Checking Admissibility in Finite Algebras

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Abstract. Checking if a quasiequation is admissible in a finite algebra is a decidable problem, but the naive approach, i.e., checking validity in the corresponding free algebra, is computationally unfeasible. We give an algorithm for obtaining smaller algebras to check admissibility and a range of examples to demonstrate the advantages of this approach.

1 Introduction

Rules and axioms are the building blocks of a logic. Axioms are the assumptions of the logic, whereas rules are used to derive new facts from previously derived facts. Rules are usually formulated as IF-THEN statements, e.g. "IF x is an integer and x is positive THEN $x+1$ is a natural number". More generally, a rule is a set of premises followed by a conclusion. In logic, the premises and the conclusion are formulas. In algebra they are usually equations, as in the cancellation rule "IF $x + y = x + z$ THEN $y = z$". Axioms are rules without a premise and can be read as, e.g., "x + y = y + x always holds". In algebra one often uses Σ to denote a finite set of equations and calles the rule "IF Σ THEN $\varphi \approx \psi$", written $\Sigma \Rightarrow \varphi \approx \psi$, a quasiequation. A quasiequation $\Sigma \Rightarrow \varphi \approx \psi$ is called valid in the finite algebra $\mathbf{A}$ if whenever every equation in Σ is true in $\mathbf{A}$ for a specific choice of elements of $\mathbf{A}$ for the variables occuring in $\Sigma \cup \{\varphi \approx \psi\}$, then also $\varphi \approx \psi$ is true in $\mathbf{A}$ for this choice.

Checking validity in finite algebras (similarly, derivability in finite-valued logics) has been studied extensively in the literature, and may be considered a "solved problem" in the sense that there exist both general methods for obtaining proof systems for checking validity (tableaux, resolution, multisequents, etc.) and standard optimization techniques for such systems (lemma generation, indexing, etc.) (see, e.g., [1, 12, 24]). A rule which can be added to a given system without producing new valid equations is called *admissible*. This notion was introduced by Lorenzen in 1955 [18], but the property of being admissible was already used by Gentzen twenty years earlier [7]. Admissibility has been studied intensively in the context of intermediate and transitive modal logics and their algebras [6, 8, 9, 13, 15, 21], leading also to proof systems for checking admissibility [2, 10, 14], and certain many-valued logics and their algebras [5, 16, 17, 19, 21], but a general theory for this latter case has so far been lacking.

Showing the admissibility of rules can play an important role in establishing completeness results. That means for example, that one proves the admissibility of the cut-rule "IF x = y and y = z THEN x = z" to show that the system can derive the same equations without the cut-rule. Moreover, in some cases adding admissible rules to a system can simplify or speed up reasoning in this system.

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Often it is possible to transform logical settings to algebraic settings and vice versa (see, e.g., [3]). In this sense rules and logics correspond to quasiequations and classes of algebras satisfying the same quasiequations, respectively. In this work we concentrate on the question, whether a given quasiequation is admissible in a finite algebra. This corresponds to the question, whether the quasiequation holds in a corresponding free algebra on countably infinitely many generators. Although it is well known that admissibility is decidable in finite algebras, the naive approach is computationally unfeasible. We give an algorithm to answer this question in a more efficient way.

This paper (based on joint research with my supervisor [20]) focuses on procedural aspects of the given problem and its solution. Necessary algebraic definitions are provided, so that also readers without experience in universal algebra are able to understand the text.

2 Validity and Admissibility

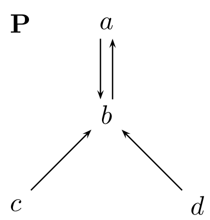
Let us first recall some basics from universal algebra. A *language* is a set of operation symbols $\mathcal{L}$ such that to each operation symbol $f \in \mathcal{L}$ a nonnegative integer $\mathrm{ar}(f)$ is assigned called the *arity of* f. An $\mathcal{L}$*-algebra* $\mathbf{A}$ is an ordered pair $\mathbf{A} = \langle \mathrm{A}, \{f_1^{\mathbf{A}}, \dots, f_k^{\mathbf{A}}\}\rangle$ such that A is a set, called the *universe of* $\mathbf{A}$, and each $f_i^{\mathbf{A}}$ is an operation on $\mathbf{A}$, corresponding to an operation symbol $f_i \in \mathcal{L}$. We often omit superscripts when describing the operations of an algebra. Let $\mathbf{A}$ and $\mathbf{B}$ be two algebras of the same language. Then $\mathbf{B}$ is a *subalgebra* of $\mathbf{A}$, written $\mathbf{B} \leq \mathbf{A}$, if $\mathrm{B} \subseteq \mathrm{A}$ and every operation of $\mathbf{B}$ is the restriction of the corresponding operation of $\mathbf{A}$. For $\{a_1, \dots, a_k\} \subseteq \mathrm{A}$ the smallest subalgebra of $\mathbf{A}$ containing $\{a_1, \dots, a_k\}$ is denoted by $\langle a_1, \dots, a_k\rangle$. We use the letters x, y, z, possibly indexed, to denote variables.

Example 1. Let $\mathcal{L} = \{\to, \mathrm{e}\}$ be a language with $\mathrm{ar}(\to) = 2$ and $\mathrm{ar}(\mathrm{e}) = 0$. Define the algebra $\mathbf{S_4^{\to e}} = \langle \{-2, -1, 1, 2\}, \to, \mathrm{e}\rangle$ with the operations

$$x \to y = \begin{cases} \max\{-x, y\} & x \leq y \\ \min\{-x, y\} & \text{otherwise} \end{cases} \qquad \text{and} \qquad \mathrm{e} = 1.$$

The algebra $\mathbf{S_2^{\to e}} = \langle \{-1, 1\}, \to, \mathrm{e}\rangle$ is a subalgebra of $\mathbf{S_4^{\to e}}$, i.e., $\mathbf{S_2^{\to e}} \leq \mathbf{S_4^{\to e}}$.

Example 2. Let $\mathcal{L}$ consist of one operation symbol $\star$ with arity 1. Then consider the algebra $\mathbf{P} = \langle \{a, b, c, d\}, \star\rangle$ where the unary operation $\star$ is described by the diagram below. The algebra $\langle \{a, b, d\}, \star\rangle$ is then clearly a subalgebra of $\mathbf{P}$.



The set $\mathrm{Tm}_{\mathcal{L}}$ of $\mathcal{L}$-terms is inductively defined: every variable is an $\mathcal{L}$-term and if $\varphi_1, \dots, \varphi_n$ are $\mathcal{L}$-terms and the operation symbol $f \in \mathcal{L}$ has arity n, then also $f(\varphi_1, \dots, \varphi_n)$ is an $\mathcal{L}$-term. We denote the term algebra over countably infinitely many variables by $\mathbf{Tm}_\mathcal{L}$ (i.e., for each $f \in \mathcal{L}$ with $\mathrm{ar}(f) = n$, $\varphi_1, \ldots, \varphi_n \in \mathrm{Tm}_\mathcal{L}$, $f^{\mathbf{Tm}_\mathcal{L}}(\varphi_1, \ldots, \varphi_n)$ is just the $\mathcal{L}$-term $f(\varphi_1, \ldots, \varphi_n)$) and let φ, ψ stand for arbitrary members of the universe $\mathrm{Tm}_\mathcal{L}$. An $\mathcal{L}$-*equation* is a pair of $\mathcal{L}$-terms, written $\varphi \approx \psi$. If Σ is a finite set of $\mathcal{L}$-equations, we call $\Sigma \Rightarrow \varphi \approx \psi$ an $\mathcal{L}$-*quasiequation.* As usual, if the language is clear from the context we may omit the prefix $\mathcal{L}$.

Example 3. Terms in the language of Example 1 are, e.g., x, $x \to x$ or $(x \to \mathrm{e}) \to y$ whereas terms corresponding to Example 2 have the form $\star(x)$ or $\star(\star(\star(y)))$. The following is a quasiequation in the language of Example 1

$$\{x \approx y \to x, \;\; x \to \mathrm{e} \approx y\} \quad \Rightarrow \quad x \approx \mathrm{e}.$$

A *homomorphism* h between two algebras $\mathbf{A}$ and $\mathbf{B}$ of the same language $\mathcal{L}$ is a map $h\colon \mathrm{A} \to \mathrm{B}$ between their universes that preserves all the operations, i.e., for all $a_1, \ldots, a_n \in \mathrm{A}$ and every operation $f \in \mathcal{L}$ with $\mathrm{ar}(f) = n$, $h(f^\mathbf{A}(a_1, \ldots, a_n)) = f^\mathbf{B}(h(a_1), \ldots, h(a_n))$. The homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ is called *surjective* if for all $b \in \mathrm{B}$, there exists an $a \in \mathrm{A}$ such that $h(a) = b$. Two algebras $\mathbf{A}$ and $\mathbf{B}$ are said to be *isomorphic*, if there exists a surjective homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ with $h(a) \neq h(b)$ for all $a \neq b$. The algebra $\mathbf{C}$ with the universe $\mathrm{C} = \{h(a) : a \in \mathrm{A}\} \subseteq \mathrm{B}$ and the restrictions of the operations of $\mathbf{B}$ to C as operations is called a *homomorphic image of* $\mathbf{A}$, written $\mathbf{C} \in \mathbb{H}(\mathbf{A})$.

We say that the quasiequation $\Sigma \Rightarrow \varphi \approx \psi$ is *valid in* $\mathbf{A}$ or "holds in $\mathbf{A}$", written $\Sigma \models_\mathbf{A} \varphi \approx \psi$, if for every homomorphism $h\colon \mathbf{Tm}_\mathcal{L} \to \mathbf{A}$, $h(\varphi') = h(\psi')$ for all $\varphi' \approx \psi' \in \Sigma$ implies $h(\varphi) = h(\psi)$.

Example 4. The quasiequation of Example 3 is not valid in $\mathbf{S_4^{\to e}}$ since the homomorphism $h\colon \mathbf{Tm}_\mathcal{L} \to \mathbf{S_4^{\to e}}$ with $h(x) = -1$ and $h(y) = 1$ satisfies $h(x) = h(y \to x)$ and $h(x \to \mathrm{e}) = h(y)$, but not $h(x) = h(\mathrm{e})$.

We also need the well-known fact (see, e.g., [4]) that taking homomorphic images and subalgebras preserves equations and quasiequations, respectively.

Lemma 1. *Let* $\mathbf{A}$ *be an algebra and* $\Sigma \cup \{\varphi \approx \psi\}$ *a finite set of equations. Then*

(a) $\models_\mathbf{A} \varphi \approx \psi$ *implies* $\models_\mathbf{B} \varphi \approx \psi$ *for all* $\mathbf{B} \in \mathbb{H}(\mathbf{A})$.
(b) $\Sigma \models_\mathbf{A} \varphi \approx \psi$ *implies* $\Sigma \models_\mathbf{B} \varphi \approx \psi$ *for all* $\mathbf{B} \leq \mathbf{A}$.

For a nonnegative integer m, let $\mathbf{F_A}(m)$ denote the *free algebra with* m *generators of the* $\mathcal{L}$*-algebra* $\mathbf{A}$, i.e., the algebra of equivalence classes $[\varphi]$ of $\mathcal{L}$-terms φ containing at most m variables $x_1, \ldots, x_m$ such that two terms φ and ψ belong to the same class if and only if $\models_\mathbf{A} \varphi \approx \psi$. The free algebra of the algebra $\mathbf{A}$ has the same language as $\mathbf{A}$ and for $\mathcal{L}$-terms $\varphi_1, \ldots, \varphi_n$ and the operation f with $\mathrm{ar}(f) = n$ we have $f^{\mathbf{F_A}(m)}([\varphi_1], \ldots, [\varphi_n]) = [f^\mathbf{A}(\varphi_1, \ldots, \varphi_n)]$.

Lemma 2 ([21], [4]). *Let* $\mathbf{A}$ *be a finite* $\mathcal{L}$*-algebra and* $\Sigma \cup \{\varphi \approx \psi\}$ *a finite set of* $\mathcal{L}$*-equations. Then*

(a) $\mathbf{F_A}(m)$ *is finite for all* $m \in \mathbb{N}$.
(b) $\models_{\mathbf{F_A}(|\mathrm{A}|)} \varphi \approx \psi$ *if and only if* $\models_\mathbf{A} \varphi \approx \psi$.
(c) $\Sigma \models_{\mathbf{F_A}(|\mathrm{A}|)} \varphi \approx \psi$ *if and only if* $\Sigma \models_{\mathbf{F_A}(\mathrm{k})} \varphi \approx \psi$, $|\mathrm{A}| \leq k \in \mathbb{N}$.

Intuitively an $\mathcal{L}$-quasiequation $\Sigma \Rightarrow \varphi \approx \psi$ is admissible in an $\mathcal{L}$-algebra $\mathbf{A}$, if every substitution (i.e., every homomorphism from the term algebra to the term algebra), that makes every equation of Σ hold in $\mathbf{A}$, also makes $\varphi \approx \psi$ hold in $\mathbf{A}$. More formally, an $\mathcal{L}$-quasiequation $\Sigma \Rightarrow \varphi \approx \psi$ is called *admissible in* $\mathbf{A}$, if for every homomorphism $\sigma\colon \mathbf{Tm}_\mathcal{L} \to \mathbf{Tm}_\mathcal{L}$:

$$\models_\mathbf{A} \sigma(\varphi') \approx \sigma(\psi') \ \text{ for all } \varphi' \approx \psi' \in \Sigma \quad \text{implies} \quad \models_\mathbf{A} \sigma(\varphi) \approx \sigma(\psi).$$

Quasiequations admissible in the n-element algebra $\mathbf{A}$ are, equivalently, quasiequations valid in $\mathbf{F_A}(n)$.

Lemma 3 ([19]). *$\Sigma \Rightarrow \varphi \approx \psi$ is admissible in $\mathbf{A}$ iff $\Sigma \models_{\mathbf{F_A}(|\mathrm{A}|)} \varphi \approx \psi$.*

If a quasiequation $\Sigma \Rightarrow \varphi \approx \psi$ is valid in an algebra $\mathbf{A}$, then it is also admissible in $\mathbf{A}$. However, the other direction is not true in general. We say that $\mathbf{A}$ is *structurally complete*, if admissibility and validity coincide for $\mathbf{A}$, i.e., $\Sigma \Rightarrow \varphi \approx \psi$ is admissible in $\mathbf{A}$ if and only if $\Sigma \models_\mathbf{A} \varphi \approx \psi$.

Example 5. Consider the two-valued Boolean algebra $\mathbf{2} = \langle \{0,1\}, \wedge, \vee, \neg, 1, 0\rangle$. Suppose that a $\{\wedge, \vee, \neg, 1, 0\}$-quasiequation $\Sigma \Rightarrow \varphi \approx \psi$ is not valid in $\mathbf{2}$, i.e., there exists a homomorphism $h\colon \mathbf{Tm}_\mathcal{L} \to \mathbf{2}$ such that $h(\varphi') = h(\psi')$ for all $\varphi' \approx \psi' \in \Sigma$ and $h(\varphi) \neq h(\psi)$. Define the homomorphism $\sigma\colon \mathbf{Tm}_\mathcal{L} \to \mathbf{Tm}_\mathcal{L}$ by sending each variable x to 1, if $h(x) = 1$ and to 0, if $h(x) = 0$. It follows immediately that $\models_\mathbf{2} \sigma(\varphi') \approx \sigma(\psi')$ for all $\varphi' \approx \psi' \in \Sigma$, but $\not\models_\mathbf{2} \sigma(\varphi) \approx \sigma(\psi)$. So $\Sigma \Rightarrow \varphi \approx \psi$ is not admissible in $\mathbf{2}$, hence $\mathbf{2}$ is structurally complete.

3 A Procedure to Check Admissibility

To check if a given quasiequation $\Sigma \Rightarrow \varphi \approx \psi$ is admissible in a finite algebra $\mathbf{A}$, it suffices by Lemma 3 to check whether the quasiequation is valid in the free algebra $\mathbf{F_A}(|\mathrm{A}|)$, which we know is always finite. The validity of quasiequations in finite algebras is well studied and decidable (see, e.g., [1, 12, 24]). However, even free algebras on a small number of generators can be very large. E.g., the free algebra $\mathbf{F}_{\mathbf{S}_\mathbf{4}^{\to e}}(2)$ has 453 elements, where $\mathbf{S}_\mathbf{4}^{\to e}$ is the algebra of Example 1. We therefore seek smaller algebras $\mathbf{B}$ such that, as for $\mathbf{F_A}(|\mathrm{A}|)$:

$$\Sigma \Rightarrow \varphi \approx \psi \text{ is admissible in } \mathbf{A} \quad \Longleftrightarrow \quad \Sigma \models_\mathbf{B} \varphi \approx \psi.$$

Proposition 1. *Let $\mathbf{A}, \mathbf{B}$ be $\mathcal{L}$-algebras such that $\mathbf{B}$ is a subalgebra of $\mathbf{F_A}(|\mathrm{A}|)$ and $\mathbf{A}$ is a homomorphic image of $\mathbf{B}$. Then $\Sigma \Rightarrow \varphi \approx \psi$ is admissible in $\mathbf{A}$ iff $\Sigma \models_\mathbf{B} \varphi \approx \psi$.*

Proof. Let $\Sigma \Rightarrow \varphi \approx \psi$ be admissible in $\mathbf{A}$. So $\Sigma \models_{\mathbf{F_A}(|\mathrm{A}|)} \varphi \approx \psi$ by Lemma 3 and then $\Sigma \models_\mathbf{B} \varphi \approx \psi$ by Lemma 1. For the other direction suppose that $\Sigma \models_\mathbf{B} \varphi \approx \psi$ and $\models_\mathbf{A} \sigma(\varphi') \approx \sigma(\psi')$ for all $\varphi' \approx \psi' \in \Sigma$. Then $\models_{\mathbf{F_A}(\mathrm{n})} \sigma(\varphi') \approx \sigma(\psi')$ for all $\varphi' \approx \psi' \in \Sigma$ by Lemma 2 and therefore $\models_\mathbf{B} \sigma(\varphi') \approx \sigma(\psi')$ for all $\varphi' \approx \psi' \in \Sigma$ by Lemma 1. But then $\models_\mathbf{B} \sigma(\varphi) \approx \sigma(\psi)$ and since $\mathbf{A}$ is a homomorphic image of $\mathbf{B}$, $\models_\mathbf{A} \sigma(\varphi) \approx \sigma(\psi)$ by Lemma 1. ☐

Note that every subalgebra $\mathbf{B}$ of a subalgebra $\mathbf{C}$ of the free algebra $\mathbf{F_A}(|\mathrm{A}|)$, i.e., $\mathbf{B} \leq \mathbf{C} \leq \mathbf{F_A}(|\mathrm{A}|)$, is a subalgebra of $\mathbf{F_A}(|\mathrm{A}|)$. So since $\mathbf{F_A}(m_1) \leq \mathbf{F_A}(m_2)$ for all $m_1 \leq m_2$ (see, e.g., [4]), we possibly do not need $|\mathrm{A}|$ generators. This suggests the following procedure when $\mathbf{A}$ is finite:

(i) Find the smallest free algebra $\mathbf{F_A}(m)$ such that $\mathbf{A} \in \mathbb{H}(\mathbf{F_A}(m))$.

(ii) Compute subalgebras $\mathbf{B}$ of $\mathbf{F_A}(m)$, increasing in their size, and check for each whether $\mathbf{A} \in \mathbb{H}(\mathbf{B})$.

(iii) Derive a proof system for a smallest $\mathbf{B}$ with the properties of (ii).

Steps (i) and (ii) of the procedure have been implemented using macros implemented for the Algebra Workbench [23]. Step (iii) can be implemented directly making use of a system such as MUltlog/MUltseq [11, 22].

We now give some explanation how to implement the first two steps of the procedure. For a given $\mathbf{A}$, we want to find the smallest free algebra $\mathbf{F_A}(m)$ ($m \leq |\mathrm{A}|$) such that $\mathbf{A}$ is a homomorphic image of $\mathbf{F_A}(m)$. The idea is to calculate first $\mathbf{F_A}(0)$ and to check whether $\mathbf{A} \in \mathbb{H}(\mathbf{F_A}(0))$. Stop if this is the case, otherwise calculate $\mathbf{F_A}(1)$ and check whether $\mathbf{A} \in \mathbb{H}(\mathbf{F_A}(1))$ and so on.

Suppose that, given a finite algebra $\mathbf{A} = \langle\{a_1, \dots, a_n\}, f_1, \dots, f_k\rangle$, we want to calculate the elements of $\mathbf{F_A}(m)$. Recall that the elements of the free algebra can be seen as equivalence classes of terms. Therefore we need to know all the terms that are definable using the given generators. To decide whether two terms φ and ψ are the same, i.e., $\models_\mathbf{A} \varphi \approx \psi$, we would have to check all the possible homomorphisms $h\colon \mathbf{Tm}_\mathcal{L} \to \mathbf{A}$. So we simulate the truth table checking by storing the elements by sequences of elements of $\mathbf{A}$.

We represent the m generators of $\mathbf{F_A}(m)$ by sequences $\langle \pi_i(\bar{a}_1), \dots, \pi_i(\bar{a}_{n^m})\rangle$ where $\bar{a}_1, \dots, \bar{a}_{n^m}$ are the elements of $\mathbf{A}^m$ and π_i, $i = 1, \dots, m$ the i-th projection-map from $\mathbf{A}^m$ to $\mathbf{A}$ and collect them in a set G. Then we run the function DefinableTerms (see Fig. 1) to get the elements of $\mathbf{F_A}(m)$, stored as sequences of length n^m in the set F.

```
function DefinableTerms(G, {f_1, ..., f_k})
    F ← G
    repeat
        F_0 ← F
        for all f ∈ {f_1, ..., f_k} do
            F ← F ∪ { ⟨f(g_1), ..., f(g_ar(f))⟩ : g_1, ..., g_ar(f) ∈ F }
        end for
    until F_0 == F
    return F
end function
```

Fig. 1. Algorithm to generate all the definable terms, given a set of generators G

Example 6. Suppose that we want to calculate the elements of the free algebra for the algebra $\mathbf{P} = \langle\{a, b, c, d\}, \star\rangle$ defined in Example 2. We certainly need generators for the free algebra since $\mathbf{P}$ has no constants, i.e., no nullary operations. So the first step will be to calculate $\mathbf{F_P}(1)$: our generator is the sequence (a, b, c, d). Running the function DefinableTerms$((a, b, c, d), \{\star\})$ gives us $F = \{(a, b, c, d), (b, a, b, b), (a, b, a, a)\}$. It is easy to see that there cannot be a surjective homomorphism from $\mathbf{F_P}(1)$ to $\mathbf{A}$ since $\mathbf{A}$ has four elements. So we have to calculate the algebra $\mathbf{F_P}(2)$ with generators $(a, a, a, a, b, b, b, b, c, c, c, c, d, d, d, d)$ and $(a, b, c, d, a, b, c, d, a, b, c, d, a, b, c, d)$, which will give us a six element algebra that fulfills the requirement of the homomorphism.

The second step of the procedure requires us to calculate subalgebras of the free algebra, increasing in their size. We could, at least theoretically, check $\mathbf{A} \in \mathbb{H}(\mathbf{B})$ for all $\mathbf{B} \leq \mathbf{F_A}(m)$. But since we are interested in the smallest algebras with this property, we generate the subalgebras by increasing their size and always testing whether they satisfy the property. The principles for the calculation of subalgebras are those of DefinableTerms defined in Fig. 1. Using the generating elements as arguments for the operations, we increase the set of "reached" elements as long as we get new elements.

We first calculate all the one-generated subalgebras of the free algebra $\mathbf{F_A}(m)$, i.e., $\langle\varphi\rangle$ for $\varphi \in \mathrm{F}_\mathbf{A}(m)$ and store their sizes $|\langle\varphi\rangle|$. Now we know that the size of the two-generated subalgebra $\langle\varphi_1, \varphi_2\rangle$ of $\mathbf{F_A}(m)$ is at least $\max\{|\langle\varphi_1\rangle|, |\langle\varphi_2\rangle|\}$. Suppose that $k = \min\{|\langle\varphi\rangle| \colon \varphi \in \mathrm{F}_\mathbf{A}(m)\}$. If there is more than one $\varphi \in \mathrm{F}_\mathbf{A}(m)$ with $|\langle\varphi\rangle| = k$, then we generate all the algebras (increasing the number of generators) $\langle\varphi_1, \ldots, \varphi_r\rangle$ with $\max\{|t_1|, \ldots, |t_r|\} \leq k$, again testing if there exists a surjective homomorphism to $\mathbf{A}$ and storing their sizes. We then proceed similarly for $k' = \min\{|\langle\varphi\rangle| \colon \varphi \in \mathrm{F}_\mathbf{A}(m), |\langle\varphi\rangle| > k\}$. As soon as we find an algebra $\mathbf{B}$ with $\mathbf{A} \in \mathbb{H}(\mathbf{B})$ we have an upper-bound for the size of the algebras to test (note that this upper-bound always exists since it cannot exceed the size of the free algebra $\mathbf{F_A}(m)$). However, we then have to continue until we know that every combination of generators will lead to a subalgebra $\mathbf{B}'$ with $\mathrm{B} \leq \mathrm{B}'$.

It is not hard to see that step (i) is sound and terminating since we use the operations of the generating algebra $\mathbf{A}$ to calculating new, finite sequences of elements of $\mathbf{A}$. So there are at most $|\mathrm{A}|^{|\mathrm{A}|^m}$ sequences. The soundness of step (ii) is, similar to the previous step, given by the construction of the subalgebras and the used bounds of the cardinalities of the algebras. The algorithm terminates since there are only finitely many subalgebras of the free algebra.

Example 7. Consider the algebra $\mathbf{S_3^{\to\neg}} = \langle\{-1, 0, 1\}, \to, \neg\rangle$, where $\to$ is defined as in Example 1 and $\neg x = -x$. Note that an equation of the form $\varphi \approx \varphi \to \varphi$ holds in $\mathbf{S_3^{\to\neg}}$ iff φ is a theorem of the $\{\to, \neg\}$-fragment of the logic RM. Now, following our procedure, we obtain:

(i) $\mathbf{S_3^{\to\neg}} \notin \mathbb{H}(\mathbf{F}_{\mathbf{S_3^{\to\neg}}}(1))$, but $\mathbf{S_3^{\to\neg}} \in \mathbb{H}(\mathbf{F}_{\mathbf{S_3^{\to\neg}}}(2))$.
(ii) $\mathbf{F}_{\mathbf{S_3^{\to\neg}}}(2)$ has 264 elements and the smallest subalgebras $\mathbf{B} \leq \mathbf{F}_{\mathbf{S_3^{\to\neg}}}(2)$ with $\mathbf{S_3^{\to\neg}} \in \mathbb{H}(\mathbf{B})$ have 6 elements.

The fact that we first check the smaller subalgebras is useful here: We only had to check the 264 one-generated algebras, 15 two-generated and 3 three-generated algebras rather than all the 5134 possible subalgebras of $\mathbf{F}_{\mathbf{S_3^{\to\neg}}}(2)$.

Example 8. Small changes in the universe or language of an algebra can dramatically change the size and structure of its free algebra. Consider the algebra $\mathbf{S_4^{\to\neg e}} = \langle\{-2, -1, 1, 2\}, \to, \neg, \mathrm{e}\rangle$ where $\to$ and $\neg$ are defined as in Example 7 and $\mathrm{e}=1$. Although this algebra is only slightly different to $\mathbf{S_4^{\to e}}$ and $\mathbf{S_3^{\to\neg}}$ of the Examples 1 and 7, the appropriate free algebra is much smaller and only needs one generator:

(i) $\mathbf{S_4^{\to\neg e}} \notin \mathbb{H}(\mathbf{F}_{\mathbf{S_4^{\to\neg e}}}(0))$, but $\mathbf{S_4^{\to\neg e}} \in \mathbb{H}(\mathbf{F}_{\mathbf{S_4^{\to\neg e}}}(1))$.
(ii) $\mathbf{F}_{\mathbf{S_4^{\to\neg e}}}(1)$ has 18 elements and the smallest subalgebras $\mathbf{B} \leq \mathbf{F}_{\mathbf{S_4^{\to\neg e}}}(1)$ with $\mathbf{S_4^{\to\neg e}} \in \mathbb{H}(\mathbf{B})$ have 6 elements.

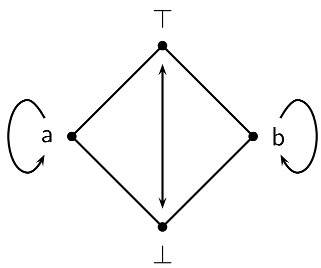
Example 9. In some cases, it is possible to establish structural completeness results for the algebra $\mathbf{A}$ using the described procedure. The smallest $\mathbf{B} \leq \mathbf{F_A}(m)$ with $\mathbf{A} \in \mathbb{H}(\mathbf{B})$ may be an isomorphic copy of $\mathbf{A}$ itself. In particular, known structural completeness results have been confirmed for

$\mathbf{Ł_3^{\rightarrow}} = \langle \{0, \frac{1}{2}, 1\}, \rightarrow_{\text{L}} \rangle$	the 3-element Komori C-algebra
$\mathbf{B_1} = \langle \{0, \frac{1}{2}, 1\}, \min, \max, \neg_{\text{G}} \rangle$	the 3-element Stone algebra
$\mathbf{G_3} = \langle \{0, \frac{1}{2}, 1\}, \min, \max, \rightarrow_{\text{G}} \rangle$	the 3-element positive Gödel algebra
$\mathbf{S_3^{\rightarrow}} = \langle \{-1, 0, 1\}, \rightarrow_{\text{S}} \rangle$	the 3-element implicational Sugihara monoid

where $x \rightarrow_{\text{L}} y = \min(1, 1 - x + y)$, $x \rightarrow_{\text{G}} y$ is y if $x > y$, otherwise 1, $\neg_{\text{G}} x = x \rightarrow_{\text{G}} 0$, and $\rightarrow_{\text{S}}$ is the operation $\rightarrow$ of $\mathbf{S_3^{\rightarrow\neg}}$ from Example 7. A new structural completeness result has also been established for the psuedocomplemented distributive lattice $\mathbf{B_2}$ obtained by adding a top element to the 4-element Boolean algebra.

Example 10. There are even cases where we do not need any generators for the free algebra since the constants alone suffice. Consider the 5-valued Post algebra $\mathbf{P_5} = \langle \{0, 1, 2, 3, 4\}, \wedge, \vee, ', 0, 1 \rangle$ where $\langle \{0, 1, 2, 3, 4\}, \wedge, \vee \rangle$ builds a 5-chain with $0 < 4 < 3 < 2 < 1$ and $0' = 1, 1' = 2, 2' = 3, 3' = 4, 4' = 0$. This algebra builds the algebraic counterpart to the Post logic P_5. Running our procedure we recognize that $\mathbf{P_5}$ is isomorphic to $\mathbf{F_{P_5}}(0)$, i.e., $\mathbf{P_5}$ is also structurally complete.

Example 11. Consider the algebra $\mathbf{D_4} = \langle \{\bot, a, b, \top\}, \wedge, \vee, \neg, \bot, \top \rangle$, called the 4-element De Morgan algebra, consisting of a distributive bounded lattice with an involutive negation defined as shown below and also its constant-free case called the 4-element De Morgan lattice $\mathbf{D_4^L} = \langle \{\bot, a, b, \top\}, \wedge, \vee, \neg \rangle$.



Following our procedure, we obtain:

(i) $\mathbf{D_4} \notin \mathbb{H}(\mathbf{F_{D_4}}(1))$ and $\mathbf{D_4^L} \notin \mathbb{H}(\mathbf{F_{D_4^L}}(1))$, but $\mathbf{D_4} \in \mathbb{H}(\mathbf{F_{D_4}}(2))$ and $\mathbf{D_4^L} \in \mathbb{H}(\mathbf{F_{D_4^L}}(2))$.

(ii) $\mathbf{F_{D_4}}(2)$ has 168 elements, $\mathbf{F_{D_4^L}}(2)$ has 166 elements and the smallest subalgebras of the free algebras, for which $\mathbf{D_4}$ and $\mathbf{D_4^L}$ are homomorphic images, have 10 and 8 elements, respectively.

Example 12. Similar results were also obtained in [19] for Kleene algebras and lattices generated by the 3-element chains $\mathbf{C_3} = \langle \{\top, a, \bot\}, \wedge, \vee, \neg, \bot, \top \rangle$ and $\mathbf{C_3^L} = \langle \{\top, a, \bot\}, \wedge, \vee, \neg \rangle$ where $\neg$ swaps $\bot$ and $\top$ and leaves a fixed. In both cases the smallest subalgebra of the free algebra, for which $\mathbf{C_3}$ and $\mathbf{C_3^L}$ are homomorphic images, is a 4-element chain.

Example 13. Consider the 3-valued Łukasiewicz algebra $\mathbf{Ł_3} = \langle \{0, \frac{1}{2}, 1\}, \rightarrow, \neg \rangle$ with $x \rightarrow y = \min(1, 1 - x + y)$ and $\neg x = 1 - x$. Following our procedure:

(i) $\mathbf{Ł_3} \notin \mathbb{H}(\mathbf{F_{Ł_3}}(0))$, but $\mathbf{Ł_3} \in \mathbb{H}(\mathbf{F_{Ł_3}}(1))$.

$\mathbf{A}$	$\lvert\mathbf{A}\rvert$	Quasivariety $\mathbb{Q}(\mathbf{A})$	$\lvert$Free algebra$\rvert$	$\lvert$Output algebra$\rvert$
$\mathbf{Ł_3}$	3	algebras for $Ł_3$ (Ex. 13)	$\lvert\mathbf{F_A}(1)\rvert = 12$	6
$\mathbf{B_1}$	3	Stone algebras (Ex. 9)	$\lvert\mathbf{F_A}(1)\rvert = 6$	3
$\mathbf{C_3}$	3	Kleene algebras (Ex. 12)	$\lvert\mathbf{F_A}(1)\rvert = 6$	4
$\mathbf{Ł_3^{\rightarrow}}$	3	algebras for $Ł_3^{\rightarrow}$ (Ex. 9)	$\lvert\mathbf{F_A}(2)\rvert = 40$	3
$\mathbf{C_3^L}$	3	Kleene lattices (Ex. 12)	$\lvert\mathbf{F_A}(2)\rvert = 82$	4
$\mathbf{S_3^{\rightarrow\neg}}$	3	algebras for $\mathrm{RM}^{\rightarrow\neg}$ (Ex. 7)	$\lvert\mathbf{F_A}(2)\rvert = 264$	6
$\mathbf{S_3^{\rightarrow}}$	3	algebras for $\mathrm{RM}^{\rightarrow}$ (Ex. 9)	$\lvert\mathbf{F_A}(2)\rvert = 60$	3
$\mathbf{G_3}$	3	algebras for G_3 (Ex. 9)	$\lvert\mathbf{F_A}(2)\rvert = 18$	3
$\mathbf{D_4^L}$	4	De Morgan lattices (Ex. 11)	$\lvert\mathbf{F_A}(2)\rvert = 166$	8
$\mathbf{D_4}$	4	De Morgan algebras (Ex. 11)	$\lvert\mathbf{F_A}(2)\rvert = 168$	10
$\mathbf{P}$	4	$\mathbb{Q}(\mathbf{P})$ (Ex. 2)	$\lvert\mathbf{F_A}(2)\rvert = 6$	6
$\mathbf{S_4^{\rightarrow\neg e}}$	4	$\mathbb{Q}(\mathbf{S_4^{\rightarrow\neg e}})$ (Ex. 8)	$\lvert\mathbf{F_A}(1)\rvert = 18$	6
$\mathbf{B_2}$	5	$\mathbb{Q}(\mathbf{B_2})$ (Ex. 9)	$\lvert\mathbf{F_A}(1)\rvert = 7$	5
$\mathbf{P_5}$	5	algebras for P_5 (Ex. 10)	$\lvert\mathbf{F_A}(0)\rvert = 5$	5

Table 1. Algebras for checking admissibility

(ii) $\mathbf{F_{Ł_3}}(1)$ has 12 elements and the smallest subalgebras $\mathbf{B} \leq \mathbf{F_{Ł_3}}(1)$ with $\mathbf{Ł_3} \in \mathbb{H}(\mathbf{B})$ have 6 elements.

Note that our procedure does not necessarily find the smallest algebra $\mathbf{B}$ for checking admissibility in $\mathbf{A}$. I.e., there may be an algebra $\mathbf{C}$ with $\mathrm{C} \leq \mathrm{B}$ and

$$\Sigma \Rightarrow \varphi \approx \psi \text{ is admissible in } \mathbf{A} \quad \iff \quad \Sigma \models_{\mathbf{C}} \varphi \approx \psi.$$

Example 14. Following our procedure for the algebra $\mathbf{P}$ defined in Example 2:

(i) $\mathbf{P} \notin \mathbb{H}(\mathbf{F_P}(1))$, but $\mathbf{P} \in \mathbb{H}(\mathbf{F_P}(2))$.
(ii) $\mathbf{F_P}(2)$ has 6 elements and the smallest subalgebra $\mathbf{B} \leq \mathbf{F_P}(2)$ with $\mathbf{P} \in \mathbb{H}(\mathbf{B})$ is $\mathbf{F_P}(2)$ itself.

However, $\mathbf{P}$ can be embedded into $\mathbf{F_P}(1) \times \mathbf{F_P}(1)$; that is, $\mathbf{P}$ is structurally complete (see, e.g., [5]). But this means that a quasiequation $\Sigma \Rightarrow \varphi \approx \psi$ is admissible in $\mathbf{P}$ iff $\Sigma \Rightarrow \varphi \approx \psi$ is valid in $\mathbf{P}$.

This last issue, but also possibilities of improving the given procedure for checking admissibility (e.g., ruling out symmetric cases of generators when calculating a free algebra) will be the subject of future work. The given examples for our procedure are summarized in Table 1.

References

1. M. Baaz, C. G. Fermüller, and G. Salzer. Automated Deduction for Many-Valued Logics. In *Handbook of Automated Reasoning*, volume II, chapter 20, pages 1355–1402. Elsevier Science B.V., 2001.
2. S. Babenyshev, V. Rybakov, R. A. Schmidt, and D. Tishkovsky. A Tableau Method for Checking Rule Admissibility in S4. In *Proceedings of UNIF 2009*, volume 262 of *ENTCS*, pages 17–32, 2010.

3. W. J. Blok and D. Pigozzi. *Algebraizable Logics*. Number 396 in Memoirs of the American Mathematical Society volume 77. American Mathematical Society, 1989.
4. S. Burris and H. P. Sankappanavar. *A Course in Universal Algebra*, volume 78 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1981.
5. P. Cintula and G. Metcalfe. Structural Completeness in Fuzzy Logics. *Notre Dame Journal of Formal Logic*, 50(2):153–183, 2009.
6. P. Cintula and G. Metcalfe. Admissible Rules in the implication-negation fragment of intuitionistic logic. *Annals of Pure and Applied Logic*, 162(10):162–171, 2010.
7. G. Gentzen. Untersuchungen über das Logische Schliessen. *Math. Zeitschrift*, 39:176–210,405–431, 1935.
8. S. Ghilardi. Unification in Intuitionistic Logic. *Journal of Symbolic Logic*, 64(2):859–880, 1999.
9. S. Ghilardi. Best Solving Modal Equations. *Annals of Pure and Applied Logic*, 102(3):184–198, 2000.
10. S. Ghilardi. A resolution/tableaux algorithm for projective approximations in IPC. *Logic Journal of the IGPL*, 10(3):227–241, 2002.
11. A. J. Gil and G. Salzer. Homepage of MUltseq. `http://www.logic.at/multseq`.
12. R. Hähnle. *Automated Deduction in Multiple-Valued Logics*. OUP, 1993.
13. R. Iemhoff. On the Admissible Rules of Intuitionistic Propositional Logic. *Journal of Symbolic Logic*, 66(1):281–294, 2001.
14. R. Iemhoff and G. Metcalfe. Proof Theory for Admissible Rules. *Annals of Pure and Applied Logic*, 159(1–2):171–186, 2009.
15. E. Jeřábek. Admissible Rules of Modal Logics. *Journal of Logic and Computation*, 15:411–431, 2005.
16. E. Jeřábek. Admissible rules of Łukasiewicz logic. *Journal of Logic and Computation*, 20(2):425–447, 2010.
17. E. Jeřábek. Bases of admissible rules of Łukasiewicz logic. *Journal of Logic and Computation*, 20(6):1149–1163, 2010.
18. P. Lorenzen. *Einführung in die operative Logik und Mathematik*, volume 78 of *Grundlehren der mathematischen Wissenschaften*. Springer, 1955.
19. G. Metcalfe and C. Röthlisberger. Admissibility in De Morgan algebras. *Soft Computing*, to appear.
20. G. Metcalfe and C. Röthlisberger. Unifiability and Admissibility in Finite Algebras. In *Proceedings of CiE 2012*, volume 7318 of *LNCS*, pages 485–495. Springer, 2012.
21. V.V. Rybakov. *Admissibility of Logical Inference Rules*, volume 136 of *Studies in Logic and the Foundations of Mathematics*. Elsevier, Amsterdam, 1997.
22. G. Salzer. Homepage of MUltlog. `http://www.logic.at/multlog`.
23. M. Sprenger. Algebra Workbench. `http://www.algebraworkbench.net`.
24. R. Zach. Proof theory of finite-valued logics. Master's thesis, Technische Universität Wien, 1993.