mathcal{K}$, as desired. Also, $\mathfrak{ue}\mathfrak{M}$ is a model based on the ultrafilter extension $\mathfrak{ue}\mathfrak{F}$ and we have $\mathfrak{ue}\mathfrak{M} \Vdash \neg\varphi$, which proves $\mathfrak{ue}\mathfrak{F} \notin \mathcal{K}$.

For the converse, let $\mathcal{K}$ be an elementary class of frames that is closed under surjective bounded morphisms and reflects generated subframes and ultrafilter extensions. Denote by Σ the set of all formulas that are satisfiable in all models based on all frames in $\mathcal{K}$. Then $\mathcal{K} \subseteq \mathrm{Fr}_\exists(\Sigma)$ and it remains to prove the reverse inclusion.

Let $\mathfrak{F} = (W, R) \in \mathrm{Fr}_\exists(\Sigma)$. Let Φ be a set of propositional variables that contains a propositional variable p_A for each $A \subseteq W$. Let $\mathfrak{M} = (W, R, V)$, where $V(p_A) = A$ for all $A \subseteq W$. Denote by Δ the set of all modal formulas over Φ which are globally true on $\mathfrak{M}$. Now, for any finite $\delta \subseteq \Delta$ there is $\mathfrak{F}_\delta \in \mathcal{K}$ and a model $\mathfrak{M}_\delta$ based on $\mathfrak{F}_\delta$ such that $\mathfrak{M}_\delta \Vdash \delta$. Otherwise, since Δ is closed under conjunctions, there is $\varphi \in \Delta$ such that $\neg\varphi \in \Sigma$, thus $\neg\varphi$ is satisfiable in $\mathfrak{M}$, which contradicts $\mathfrak{M} \Vdash \Delta$.

Now, let I be the family of all finite subsets of Δ. For each $\varphi \in \Delta$, put $\hat{\varphi} = \{\delta \in I : \varphi \in \delta\}$. The family $\{\hat{\varphi} : \varphi \in \Delta\}$ clearly has the finite intersection property, so it can be extended to an ultrafilter U over I. But for all $\varphi \in \Delta$ we have $\{\delta \in I : \mathfrak{M}_\delta \Vdash \varphi\} \supseteq \hat{\varphi}$ and $\hat{\varphi} \in U$, thus $\{\delta \in I : \mathfrak{M}_\delta \Vdash \varphi\} \in U$, so the Proposition 2 implies $\prod_U \mathfrak{M}_\delta \Vdash \varphi$. The model $\prod_U \mathfrak{M}_\delta$ is based on the frame $\prod_U \mathfrak{F}_\delta$. Since $\mathcal{K}$ is elementary, it is also closed under ultraproducts, so $\prod_U \mathfrak{F}_\delta \in \mathcal{K}$. It remains to prove that there is a surjective bounded morphism from some ultrapower of $\prod_U \mathfrak{F}_\delta$ to a generated subframe of $\mathfrak{ue}\mathfrak{F}$. Then the assumed properties of $\mathcal{K}$ imply that $\mathfrak{F} \in \mathcal{K}$, as desired.

The classical model theory provides that there is an ω-saturated ultrapower of $\prod_U \mathfrak{M}_\delta$ (cf. [2]). Let $\mathfrak{M}_\Delta$ be such an ultrapower. We have that $\mathfrak{M}_\Delta$ is modally saturated. Also, it is elementarily equivalent to $\prod_U \mathfrak{M}_\delta$, so using standard translation we obtain $\mathfrak{M}_\Delta \Vdash \Delta$. The model $\mathfrak{M}_\Delta$ is based on a frame $\mathfrak{F}_\Delta$, which is an ultrapower of $\prod_U \mathfrak{F}_\delta$. Now define a mapping from $\mathfrak{F}_\Delta$ to $\mathfrak{ue}\mathfrak{F}$ by putting $f(w) = \{A \subseteq W : \mathfrak{M}_\Delta, w \Vdash p_A\}$.

First we need to prove that f is well-defined, i. e. that $f(w)$ is indeed an ultrafilter over W.

(1) We easily obtain $W \in f(w)$, since $p_W \in \Delta$ by the definition of V.

(2) If $A, B \in f(w)$, then $\mathfrak{M}_\Delta, w \Vdash p_A \wedge p_B$. Clearly, $\mathfrak{M} \Vdash p_A \wedge p_B \leftrightarrow p_{A\cap B}$. Thus $\mathfrak{M}_\Delta \Vdash p_A \wedge p_B \leftrightarrow p_{A\cap B}$, so $\mathfrak{M}_\Delta, w \Vdash p_{A\cap B}$, i. e. $A \cap B \in f(w)$.

(3) If $A \in f(w)$ and $A \subseteq B \subseteq W$, then from the definition of V it follows $\mathfrak{M} \Vdash p_A \to p_B$. But then also $\mathfrak{M}_\Delta \Vdash p_A \to p_B$, hence $\mathfrak{M}_\Delta, w \Vdash p_B$, so $B \in f(w)$.

(4) For all $A \subseteq W$ we have $\mathfrak{M} \Vdash p_A \leftrightarrow \neg p_{W\setminus A}$, which similarly as in the previous points implies $A \in f(w)$ if and only if $W \setminus A \notin f(w)$, as desired.

Assume for the moment that we have: $u = f(w)$ if and only if $(\mathfrak{ue}\mathfrak{M}, u)$ and $(\mathfrak{M}_\Delta, w)$ are modally equivalent. Since $\mathfrak{ue}\mathfrak{M}$ and $\mathfrak{M}_\Delta$ are modally saturated, the modal equivalence between their points is a bisimulation. So f is a bisimulation, but it is also a function, which means that it is a bounded morphism from $\mathfrak{F}_\Delta$ to $\mathfrak{ue}\mathfrak{F}$. But then the corestriction of f to its image is a surjective bounded morphism from an ultrapower of $\prod_U \mathfrak{F}_\delta$ to a generated subframe of $\mathfrak{ue}\mathfrak{F}$, which we needed.

So to conclude the proof, it remains to show that $u = f(w)$ holds if and only if $(\mathfrak{ue}\mathfrak{M}, u)$ and $(\mathfrak{M}_\Delta, w)$ are modally equivalent. Let $u = f(w)$. Then we have $\mathfrak{ue}\mathfrak{M}, u \Vdash \varphi$ if and only if $V(\varphi) \in u$, which is by the definition of f equivalent to $\mathfrak{M}_\Delta, w \Vdash p_{V(\varphi)}$. But the definition of V clearly implies $\mathfrak{M} \Vdash \varphi \leftrightarrow p_{V(\varphi)}$, so also $\mathfrak{M}_\Delta \Vdash \varphi \leftrightarrow p_{V(\varphi)}$, which provides the needed modal equivalence.

For the converse, the assumption implies that we have $\mathfrak{ue}\mathfrak{M}, u \Vdash p_A$ if and only if $\mathfrak{M}_\Delta, w \Vdash p_A$, for all $A \subseteq W$. This means that $V(p_A) = A \in u$ if and only if $A \in f(w)$, i. e. $u = f(w)$.

□

5 Conclusion: link to the universal modality

Although the approach of this paper is to define ∃-definability as a metalingual notion, it should be noted that it can be included in the language itself. That is, the satisfiability of a modal formula under any valuation on a frame can be expressed by a formula of the modal language enriched with the universal modality (BMLU). The syntax is an extension of the basic modal language by new modal operator $A\varphi$, and we can also define its dual $E\varphi := \neg A \neg \varphi$. We call A the *universal modality*, and E the *existential modality.* The semantics of the new operators is standard modal semantics, with respect to the universal binary relation $W \times W$ on a frame $\mathfrak{F} = (W, R)$. This means that the standard translation of universal and existential operators is as follows (cf. [4] and [8]):

$ST_x(E\varphi) = \exists y ST_y(\varphi)$,
$ST_x(A\varphi) = \forall y ST_y(\varphi)$.

Now, let $\mathcal{K}$ be a class of Kripke frames. Clearly, $\mathcal{K}$ is modally $\exists$-definable if and only if it is definable by a set of formulas of the existential fragment of BMLU, i. e. by a set of formulas of the form $E\varphi$, where φ is a formula of BML. This immediately follows from the clear fact that for any frame $\mathfrak{F}$ and any φ we have $\mathfrak{F} \Vdash E\varphi$ if and only if $\mathfrak{F} \models \forall P_1 \dots \forall P_n \exists y ST_y(\varphi)$, where $P_1, \dots, P_n$ correspond to propositional variables that occur in φ, and the letter holds if and only if φ is satisfiable under any valuation on $\mathfrak{F}$.

Goranko and Passy [4] gave a characterization that an elementary class is modally definable in BMLU if and only if it is closed under surjective bounded morphisms and reflects ultrafilter extension. So, from the main theorem of this paper we conclude that reflecting generated subframes, not surprisingly, is what distinguishes existential fragment within this language, at least with respect to elementary classes. Also, the usual notion of modal definability clearly coincides with the universal fragment of BMLU, hence the Goldblatt-Thomason Theorem tells us that closure under generated subframes and disjoint unions is essential for this fragment.

As for some further questions that might be worth exploring, for example, similarly to the notion of $\pm$-definability from [5], we can say that a class of frames is *modally $\forall\exists$-definable* if there is a pair (Σ_1, Σ_2) of sets of formulas such that a class consists exactly of frames on which every formula from Σ_1 is valid and every formula from Σ_2 is satisfiable under any valuation, and try to obtain a characterization theorem. This also coincides with a fragment of BMLU, and generalizes both usual modal definability and $\exists$-definability. Furthermore, we may be able to obtain general characterization theorems for these fragments, without the assumption of the first-order definability.

On the other hand, a question to be addressed is which modally $\exists$-definable classes are elementary, and is there an effective procedure analogous to the one for Sahlqvist formulas, to obtain a first-order formula equivalent to a second-order translation $\forall P_1 \dots \forall P_n \exists x ST_x(\varphi)$ for some sufficiently large and interesting class of modal formulas.

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