Computing with Numbers, Cognition and the Epistemic Value of Iconicity

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Abstract. *In my paper I will rely on research by Grosholz (2007) considering, in particular, her thesis of the irreducibility of iconic representation in mathematics. Against this background, my aim is to discuss the epistemic value of iconicity in the case of different representations of numbers in elementary arithmetic. In order to make my point, I will bring in a case study selected from Leibniz's work with notations and the lessons Leibniz draws in the context of his number-theoretic considerations.*

Keywords: Iconic Representation, Notations, Visual Reasoning, Numbers, Epistemic Value

1 Introduction

According to the standard view in twentieth century philosophy of logic and mathematics, reasoning in the formal sciences is best characterized as purely syntactic so that 1) the body of mathematical knowledge can be seen as built up systematically and 2) a purely syntactic presentation guarantees formal rigor and transparency by making *explicit* all relevant epistemic contents in proving mathematical results. Full-explicitness goes hand in hand with the elimination of any reference to the context of work.[1] From such assumptions follows that figures and, more generally, iconic ingredients are to be eliminated from a fully systematic presentation.

This view of the role of representation in the formal sciences has been called into question by a number of recent investigations. According to a critical line of research developed by Grosholz (2007), the research mathematician engages in formal reasoning that is broadly conceived as sets of problem-solving activities which make use of an irreducible variety of modes of representation or working tools such as notations, tables, figures, etc. but also discursive reasoning in natural language. Such modes of representation depend on the specific context of work as well as acquired cognitive abilities of the agent which include background knowledge that remains largely *implicit*. As the critics point out "syntactic reconstructions" of formal reasoning may be complemented by formal semantics but the price to pay is the elimination of discursive language and forms of know-how relevant to the successful use of notations and other working tools, as well as rich dialogical aspects of mathematical practice leading to innovation. In particular, the syntactic view requires that figures be eliminated in favor of a one dimensional formal reconstruction which is purely symbolic.

[1] For a discussion of the standard view see [6], [5], and [4].

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2 Aim of My Paper

In my paper I will rely on research by Grosholz (2007) considering, in particular, her thesis of the irreducibility of iconic representation in mathematics. Against this background, my aim is to discuss the epistemic value of iconicity in the case of different representations of numbers in elementary arithmetic. In order to make my point, I will bring in a case study selected from Leibniz's work with notations and the lessons Leibniz draws in the context of his number-theoretic considerations. The reason for my selection of this particular case study is twofold. Firstly, throughout his work as a mathematician, Leibniz emphasizes the importance of visual reasoning while relying on a variety of tools which display rich iconic aspects in the implementation of problem-solving activities. Secondly, Leibniz discusses the peculiar iconicity of representations of numbers; in particular, he illuminates the issue of the epistemic value of different numerical systems by discussing the cognitive benefits of the binary system vis-à-vis the system of Arabic numerals. For Leibniz some notations are more fruitful than others, moreover, simplicity and economy is amongst the epistemic virtues of notational systems. In the case under consideration – my case study – Leibniz argues for the view that the iconic aspects present in binary notation reveal structural relations of natural numbers that are concealed in other numerical modes of representation such as the system of Arabic numerals.

3 The Idea of Iconic Representation

Let me start by focusing on the idea of iconic representation. Representations may be iconic, symbolic and indexical depending upon their role in reasoning with signs in specific contexts of work.[2] According to the received view representations are iconic when they resemble the things they represent. In the case of arithmetic this characterization appears as doubtful because of its appeal to a vague idea of similarity which would seem untenable when representations of numbers are involved. But Grosholz argues that in mathematics iconicity is often an irreducible ingredient, as she writes,

> In many cases, the iconic representation is indispensable. This is often, though not always, because shape is irreducible; in many important cases, the canonical representation of a mathematical entity is or involves a basic geometrical figure. At the same time, representations that are 'faithful to' the things they represent may often be quite symbolic, and the likenesses they manifest may not be inherently visual or spatial, though the representations are, and articulate likeness by visual or spatial means [6, p. 262].

In order to determine whether a representation is iconic or symbolic, the discursive context of research needs to be taken into account in each particular case, in other words, iconicity cannot simply be read off the representation in isolation of the context of use. We find here a more subtle understanding of "iconicity" than the traditional view. Let me focus on the idea that representations "articulate likeness by visual or spatial means" in the case of arithmetic. Grosholz suggests that even highly abstract symbolic reasoning goes hand in hand with certain forms of visualizations. To many this sounds polemical at best. Granted to the critic that "shape is irreducible" in

[2]This tripartite distinction goes back to Charles Peirce's theory of signs. For a discussion of the distinction, see [6, p. 25].

geometry as it is the case with geometrical figures. But what is the role of iconicity in the representation of numbers, and more generally, what is involved in visualizing in arithmetic?

Giardino (2010) offers a useful characterization of the cognitive activity of "visualizing" in the formal sciences. In visualizing, she explains, we are decoding articulated information which is embedded in a representation, such articulation is a specific kind of spatial organization that lends unicity to a representation turning it intelligible. In other words, spatial organization is not just a matter of physical display on the surface (paper or table) but "intelligible spatiality" which may require substantial background knowledge:

> (...) to give a visualization is to give a contribution to the organization of the available information (...) in visualizing, we are referring also to background knowledge with the aim of getting to a global and synoptic representation of the problem [3, p. 37].

According to this perspective, the ability to read off what is referred to in a representation depends on some background knowledge and expertise of the reader. Such cognitive act is successful only if the user is able to decode the encrypted information of a representation while establishing a meaningful relationship between the representation and the relevant background knowledge which often remains implicit. The starting poing of this process is brought about by representations that are iconic in a rudimentary way, namely, they have spatial isolation and organize information by spatial and visual means; and they are indivisible things. Borrowing Goodman's terms we may say that representations have 'graphic suggestiveness'.

4 The Role of Iconicity in the Representation of Numbers

Against this background, I will bring in my case study in order to consider the role that iconicity plays in the representation of natural numbers both by Arabic numerals $(0, 1, \ldots, 9)$ and by binary numerals $(0, 1)$. In a number of fragments, Leibniz discusses both notational systems. He compares them with regard to usefulness for computation and heuristic value leading to discovery of novelties. I begin by asking about the interest in choosing a particular representation of numbers in the context of arithmetical problem-solving activities. Once more, I will rely here on the Leibnizian view as discussed by Grosholz. According to this view the use of different modes of representation in the formal sciences allows us to explore "productive and explanatory conditions of intelligibility for the things we are thinking about" [7, p. 333].[3] The objects of study of mathematics are abstract ("intelligible"), but they are not transparent but problematic, and they are inexhaustible requiring mathematical analysis which will clarify and further develop conceptual structures aided by the appropriate working tools.[4] In the case of number-theoretic research different modes of representation may reveal different aspects of things leading to discovery of new properties and the design of more elaborated tools.

[3] The roots of this perpective are, as Grosholz recognizes, in Leinbiz's theory of expression developed around 1676-1684.

[4] See [6, p. 47 and p. 130].

5 Representation of Numbers

From this perspective, let us now consider the case of the representation of natural numbers. A natural number is, Grosholz writes, "either the unit or a multiplicity of units in one number" [6, p. 262]. Following this idea a very iconic representation of six could look like this:

$$//////$$

On the one hand, this "picture" contains – or shows – the multiplicity of units contained in the number six. On the other hand, the unity of the number six is expressed by some strategy of differentiation from the rest of objects surrounding it on the page (or surface). In this particular case, the six strokes are spatially organized to achieve this aim. But this works only with small numbers and if we want to represent, for instance, the number twenty-four, the iconicity of the strokes collapses partly because the reader cannot easily visualize the number that was meant to be thereby represented. In opposition to this rudimentary representation of numbers by means of strokes, take the Arabic numeral "6" which does not reveal the multiplicity of units contained in the natural number six but exhibits instead the unity of the number itself. Arabic numerals exhibit each number as a unity and just for this reason they are iconic too. If what is at stake are big numbers, representation by strokes becomes useless and the weaker iconicity of Arabic numerals appears as more productive in problem solving activities such as basic computing with numbers. In other words, iconicity is a matter of degree depending on each context of work as well as background knowledge relevant to the problem-solving activity. When it comes to arithmetical operations the "graphic suggestiveness" of strokes may not be the last word while the "maneuverability" of Arabic numerals seems more decisive as it allows us to operate with precision and easiness. In Arabic notation each individual mark is "dense" in the sense that in each character there is a lot of information codified in highly compressed way.

6 Leibniz and His Preference for the Binary System

In this section, I will look at Leibniz's discussion of the binary notation broadly exposed in "Explication de l'arithmetique binaire" (1703). I consider this case study of great interest because it brings to light specific aspects which are central to the topic of my paper, in particular, the epistemic value of iconicity in the representation of number systems in specific problem-solving contexts of work. Leibniz emphasizes what he sees as the cognitive virtues of his favorite notational system in arithmetic, the binary notation. This notation displays some properties of numbers by means of only two characters, namely, 0 and 1 and the following four rules: $1+1=10, 1+0=1$ (addition) and $1\cdot 1=1, 1\cdot 0=0$ (multiplication). In binary notation when we reach two, we must start again; thus, in this notation two is expressed by 10, four by 100, eight by 1000 and so on.

Time and again Leibniz points to the values of this binary notation, economy and simplicity of the system. All of arithmetic can be expressed by only two characters and a few rules for the manipulation of them. Having presented the law for the construction of the system, Leibniz explains the benefits of it comparing it with the more familiar decimal system:

> Mais au lieu de la progression de dix en dix, j'ai employé depuis plusieurs années la progression la plus simple de toutes, qui va de deux en deux; ayant trouvé qu'elle sert à la perfection de la science des Nombres [10, Vol. VII, p. 223].

For Leibniz, the economy and simplicity of his binary system seems to run contrary to the decimal system of Arabic numeration. Simplicity and easiness go hand in hand, as in every operation with binary notation the elements of the system are made fully explicit, while in Arabic notation we must always appeal to memory:

> Et toutes ces opérations sont si aisées (...) [o]n n'a point besoin non plus de rien apprendre par coeur ici, comme il faut faire dans le calcul ordinaire, ou il faut scavoir, par exemple, que 6 et 7 pris ensemble font 13; et que 5 multiplié par 3 donne 15 (...) Mais ici tout cela se trouve et se prouve de source... [10, Vol. VII, p. 225].

Consider the case of three multiplied by three. In order to solve this case by means of the decimal system, Leibniz argues, we need to appeal to memory; we must recall the multiplication table for 3 which gives us the correct outcome, and the same goes for any numeral of the Arabic system from 0 to 9. In contrast, the same operation made within the binary system always makes explicit all applications of the rules for any operation we perform. In this case we do not need to rely upon memory but only on the combination of characters fully deployed on the page. Thus, in decimal we know by memory that $3 \cdot 3 = 9$ while in binary notation we "see" it (Fig. 1), where "11" expresses the natural number three and "1001" stands for the number nine.

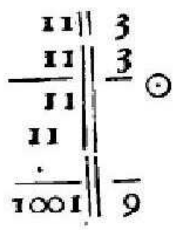

Fig. 1. Leibniz, *Mathematische Schriften*, VII, ed. Gerhardt, p. 225.

Moreover, Leibniz insists, the binary system reveals structural relations among characters, and in the same text "Explication de l'arithmetique binaire", Leibniz goes on to note that in virtue of the economy and simplicity of the binary system we are able to easily *visualize* structural relations between numbers thus uncovering novelties. The example he presents here to illustrate his point is that of geometric progression of ratio two:

> On voit ici d'un coup d'oeil la raison d'une *propriété célébre de la progression Géométrique double* en Nombres entiers. . . [10, Vol. VII, p. 224, italics included in quoted edition].

Let us take the following geometric progression "deux-en-deux" of natural numbers $7, 14, 28$. Next, we express those numbers as sums of powers of two. Accordingly, within the decimal system of Arabic numerals we then have the following progressions:

a)
$$4 + 2 + 1 = 7$$
b)
$$8 + 4 + 2 = 14$$
c)
$$16 + 8 + 4 = 28$$

Finally, we proceed to decompose a), b) and c) into powers of two:

a')
$$2^0 + 2^1 + 2^2 = 2^0 \cdot 7$$
b')
$$2^1 + 2^2 + 2^3 = 2^1 \cdot 7$$
c')
$$2^2 + 2^3 + 2^4 = 2^2 \cdot 7$$

In the first case, the three lines do not provide any information about the pattern that lead to a), b), c). Instead, those lines require familiarity with all elements of the system as well as familiarity with the operation in question (addition).

Similarly in the second case, the three lines do not provide any information about the pattern underlying the progression. In each of the three lines, the right side of a'), b'), c') does not indicate the outcome. We must find out how to express the outcome as power of two. In all three cases we find that the outcome cannot be expressed as power of two and that it is therefore necessary to introduce a new element: the factor 7. Of course, we must know the multiplication table for seven as well.

Let us now go back to the binary notation and consider how the spatial distribution of a), b), c) is expressed by binaries (see Fig. 2).

In opposition to the decimal system of Arabic numerals, within the binary system it is unnecessary to analyze the case in two separate steps. This is so because the characters and the order they exhibit on the page make visible the pattern underlying the progression. We only need to know the rules for addition and the system characters ("0" and "1").

Finally, Leibniz points to another feature of binaries in relation with the construction of the system. It is the simplicity and economy of the binary that according to him brings forth a remarkable periodicity and order. In making this point the author again emphasizes visual aspects and spatial configuration of characters:

> (. . .) les nombres étant réduits aux plus simples principes, comme 0 & 1, il paroît partout un ordre merveilleux. Par exemple, dans la *Table* même *des Nombres*, on voit en chaque colonne régner des périodes qui recommencent toujours [10, Vol. VII, p. 226, italics included in quoted edition].

100	1000	10000
10	100	1000
1	10	100
——	——	——
111	1110	11100

Fig. 2. Geometric progression of ratio two as expressed by binaries.

Leibniz groups together numbers that fall under $2^1, 2^2, 2^3$, etc. I include below a segment of a larger table used by Leibniz to show three groups of numbers (surrounded by vertical and horizontals lines), namely, 0, 1; 2, 3 and 4 – 7. Each group is a cycle which is iterated in the next cycle, and so on ad infinitum as we can easily see.[5]

7 Conclusion

For Leibniz mathematical research starts with the search for suitable signs ("characters") and the design of good notations (or "characteristics") by means of which structural relations of intelligible objects of study could be explored; a good "characteristic" should allow us to uncover different aspects of things by means of a sort of reasoning with signs. When Leibniz remarks this in a brief text of 1683-1684 his example of a "more perfect characteristic" is the binary notation vis-à-vis the decimal system.[6] In order to understand Leibniz preference for the binary system, we recall here, it is useful to focus on the importance of visual reasoning in problem-solving contexts of work. According to Leibniz, problem-solving activities and the discovery of new properties is the goal of mathematical analysis in the case of number theory, one of the areas of research he was most interested in pursuing. Such goal strongly motivates the design of notational systems with a view to obtain fruitful results. As already pointed out, for Leibniz the binary system is characterized by its simplicity and economy so that in each operation every element ("0", "1") is displayed for the eye to see without any need to rely on memory as in the case of operating with Arabic numerals. Leibniz observed that in certain contexts of work the spatial distribution of such binary elements reveals patterns which are relevant to the resolution of the prob-

[5] [10, Vol. VII, p. 224].

[6] Exempli gratia perfectior est característica numerorum bimalis quam decimales vel alia quaecunque, quia in bimali – ex characteribus – omnia demonstrari possunt quae de numeris asseruntur, in decimali vero non item. [2, p. 284].

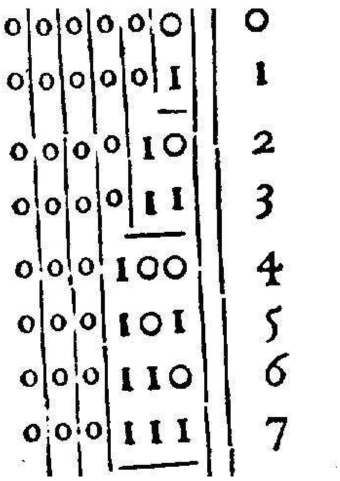

Fig. 3. Leibniz, *Mathematische Schriften*, VII, ed. Gerhardt, p. 225.

lem under consideration, or to the discovery of new properties that would otherwise remain hidden. Such is the case of the geometrical progression "deux-en-deux", which as we have just noted, can be easily seen only when expressed by means of the binary system. No doubt that for practical considerations of everyday life the decimal system of Arabic numerals may be easier to calculate with, nonetheless Leibniz was fascinated by the binary as facilitating algorithmic structures which like the calculus engaged the issue of infinite series.

To conclude, one of the things we can learn from my case study is that in computing with numbers – a way of "reasoning with signs" – we always require systems of signs or characters but some of them are more fruitful than others, some are easier to calculate with but beyond the specific epistemic virtues they may have, all of them include important iconic features that are most relevant to cognition. This conclusion, in particular, calls into question the old idea that working with algorithmic structures – computing with numbers – is a purely mechanical affaire which excludes iconicity.

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References

1. Bender, A.: Fingers as a Tool for Counting – Naturally Fixed or Culturally Flexible? *Front. Psychology* 2:256. doi: 10.3389/fpsyg.2011.00256 (2011)
2. Couturat, L.: *Leibniz, Opuscules et Fragments Inédits.* Paris, Alcan (1903). Reprinted by Olms, Hildesheim (1961)
3. Giardino, V.: Intuition and Visualization in Mathematical Problem Solving. Published online: 9 February 2010 – Springer Science+Business Media B. V. Topoi, 29:29–39 (2010)

4. Goethe, N. B., Friend, M.: Confronting Ideals of Proof with the Ways of Proving of the Research Mathematician. Studia Logica 96 (2), Dordrecht: Springer, p. 273–288 (2010)
5. Goethe, N. B.: Revisiting the Question about Proof: Philosophical Theory, History, and Mathematical Practice. Manuscrito, Vol. 31 (1), CLE, Campinas S.P. p. 361–386 (2008)
6. Grosholz, E.: *Representation and productive ambiguity in mathematics and the sciences.* Oxford New York (2007)
7. Grosholz, E.: Reference and Analysis: The Representation of Time in Galileo, Newton, and Leibniz. *Arthur O. Lovejoy Lecture* , Journal of the History of Ideas, Volume 72, Number 3 (July 2011) pp. 333–350 (2010)
8. Ippoliti, E.: *Inferenze Ampliative. Visualizzazione, Analogia e Rappresentazioni Multiple.* Morrisville, North Carolina, USA (2008)
9. Lakoff, G., Núñez.: *Where Mathematics Comes From. How the Embodied Mind Brings Mathematics into Being.* Basic Books, The Perseus Books Group (2000)
10. Leibniz GW.: E.: *Mathematische Schriften.* Vol. I – VII (1849–1863). Reprinted by Olms, Hildesheim (1962)