

Looking for analogical proportions in a formal concept analysis setting

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Abstract. Categorization and analogical reasoning are two important cognitive processes, for which there exist formal counterparts (at least they may be regarded as such): namely, formal concept analysis on the one hand, and analogical proportions (modeled in propositional logic) on the other hand. This is a first attempt aiming at relating these two settings. The paper presents an algorithm that takes advantage of the lattice structure of the set of formal concepts for searching for analogical proportions that may hold in a formal context. Moreover, properties linking analogical proportions and formal concepts are laid bare.

1 Introduction

Categorization and analogical reasoning play important roles in cognitive processes. They both heavily rely on the ideas of similarity and dissimilarity. Items belonging to the same category should be similar, while they are dissimilar with respect to items belonging to other categories. Analogical proportions, which are statements of the form ‘ a is to b as c is to d ’, express the similarity of the relations linking a and b with the relations linking c and d (note that however a and b may be somewhat dissimilar (as well as c and d). In a Boolean setting, where items are described in terms of binary attributes, similarity amounts to the identity of properties, while dissimilarity refers to the presence of properties for an item which are absent in the other considered item.

Among formal approaches aiming at categorizing items, Formal Concept Analysis (FCA) provides a way for characterizing concepts both extensionally in terms of the objects that they cover and intensionally in terms of the properties that these objects share. FCA is known as a lattice-theoretic framework devised for knowledge extraction from Boolean data tables called *formal contexts* that relate objects and properties. Introduced under this name by Wille [13], FCA has been developed by Ganter and Wille [7] and their followers for thirty years.

Besides, there has been a renewal of interest for analogical proportions in the last decade, firstly in relation with computational linguistic concerns. Set-based, algebraic and logical models have been proposed [8, 12, 1, 9]. In the following, we more particularly use the Boolean view [9] of analogical proportions that is directly relevant for application to formal contexts. Then, it makes sense to look for analogical proportions in Boolean contexts, and to try to understand what formal concepts and analogical proportions may have in common.

The paper is organized as follows. We first provide a short background on analogical proportions in Section 2. Then in Section 3, after a brief reminder of basic definitions in FCA, we present an efficient algorithm able to discover analogical proportions in a formal context by using the lattice of formal concepts. In Section 4, we further investigate the theoretical relations between FCA and analogical proportions, by showing how formal concepts are involved in analogical proportions, before indicating lines for further research and concluding.

2 Analogical proportions

An analogical proportion ‘ a is to b as c is to d ’, usually denoted $a : b :: c : d$, expresses that the way a and b differ is the same as the way c and d differ [9]. This leads to the following definitions, here stated for three closely related kinds of items: subsets of a finite set, Boolean truth values, and objects defined by Boolean properties (also called binary *attributes*).

Analogical proportion between sets First, let us consider four sets A, B, C and D , all subsets of some set X . The dissimilarity between A and B is evaluated by $A \cap \overline{B}$ and by $\overline{A} \cap B$, where \overline{A} denotes the complement of A in X , while the similarity corresponds to $A \cap B$ and $\overline{A} \cap \overline{B}$. Viewing an analogical proportion as expressing that the differences between A and B and between C and D are the same, we get the following definition [9]:

Definition 1 *Four subsets A, B, C and D of a finite set X are in analogical proportion in this order when $A \cap \overline{B} = C \cap \overline{D}$ and $\overline{A} \cap B = \overline{C} \cap D$.*

Analogical proportion between Boolean objects This expression has an immediate logical counterpart when a, b, c , and d now denote Boolean variables:

$$((a \wedge \neg b) \equiv (c \wedge \neg d)) \wedge ((\neg a \wedge b) \equiv (\neg c \wedge d))$$

This formula is true for the 6 truth value assignments of a, b, c, d appearing in Table 1, and is false for the $2^4 - 6 = 10$ remaining possible assignments.

It can be checked that the above definitions of an analogical proportion satisfies the following characteristic postulates [8]:

- $a : b :: a : b$ (identity)
- $a : b :: c : d \implies c : d :: a : b$ (global symmetry)
- $a : b :: c : d \implies a : c :: b : d$ (central permutation)
- $a : b :: c : d$ and $\neg(b : a :: c : d)$ are consistent (local dissymmetry)

a	×			×		×
b	×		×			×
c	×	×		×	×	
d	×		×		×	×

Table 1. The six Boolean 4-tuples that are in analogical proportion. The Boolean truth-values True and False are written as a cross and a blank.

Objects defined by Boolean properties Let us suppose that the objects (or *items*) a, b, c, d are described by sets of binary properties belonging to a set $Prop$. Then, each item can be viewed as a subset of $Prop$, made of the attributes that hold true on this item³. Then, we can apply Definition 1, namely $a \cap \bar{b} = c \cap \bar{d}$ and $\bar{a} \cap b = \bar{c} \cap d$. Another way of seeing this analogical proportion is given by the equivalent definition:

Definition 2 *Four objects (a, b, c, d) defined by binary properties are in analogical proportion iff the truth-values of each property on these objects make a 4-tuple of binary values corresponding to one of the six 4-tuples displayed in Table 1.*

Analogical dissimilarity We now introduce the concept of *analogical dissimilarity* (AD) between four objects defined by binary attributes. It is simply the sum on all the attributes of the analogical dissimilarity per attribute. The latter is defined according to the following table:

a						×	×	×	×	×	×	×	×
b			×	×	×	×					×	×	×
c		×	×			×	×		×	×		×	×
d	×	×		×	×		×	×		×	×		×
$AD =$	0	1	1	0	1	0	2	1	1	2	0	1	0

AD per attribute is merely the minimal number of bit(s) that has/have to be flipped in order to turn the four bits into an analogical proportion, according to Table 1. Notice that any 4-tuple with a zero AD is an analogical proportion, and vice-versa. For example, the four first objects of the formal context that we will later call \mathcal{BASE}_{lm} (see Figure 2) are such that $AD(\text{leech, bream, frog, dog}) = 0 + 1 + 0 + 0 + 0 + 0 + 0 + 1 + 1 = 3$.

3 Searching for analogical proportions in formal concepts

This section is devoted to the following problem: given a formal context with n objects and d properties (or attributes), is it possible to discover 4-tuples of objects in analogical proportion, without running an $\mathcal{O}(n^4 \cdot d)$ algorithm? We give in this section an heuristic algorithm which uses the lattice of formal concepts, and has shown experimentally its efficiency for discovering analogical proportions. We start with a brief reminder on FCA.

³For an object x , this subset is called $R^{\uparrow}(x)$ in Formal Concept Analysis, see Sect. 3.1.

3.1 Formal concept analysis (FCA)

FCA starts with a binary relation R , called *formal context*, defined between a set Obj of objects and a set $Prop$ of Boolean properties. The notation $(x, y) \in R$ means that object x has property y . $R^\uparrow(x) = \{y \in Prop | (x, y) \in R\}$ is the set of properties of object x . Similarly, $R^\downarrow(y) = \{x \in Obj | (x, y) \in R\}$ is the set of objects having property y .

Given a set Y of properties, one can define the set of objects [7]: $R^\downarrow(Y) = \{x \in Obj | R^\uparrow(x) \supseteq Y\}$.

This is the set of objects sharing all properties in Y (and having maybe some others). Then a *formal concept* is defined as a pair made of its *extension* X and its *intension* Y such that

$$R^\downarrow(Y) = X \text{ and } R^\uparrow(X) = Y,$$

where $(X, Y) \subseteq Obj \times Prop$, and $R^\uparrow(X)$ is similarly defined as $\{y \in Prop | R^\downarrow(y) \supseteq X\}$. It can be also shown that formal concepts are maximal pairs (X, Y) (in the sense of inclusion) such that $X \times Y \subseteq R$.

Moreover, the set of all formal concepts is equipped with a partial order (denoted \preceq) defined as: $(X_1, Y_1) \preceq (X_2, Y_2)$ iff $X_1 \subseteq X_2$ (or, equivalently, $Y_2 \subseteq Y_1$), and forms a complete lattice, called the concept lattice of R .

Let us consider an example where R is a relation that defines links between eight objects $Obj = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and nine properties $Prop = \{a, b, c, d, e, f, g, h, i\}$. There is a “ \times ” in the cell corresponding to an object x and to a property y if the object x has property y , in other words the “ \times ”s describe the relation R (or context). An empty cell corresponds to the fact that $(x, y) \notin R$, i.e., it is known that object x has not property y . The relation R in the example is given in Figure 1. There are 5 formal concepts. For instance, consider $X = \{a, b, c, d, e\}$. Then $R^{\uparrow\Delta}(X) = \{7, 8\}$; likewise if $Y = \{7, 8\}$. Then $R^{\downarrow\Delta}(Y) = \{a, b, c, d, e\}$.

	a	b	c	d	e	f	g	h	i
1							\times		
2							\times		
3							\times	\times	\times
4							\times	\times	\times
5			\times	\times	\times	\times			
6			\times	\times	\times	\times			
7	\times	\times	\times	\times	\times				
8	\times	\times	\times	\times	\times				

Fig. 1. R_2 : a relation with 5 formal concepts (and 2 sub-contexts)

3.2 Organizing the search

The basic idea of our algorithm is to start from some 4-tuple of objects, to observe the attributes that contribute to make AD non-zero for this 4-tuple, and to replace one of the four objects by another object. Then we iterate the process. Two important features have been added to avoid a random walk in the space of the 4-tuples.

1. The replacement of one object by another is done according to the observation of the lattice of concepts. The idea is to try to decrease the value of AD . This point will be explained in the next section.
2. All 4-tuples of objects that are created are stored in a list, ordered by increasing value of AD . The next 4-tuple to be chosen is the first in the list.

This algorithm can therefore be seen as an optimization procedure, more precisely as a *best-first* version of the *GRAPHSEARCH* algorithm ([11]). The ordered list of 4-tuples is an *Open* list in this interpretation.

3.3 Decreasing the analogical dissimilarity

Let us come now to the heart of the algorithm, namely the replacement of one object by another. Can we find some information in the lattice that leads us to choose both an object in the 4-tuple and another object to replace it? Remember that we are looking for a replacement that makes the AD decrease.

Now, let us consider the following situation, taken from $BAS\mathcal{E}_{lm}$ (see Figure 2). Suppose that we are studying the 4-tuple of objects $(3, 4, 9, 12)$, with $AD = 1$. Attribute c is the only one to contribute in the AD of this 4-tuple. Actually, c has the values $(1, 1, 0, 1)$ on the 4-tuple $(3, 4, 9, 12)$. We notice now that there are two interesting concepts in the lattice with respect to c , the first one being $(\{b, c, h, g, a\}, \{3\})$ and the second being $(\{b, h, g, a\}, \{2, 3\})$, which are directly connected. What can we deduce from this pair of connected concepts?

- Attribute c has value 1 on object 3.
- Attribute c has value 0 on object 2.
- Attribute c is the only attribute to switch from 1 to 0 to transform concept $(\{b, c, h, g, a\}, \{3\})$ into concept $(\{b, h, g, a\}, \{2, 3\})$.

We can conclude from this evidence that replacing object 3 by object 2 will decrease by 1 the value of AD , since c will take values $(0, 1, 0, 1)$ on the 4-tuple $(2, 4, 9, 12)$, and therefore that $(2, 4, 9, 12)$ is an analogical proportion.

Unfortunately this argument does not lead to a greedy algorithm, since there no insurance, given a 4-tuple, that there exists a couple of concepts having the three above properties. Most of the time, actually, there is more than one attribute switching to 1 between two connected concepts, and only one insuring the decreasing of AD .

Let us take another example from the same data base. The 4-tuple $(5, 4, 9, 12)$ has a AD of 4, because of the attributes d, f, g and h . Two interesting connected

concepts are $(\{c, f, d, e, i, a\}, \{12\})$ and $(\{c, d, e, a\}, \{7, 12\})$, since f switching from 1 to 0 will decrease the AD (see the table below). But replacing 12 by 7 in the 4-tuple $(5, 4, 9, 12)$ will switch not only f but also the attribute i , and we don't know what will happen when switching i : it may decrease as well as increase the AD . Actually, in this case, it increases the AD . Hence, the 4-tuple $(5, 4, 9, 7)$ has the same AD of 4.

	a	b	c	d	e	f	g	h	i
5	×	×		×		×			
4	×		×					×	×
9	×	×		×	×	×			
12	×		×	×	×				×

$AD = 4$

	a	b	c	d	e	f	g	h	i
5	×	×		×		×			
4	×		×					×	×
9	×	×		×	×	×			
7	×		×	×	×				

$AD = 4$

By generalizing these examples, we propose now an heuristic to try to decrease the AD that we call **h-Doap**.

Heuristic h-Doap. Let a couple of connected concepts be $(A \cup B, Z)$ and $(B, Z \cup Y)$, A and B being subsets of attributes and Y and Z being subsets of objects such that $A \cap B = \emptyset$ and $Y \cap Z = \emptyset$. If there is a 4-tuple with one of its four objects in Z and if there is an attribute in B that decreases the AD of this 4-tuple when switching from 1 to 0, then create a new 4-tuple by replacing the object in Z by an object in Y .

Next section shows how this heuristic can be used to discover 4-tuples of objects with null AD in a formal context, i.e. analogical proportions of objects.

3.4 Algorithm

Discovering one analogical proportion. We explain in this section the algorithm used to discover one analogical proportion in a formal context. We call it 'Discover One Analogical Proportion', in short **Doap**. As already stated, it is a simple version of *Graphsearch*, where the nodes to be explored are 4-tuples of objects. We denote *Start* the 4-tuple of objects chosen to begin, *Open* the current set of 4-tuples to be processed and *Closed* the set of 4-tuples already processed.

The choice of *Start* is either done randomly or by selecting objects which appear in small subsets of objects in the lattice. We also require that the explored 4-tuples are composed of four different objects, since we do not want to converge towards 4-tuples trivially in proportion, such as $(1, 3, 1, 3)$ or even $(2, 2, 2, 2)$.

The algorithm stops either by discovering an analogical proportion, or in failure. One has to notice that its failure does not insure that there is no analogical proportion, since there is no guarantee given by the heuristic. We have never met this failure case, but our experiments are very limited, as explained in section 3.5.

```

1: Algorithm Doap(Start)
2: begin
3: Closed  $\leftarrow \emptyset$ 
4: Open  $\leftarrow \{Start\}$ 
5: while Open  $\neq \emptyset$  do
6:   x  $\leftarrow$  the 4-tuple in Open having the lowest AD value
7:   if AD(x) = 0 then
8:     return x
9:   else
10:    Open  $\leftarrow$  Open  $\setminus \{x\}$  ; Closed  $\leftarrow$  Closed  $\cup \{x\}$ 
11:    decision  $\leftarrow$  1
12:    while decision = 1 do
13:      Use heuristic h-Doap to construct a new 4-tuple y from x
14:      if y is composed of four different objects and y  $\notin$  Closed and y  $\notin$ 
        Open then
15:        Open  $\leftarrow$  Open  $\cup \{y\}$  ; decision  $\leftarrow$  0
16:      end if
17:    end while
18:  end if
19: end while
20: return failure
21: end

```

Discovering several analogical proportions. To discover more analogical proportions, the simplest manner is to imbed algorithm **Doap** in a procedure that discards the first two objects of a discovered analogical 4-tuple from the formal context before re-running the algorithm. Since the transitivity holds for analogical proportions on objects ($u : v :: w : x$ and $w : x :: y : z$ implies $u : v :: y : z$), we are losing no information on analogical 4-tuples. However, we are not insured to find all proportions in that manner, due to the fact that algorithm **Doap** may not find an existing proportion.

3.5 Experiments

The size of *Close* when the algorithm **Doap** stops is a precise indication of its practical time complexity. Notice that a random algorithm, running on n objects (without any construction of a formal lattice), in which there are q 4-tuples in analogical proportion would in average try $((n^4)/8 \cdot q)$ 4-tuples before discovering a proportion. In the previous formula, the number “8” comes from the fact that, when there is one analogical proportion in a formal context, then there are in fact exactly 8 through suitable permutations. This property stems directly from the postulates an analogical proportion (see section 2).

We have used two different formal contexts to run the algorithm **Doap**. The first one is described in [2], except that we have added four objects 9, 10, 11 et 12 in order to have (at least) the analogical proportions (3, 4, 9, 10) and (1, 8, 11, 12). This leads to the formal context called \mathcal{BASE}_{lm} , Figure 2.

		<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>i</i>
leech	1	×	×					×		
bream	2	×	×					×	×	
frog	3	×	×	×				×	×	
dog	4	×		×				×	×	×
spike-weed	5	×	×		×	×				
reed	6	×	×	×	×		×			

		<i>a</i>	<i>b</i>	<i>c</i>	<i>d</i>	<i>e</i>	<i>f</i>	<i>g</i>	<i>h</i>	<i>i</i>
bean	7	×		×	×	×				
maize	8	×		×	×	×		×		
x	9	×	×		×	×	×	×		
y	10	×			×	×	×			×
z	11	×	×			×		×		×
t	12	×		×	×	×	×			×

Fig. 2. \mathcal{BASE}_{lm} : A formal context from Bělohlávek [2] increased with four objects 9, 10, 11 et 12 in order to have (at least) the analogical proportions (3, 4, 9, 10) and (1, 8, 11, 12).

The lattice constructed on this formal context (with the In-Close free software [14]) has 31 concepts. We have run `Doap` more than 600 times. It has always terminated by finding one of the three analogical proportions in the data⁴. The average size of the *Closed* list is 63 and its median size is 28. Figure 3 gives the details.

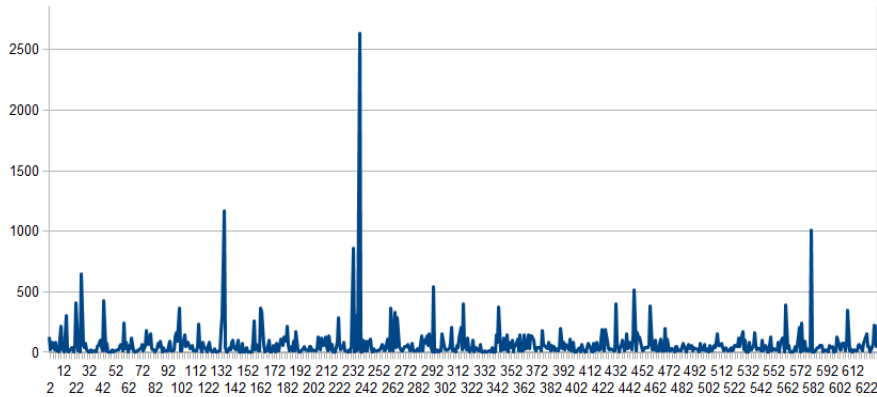


Fig. 3. Results of 622 runs of `Doap` on \mathcal{BASE}_{lm} . The size of the *Close* list for each run is on the *Y* axis.

To appreciate these results, we have compared with a random search, replacing line 13 of the `Doap` algorithm by picking a random 4-tuple. The detailed results are given in Figure 4. The average size of the *Closed* list is 253 and its median size is 174.

We also have tried to “symmetrize” the role of 0 and 1 in this formal context by adding the reverse attributes (indeed the Table 1 defining analogical proportions is left unchanged when exchanging 0 and 1). It leads to 12 objects and 18

⁴ We actually had a good surprise: `Doap` found a third proportion, namely (2, 4, 9, 12).

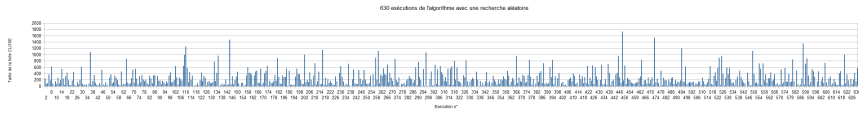


Fig. 4. Results of 630 runs of random Doap on $BASE_{lm}$. The Y axis is graduated from 0 up to 2000.

attributes instead of 9. The size of the lattice of concepts is now 94. The algorithm Doap with the same parameters examines in average 93 4-tuples before finding an analogical proportion. The median value is 31. The symmetrization does not seem to be a good idea in this case. The random Doap algorithm has failed to give complete results on these data, due to overflows in the *Close* list.

The second experiment has been run on the Lenses data base, from UCI ML Repository [6]. The nominal attributes have been transformed into binary ones by simply creating as many binary attributes as the number of modalities. The number of objects is 24 with 7 binary parameters. The size of the lattice is 43, the average number of examined 4-tuples is 77 and the median number is 20. When adding the reverse attributes, we have a lattice of size 227, an average number of 39 and a median number of 16. In that experiment, the symmetrization of the data seems clearly to have a positive effect.

A first conclusion is that our heuristic algorithm seems to perform well. In the second context the basic search space has a size over 40.000 and we examine only 77 4-tuples in average. The construction of the lattice of concepts takes in practice much more time than the discovery of analogical proportions, which seems to suggest that it is a relevant space for looking for analogical proportions.

4 Analogical proportions between formal concepts

We have seen that discovering analogical proportions in a formal context benefits from the knowledge of the associated lattice of formal concepts. Then it raises the question of understanding how formal concepts are involved in analogical proportions. Clearly, four objects in the same formal concept form an analogical proportion – in a trivial way – w.r.t. the subset of attributes involved in the formal concept. Partial answers to the question, when two formal concepts are involved in the proportion, are given in this section.

4.1 The smallest formal context in complete proportion

We are interested in this section in examining the properties of the smallest context with an analogical proportion between objects. Obviously, this context will have exactly four objects. If we want to have, only one time, each of the possible analogical proportions between attributes, we need six of them (see table 1) and we obtain $BASE_0$ (see Figure 5).

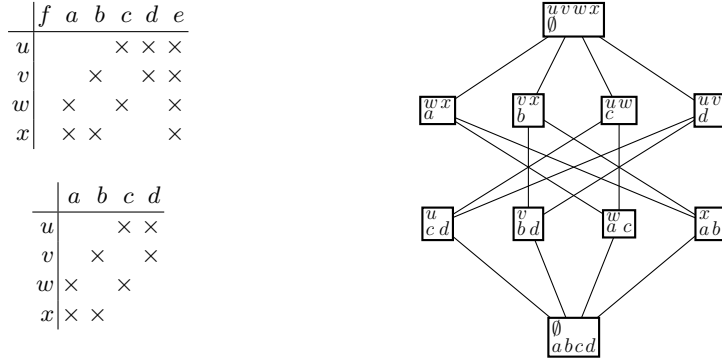


Fig. 5. \mathcal{BASE}_0 , \mathcal{BASE}_1 and the concepts lattice of \mathcal{BASE}_1 . The lattice of \mathcal{BASE}_0 is deduced from it by adding e to all subsets of attributes.

We can construct now the concept lattice of \mathcal{BASE}_0 , but it is interesting to get rid of attributes f (which will not be present in any context) and e (present in every context). We call \mathcal{BASE}_1 the reduced context, shown at Figure 5.

Its lattice is displayed in Figure 5. Note that there is a perfect symmetry between attributes and objects. The third line of the lattice expresses that $u : v :: w : x$, but also in subsets terms that $\{c, d\} : \{b, d\} :: \{a, c\} : \{a, b\}$. The second line expresses that $a : b :: c : d$ and that $\{w, x\} : \{v, x\} :: \{u, w\} : \{u, v\}$. This is not surprising: as explained in section 2, we can see an object as the set of properties that hold true for it.

4.2 Some relations between analogical proportions and lattices of concepts

Firstly, let us remark that the two following propositions are equivalent. This is immediate from section 2, in which these two equivalent definitions of analogical proportion have been presented.

1. x_1, x_2, x_3 and x_4 are four objects, in analogical proportion in this order.
2. $R^\dagger(x_1), R^\dagger(x_2), R^\dagger(x_3)$ and $R^\dagger(x_4)$ are four subsets of properties in analogical proportion in this order.

Property 1 *Let x_1, x_2, x_3 and x_4 be four objects in analogical proportion in this order. Let (X_1, Y_1) be the⁵ concept with the smallest set X_1 of objects in which x_1 is present. Let us define $(X_2, Y_2), (X_3, Y_3)$ and (X_4, Y_4) in the same way. Then the four sets of attributes Y_1, Y_2, Y_3 and Y_4 are in analogical proportion, in this order.*

Proof. Since $x_1 \in X_1$, all the attributes in Y_1 take value 1 on x_1 . Since X_1 is the smallest set of objects including x_1 , there is no attribute outside Y_1 having

⁵ If they were two, x_1 would be present in the intersection of the two.

value 1 on x_1 . Hence, Y_1 is exactly $R^\uparrow(x_1)$, the extension of x_1 , i.e. the subset of attributes that take value 1 on x_1 . This is also true for x_2, x_3 and x_4 .

We have to prove now that $x_1 : x_2 :: x_3 : x_4$ implies $R^\uparrow(x_1) : R^\uparrow(x_2) :: R^\uparrow(x_3) : R^\uparrow(x_4)$. It is immediate from the remark above. \square

For example, in \mathcal{BASE}_0 , we know that $1 : 8 :: 11 : 12$. We have $X_1 = \{1, 2, 3, 11\}$, $Y_1 = \{a, b, g\}$, $X_2 = \{6, 8, 12\}$, $Y_2 = \{a, c, d, f\}$, $X_3 = \{11\}$, $Y_3 = \{a, b, e, g, i\}$ and $X_4 = \{12\}$, $Y_4 = \{a, c, d, e, f, i\}$. The proportion $Y_1:Y_2::Y_3:Y_4$ holds, since: $\{a, b, g\}:\{a, c, d, f\}::\{a, b, e, g, i\}:\{a, c, d, e, f, i\}$.

Property 2 Let (X_1, Y_1) , (X_2, Y_2) , (X_3, Y_3) and (X_4, Y_4) be four concepts of a lattice of concepts, such that the four sets of attributes Y_1, Y_2, Y_3 and Y_4 are in analogical proportion, in this order. Let \widehat{X}_1 be the subset of X_1 composed of all objects that are in X_1 but cannot be found in any subset of X_1 belonging to a concept. We define in the same manner $\widehat{X}_2, \widehat{X}_3$ and \widehat{X}_4 . The following property holds true: $\forall x_1 \in \widehat{X}_1, x_2 \in \widehat{X}_2, x_3 \in \widehat{X}_3, x_4 \in \widehat{X}_4 : x_1 x_2 :: x_3 : x_4$.

Proof. It is the reciprocal of Property 1: Y_1 is the extension of all objects in \widehat{X}_1 , and we take x_1 in \widehat{X}_1 . We derive the conclusion from the remark above. \square

Property 3 Let x_1, x_2, x_3 and x_4 be four objects, in analogical proportion in this order.

Let $A_{1111} = \{y | y \in R^\uparrow(x_1), y \in R^\uparrow(x_2), y \in R^\uparrow(x_3), y \in R^\uparrow(x_4)\}$

Let $A_{1100} = \{y | y \in R^\uparrow(x_1), y \in R^\uparrow(x_2), y \notin R^\uparrow(x_3), y \notin R^\uparrow(x_4)\}$

Let $A_{0011} = \{y | y \notin R^\uparrow(x_1), y \notin R^\uparrow(x_2), y \in R^\uparrow(x_3), y \in R^\uparrow(x_4)\}$

Let $A_{1010} = \{y | y \in R^\uparrow(x_1), y \notin R^\uparrow(x_2), y \in R^\uparrow(x_3), y \notin R^\uparrow(x_4)\}$

Let $A_{0101} = \{y | y \notin R^\uparrow(x_1), y \in R^\uparrow(x_2), y \notin R^\uparrow(x_3), y \in R^\uparrow(x_4)\}$

Then

$(\{x_1, x_2\}, A_{1111} \cup A_{1100})$ is included into a formal concept.

$(\{x_3, x_4\}, A_{1111} \cup A_{0011})$ is included into a formal concept.

$(\{x_1, x_3\}, A_{1111} \cup A_{1010})$ is included into a formal concept.

$(\{x_2, x_4\}, A_{1111} \cup A_{0101})$ is included into a formal concept.

The result follows from the definition of the subsets of attributes considered and their clear relation with the definition of analogical proportions. The fact that we only have an inclusion in the above property should not come as a surprise. Indeed, when describing objects, attributes that are nor not relevant w.r.t. the analogical proportion may be present.

5 Lines for further research and concluding remarks

Beyond the already introduced set function, $R^\downarrow(Y) = \{x \in Obj | R^\uparrow(x) \supseteq Y\}$, which is at the core of FCA, and which leads to the definition of formal concepts, it has been noticed [5], on the basis of a parallel with possibility theory that, given a set Y of properties, four remarkable sets of objects can be defined in this setting (here the overbar denotes set complementation):

- $R^{\downarrow H}(Y) = \{x \in Obj | R^{\uparrow}(x) \cap Y \neq \emptyset\} = \cup_{y \in Y} R^{\downarrow}(y)$. This is the set of objects having at least one property in Y .
- $R^{\downarrow N}(Y) = \{x \in Obj | R^{\uparrow}(x) \subseteq Y\} = \cap_{y \notin Y} \overline{R^{\downarrow}(y)}$. This is the set of objects having no property outside Y .
- $R^{\downarrow \Delta}(Y) = R^{\downarrow}(Y) = \cap_{y \in Y} R^{\downarrow}(y)$. This is the set of objects sharing all properties in Y .
- $R^{\downarrow \nabla}(Y) = \{x \in Obj | R^{\uparrow}(x) \cup Y \neq Obj\} = \cup_{y \notin Y} \overline{R^{\downarrow}(y)}$. This is the set of objects that are missing at least one property outside Y .

It has been recently pointed out [3] that pairs (X, Y) such that $R^{\downarrow N}(Y) = X$ and $R^{\uparrow N}(X) = Y$ are characterizing independent sub-contexts (X, Y) such that $((X \times Y) + (\overline{X} \times \overline{Y})) \supseteq R$, in the sense that they do not share any object or property. Thus, in Figure 1, $(\{a, b, c, d, e, f\}, \{5, 6, 7, 8\})$ and $(\{g, h, i\}, \{1, 2, 3, 4\})$ are two formal sub-contexts.

When comparing the features underlying FCA and analogical proportions, one can notice that the same 4 “indicators” are involved from the beginning: $a \cap b$, $a \cap \bar{b}$, $\bar{a} \cap b$, and $\bar{a} \cap \bar{b}$. Indeed $R^{\downarrow H}(Y)$ is based on the condition $R^{\uparrow}(x) \cap Y \neq \emptyset$, $R^{\downarrow N}(Y)$ on the condition $R^{\uparrow}(x) \cap \overline{Y} = \emptyset$, $R^{\downarrow \Delta}(Y)$ on the condition $\overline{R^{\uparrow}(x)} \cap Y = \emptyset$, and $R^{\downarrow \nabla}(Y)$ on the condition $\overline{R^{\uparrow}(x)} \cap \overline{Y} \neq \emptyset$. Moreover, with these 4 indicators, one can define other so-called logical proportions [4], including some that are closely related to analogical proportions such as ‘paralogy’ which reads “what a and b have in common, c and d have it also” and is defined by $a \wedge b = c \wedge d$ and $\bar{a} \wedge \bar{b} = \bar{c} \wedge \bar{d}$ [10]. This more generally raises the question of the relations between FCA and these logical proportions.

Finally, the experiments with **Doap** have obviously to be scaled on larger formal contexts, in order to estimate its practical complexity more accurately. Some more thought has also to be given about the choice of the *Start* 4-tuples, especially to take advantage of the addition of the reverse attributes. An interesting point would be to be able to choose the *Start* in order to insure that every analogical proportion can be discovered. We also believe that the speed of **Doap** can be increased, since there are still a lot of parameters to tune, for example breaking ties in the head of the *Close* list in a non random fashion.

An interesting question is whether or not the construction of the lattice must precede the heuristic search. It would certainly be of great interest to construct only the parts that are required by the running of the *Doap* algorithm. This would lead to merge the two parts of the method, rather than computing the whole lattice (a very costly operation) before its exploration.

More generally, it would be clearly of interest to have an algorithm also able to find out the analogical proportions that hold in some sub-context (since as already said, irrelevant attributes may hide interesting analogical proportions), rather than in the initial formal context. This will open a machine learning point of view [1].

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