Concepts Valuation by Conjugate Möebius Inverse

Petr Gajdoš, Václav Snášel

Department of Computer Science,
VŠB - Technical University of Ostrava,
tř. 17. listopadu 15, 708 33 Ostrava-Poruba
Czech Republic
Petr.Gajdos@vsb.cz

Abstract. There is several known algorithm to construct concept lattices. The question is, how could we simplify this lattice into concepts, that are important and have selected features. According to "A Theory of Diversity", we can compute the diversity of a set of objects recursively from the pairwise dissimilarities between its elements. Using Conjugate Möebius Inverse, we can compute weights of each concept from these diversities. Determining attribute weights is a complex task, however, since there are as many potential attributes as there are non-empty subset of object. The document shows the implementation of Möebius function on concept lattices and then determinig concepts weights by pairwise between objects. We suppose, this is the way to simplify concept lattices.

Keywords: Möebius, concept lattice, diversity, dissimilarity, weight

1 Introduction

This article addresses the problem of the weighting of concepts. In "A Theory of Diversity" (Nehring and Puppe, 2002, henceforth TD), we proposed a multi-attribute approach according to which the diversity of a set of objects is determined by the number and weight of the different features (attributes) possessed by them. In some cases, the diversity of a set can be computed recursively from the pairwise dissimilarities between its elements (plus their value as singletons). Two basic models for which this is possible are the hierarchical model studied by Weitzman (1992, 1998) in the context of biodiversity and the more general line model introduced in TD. As already observed by Weitzman (1992), then hierarchical model implies that the two greatest dissimilarities between three points are always equal if singletons are equally valued. The purpose of the present paper is to show, how could we compute the weights of concepts using knowledge of this models.

Section 2 procides the necessary background from formal concepts analysis and TD. Section 3 shows the implementation of Conjugate Möebius Function to valuating concept lattice and another way to get same values by pairwise dissimilarities between objects. Last section is devoted to the poser with weighting concept lattices.

© V. Snášel, J. Pokorný, K. Richta (Eds.): Dateso 2004, pp. 74–83, ISBN 80-248-0457-3.
VŠB – Technical University of Ostrava, Dept. of Computer Science, 2004.

2 Backgound

This section shows some definitions and tools, that are important for our later valuating of concepts. First we define a context and concept lattice. Next, we summarize the basic features of the multi-attribute model developed in TD.

2.1 Context and concept lattice

Definition 1. *A* **formal concept** $C := (G, M, I)$ *concsists of two sets G and M and relation I between G and M. The elements of G are called the objects and the elements of M are called the attributes*[1] *of the context. In order to express that an object g is in a relation I with an attribute m, we write gIm or $(g, m) \in I$ and read it as "the object g has the attribute m". The relation I is also called the incidence relation of the context.*

Definition 2. *for a set $A \subset G$ of object we define*

$$A^{'} = \{m \in M \mid gIm \ for \ all \ g \in A\}$$

(the set of attributes common to the objects in A). Correspondingly, for a set B of attributes we define

$$B^{'} = \{g \in G \mid gIm \ for \ all \ m \in B\}$$

(the set of objects which have all attributes in B).

Definition 3. *A formal concept of the context (G, M, I) is a pair (A, B) with $A \subseteq G$, $B \subseteq M$, $A^{'} = B$ and $B^{'} = A$. We call A the extent and B the intent of the concept (A, B). $\mathfrak{B}(G, M, I)$ denotes the set of all concepts of context (GMI)*

Definition 4. *The concept lattice $\underline{\mathfrak{B}}(G, M, I)$ is a complete lattice in which infimim and supremum are given by:*

$$\bigwedge_{t \in T} (A_t, B_t) = \left(\bigcap_{t \in T} A_t, \left(\bigcup_{t \in T} B_t \right)^{''} \right)$$

$$\bigvee_{t \in T} (A_t, B_t) = \left(\left(\bigcup_{t \in T} A_t \right)^{''}, \bigcap_{t \in T} B_t \right).$$

We refer to [1].

---

[1] The attribute has different meaning in the Conjugate Möebius Inverse. It's a set of objects.

2.2 Diversity function

Definition 5. *Let F be the totality of all features deemed relevant in the specific context, and denote by $R \subseteq X \times F$ the "incidence" relation that describes the features possessed by each object, i.e. $(x, f) \in R$ whenever object $x \in X$ possesses feature $f \in F$. For each relevant feature $f \in F$, let $\lambda_f \geq 0$ quantify the value of realization of f. Upon normalization, λ_f can thus be thought of as the relevant importance, or weight of feature f. The diversity value of a set S is defined as*

$$v(S) = \sum_{f \in F:(x,f) \in R \ for \ some \ x \in S} \lambda_f \tag{1}$$

The diversity value of a set is given by the total weight of all different features possessed by some objects in S. Note expecially that each feature occurs at most once at sum. In particular, each single object contributes to diversity the value af all those features that are not possessed by any already existing objects.

For any subset $A \subseteq X$ of objects denote by F_A the set of features that are possessed exactly the objects in A. Each feature in F_A is possessed by all elements of A and not possessed by any element of $X \setminus A$. Then we can write

$$v(S) = \sum_{A \cap S \neq \emptyset} \sum_{f \in F_A} \lambda_f \tag{2}$$

Then, for each subset $A \subseteq X$ denote by $\lambda_A := \sum_{f \in F_A} \lambda_f$ the total weight of all features with extension A, with the convention that $\lambda_A = 0$ whenever $F_A = \emptyset$. With this notation we write

$$v(S) = \sum_{A \cap S \neq \emptyset} \lambda_A \tag{3}$$

2.3 Conjugate Möebius Inverse

Theorem 1. *For any function $v : 2^X \rightarrow R$ with $v(\emptyset) = 0$ there exists unique function $\lambda : 2^X \rightarrow R$, the Conjugate Möebius Inverse, such that $\lambda_\emptyset = 0$ and, for all S,*

$$v(S) = \sum_{A : A \cap S \neq \emptyset} \lambda_A \tag{4}$$

Furthermore, the Conjugate Möebius Inverse λ is given by the following formula. For all $A \neq \emptyset$,

$$\lambda_A = \sum_{A : A \cap S \neq \emptyset} (-1)^{|A|-|S|+1} * v(S^c), \tag{5}$$

where S^c denotes the complement of S in X.

We refer to [4].

3 Concept lattice and Möebius function

This part shows, how can we compute weights and diversity of concepts of particular concept lattice. Then we use dissimilarity and similarity function to get the same result by easier way.

Description of objects and features in incidence matrix.

C	=	cat	q	=	quadrupped (four feet)
M	=	monkey (chimpanzee)	p	=	pilli
D	=	dog	i	=	intelligence
F	=	fish (delphinus)	w	=	live in water
H	=	human	h	=	hand
W	=	whale			

Table 1. Incidence relation matrix.

	$\lambda_q = 2$ **q**	$\lambda_p = 3$ **p**	$\lambda_i = 4$ **i**	$\lambda_w = 2$ **w**	$\lambda_h = 2$ **h**
C	x	x			
M		x	x		x
D	x	x			
F			x	x	
H			x		x
W			x	x	

There are all subsets $A \subseteq X$ in the table 2. F_A presents a set of relevant features $f \in F$, which are possessed by all elements of set A, but not possessed by any element of a set $X \setminus A$. By $\lambda_A := \sum_{f \in F_A} \lambda_f$, we get the values in the table.

By $v(S) = \sum_{A: A \cap S \neq \emptyset} \lambda_A$, we compute the diversity of each subset of objects of universum X, $S \subseteq X$ in the table 3. In this time, we include all attributes and their weights to compute diversities of subsets S. The large the incidence matrix the large count of conceivable attribues and subsets of objects, so it's more difficult to compute diversity function.

Next, we consider only attributes corresponding to concepts. Sets of attributes are not sets of features of concept but they are identical to sets of objects.

We use Conjugate Möebius Inverse (5) to compute weights of attributes (concepts) from diversities in the table 4.

Table 2. All conceivable attributes.

A	F_A	λ_A	A	F_A	λ_A	A	F_A	λ_A	A	F_A	λ_A	A	F_A	λ_A
∅			FW	w	2	CMD	p	3	MFHW	i	4	CMDFH		
C			MH	h	2	CMF			CMDF			CMDFW		
M			CD	q	2	CMH			CMDH			CMDHW		
D			CM			CMW			CMDW			CMFHW		
F			CF			CDF			CMFH			CDFHW		
H			CH			CDH			CMFW			MDFHW		
W			CW			CDW			CMHW			CMDFHW		
			MD			CFH			CDFH					
			MF			CFW			CDFW					
			MW			CHW			CDHW					
			DF			MDF			CFHW					
			DH			MDH			MDFH					
			DW			MDW			MDFW					
			FH			MFH			MDHW					
			HW			MFW			DFHW					
						MHW								
						DFH								
						DFW								
						DHW								
						FHW								

Table 3. Diversities of subsets S of objects.

S	$v(S)$	S	$v(S)$	S	$v(S)$	S	$v(S)$	S	$v(S)$	S	$v(S)$
∅	0	FW	6	CMD	11	MFHW	11	CMDFH	13	CMDFHW	13
C	5	MH	9	CMF	13	CMDF	13	CMDFW	13		
M	9	CD	5	CMH	11	CMDH	11	CMDHW	13		
D	5	CM	11	CMW	13	CMDW	13	CMFHW	13		
F	6	CF	11	CDF	11	CMFH	13	CDFHW	13		
H	6	CH	11	CDH	11	CMFW	13	MDFHW	13		
W	6	CW	11	CDW	11	CMHW	13				
		MD	11	CFH	13	CDFH	13				
		MF	11	CFW	11	CDFW	11				
		MW	11	CHW	13	CDHW	13				
		DF	11	MDF	13	CFHW	13				
		DH	11	MDH	11	MDFH	13				
		DW	11	MDW	13	MDFW	13				
		FH	8	MFH	11	MDHW	13				
		HW	8	MFW	11	DFHW	13				
				MHW	11						
				DFH	13						
				DFW	11						
				DHW	13						
				FHW	8						

Table 4. Weighting by CMI

A	$S : S \subseteq A$	S^c	$(-1)^{\|A\|-\|S\|+1}$	$v(S)$	λ_A
FW	F	CMDHW	+1	13	2
	W	CMDFH	+1	13	
	FW	CMDH	−1	11	
	∅	CMDFHW	−1	13	
MH	M	CDFHW	+1	13	2
	H	CMDFW	+1	13	
	MH	CMDW	−1	11	
	∅	CMDFHW	−1	13	
CD	C	MDFHW	+1	13	2
	D	CMFHW	+1	13	
	CD	MFHW	−1	11	
	∅	CMDFHW	−1	13	
cmd	C	MDFHW	−1	13	3
	M	CDFHW	−1	13	
	D	CMFHW	−1	13	
	CM	DFHW	+1	13	
	CD	MFHW	+1	11	
	MD	CFHW	+1	13	
	CMD	FHW	−1	8	
	∅	CMDFHW	+1	13	
MFHW	M	CDFHW	+1	13	4
	F	CMDHW	+1	13	
	H	CMDFW	+1	13	
	W	CMDFH	+1	13	
	MF	CDHW	−1	13	
	MH	CDFW	−1	11	
	MW	CDFH	−1	13	
	FH	CMDW	−1	13	
	FW	CMDH	−1	11	
	HW	CMDF	−1	13	
	MFH	CDW	+1	11	
	MFW	CDH	+1	11	
	MHW	CDF	+1	11	
	FHW	CMD	+1	11	
	MFHW	CD	−1	5	
	∅	CMDFHW	−1	13	
M	M	CDFHW	−1	13	0
	∅	CMDFHW	+1	13	
∅					0
CMDFHW	all subsets	all subsets			0

Any diversity function satisfies this formula:

$$v(S \cup \{x\}) - v(S) = \sum_{A \ni x, A \cap S = \emptyset} \lambda_A \tag{6}$$

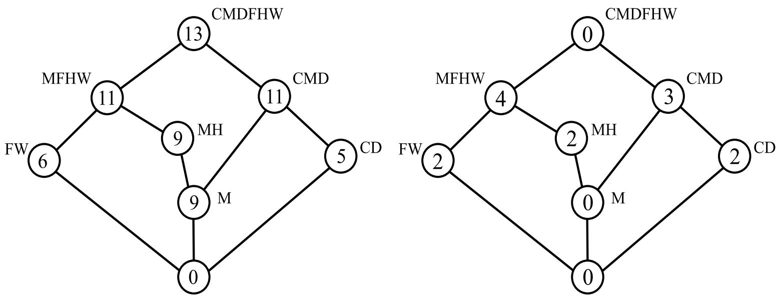

Fig. 1. a) Diversities of concepts b) Weights of concepts

Table 5. Dissimilarities

	C	**M**	**D**	**F**	**H**	**W**
C	0	2	0	5	5	5
M	6	0	6	5	3	5
D	0	2	0	5	5	5
F	6	2	6	0	2	0
H	6	0	6	2	0	2
W	6	2	6	0	2	0

By (6), the marginal diversity of an object x at a set S is given by the total weight of all attributes possessed by x but by no element of S. Accordingly, we will refer to marginal diversity also as the distinctiveness of x from S, which we denote by

$$d(x, S) := v(S \cup \{x\}) - v(S). \tag{7}$$

A diversity function naturally induces a notion of pairwise dissimilarity between objects as follows.

Definition 6. *For all* x, y

$$d(x, y) := d(x, \{y\}) = v(\{x, y\}) - v(\{y\}). \tag{8}$$

By (6), $d(x, y)$ is the weight of all attributes possessed by x but not by y. Note that, in general, d need not be symmetric. We can read it from table (5).

As we said at the beginning, there are two models in TD. The hierarchical and the more general line model. All concept lattices are hierarchical ordered. But, weighting of concepts is a difficult task. We can assign values to concepts only in small and simly lattice because of next condition.

Definition 7. *A model* $H \subseteq 2^X$ *is called a (taxonomic) hierarchy if the elements of* H *are nested in the sence that, for all* $A, B \in H$,

$$A \cap B \neq \emptyset \Rightarrow [A \subseteq B \ \vee \ B \subseteq A]. \tag{9}$$

Accordingly, we will refer to a diversity function v, as well as to the associated attribute weighting function λ, as hierarchical if the support Λ of relevant attributes forms a hierachy. Diversity function is hierarchical if and only if, for all x and S,

$$v(S \cup \{x\}) - v(S) = \min_{y \in S}[v(\{x,y\}) - v(\{y\})] \tag{10}$$

or, equivalently,

$$d(x,S) = \min_{y \in S} d(x,y) \tag{11}$$

Theorem 2. Conjugate Möebius Inverse on a hierarchy. *Let v be a diversity function with attribute weighting function λ. If v is hierarchical, then for all $A \in \Lambda$ and all $x \in A$,*

$$\lambda_A = \min_{y \in A^c} d(x,y) - \max_{y \in A} d(x,y) \tag{12}$$

Conversely, suppose that,
for all $A \in \Lambda$ and all $x \in A$, $\lambda_A = d(x, A^c) - \max_{y \in A} d(x,y)$, then λ is hierarchical.

According to (12) we compute weights of concepts in the table 6. We can see, that most of values are correct (compare with table (4)). But, some of them are not right although we have used the same formula (12). We can find hierarchical ordering in the concept lattice but it is different to hierarchy defined in (9).

Table 6. Values of concepts according to (12)

Concept	*A*	A^c	*sel. x*	$\min_{y\in A^c} d(x,y)$	$\max_{y\in A} d(x,y)$	λ_A
C1	CD	MFHW	C	2	0	2
			D	2	0	2
C2	M	CDFHW	M	0	0	0
C3	FW	CMDH	F	2	0	2
			W	2	0	2
C4	MH	CDFW	M	5	3	2
			H	2	0	2
C5	CMD	FHW	M	3	6	-3
			C	5	2	3
			D	5	2	3
C6	MFHW	CD	M	6	5	1
			F	6	2	4
			H	6	2	4
			W	6	2	4
C7	CMDFHW	∅	C	0	0	0
C8	∅	CMDFHW	∅	0	0	0

Proof. Suppose that Λ is a hierarchy. Let $x \in A \in \Lambda$, and define $z^* := argmax_{z \in A} d(x,z)$. Since $\Lambda_x = \{B \in \Lambda : x \in B\}$ is a chain, one has $B \subset A \Leftrightarrow z^* \notin B$ for all $B \in \Lambda_x$, where "$\subset$" denotes the proper subsethood relation.
Hence,

$$\begin{aligned}
&\min_{z \in A^c} d(x,z) - \max_{z \in A} d(x,z) \\
&\quad = v(\{x\} \cup A^c) - v(A^c) - d(x,z^*) \\
&\quad = \lambda(\{B : x \in B \subseteq A\}) - \lambda(\{B : x \in B, z^* \notin B\}) \\
&\quad = \lambda_A + \lambda(\{B : x \in B \subset A\}) - \lambda(\{B : x \in B, z^* \notin B\}) \\
&\quad = \lambda_A.
\end{aligned}$$

Conversely, suppose that Λ is not a hierarchy, i.e. suppose there exists $A, C \in \Lambda$ such that $A \cap C$, $A \setminus C$, and $C \setminus A$ are all non-empty. Let $x \in A \cap C$. Without loss of generality we may assume that A is an minimal element of Λ satisfying $x \in A$ and $A \setminus C \neq \emptyset$, i.e. for no proper subset $A^{'}$ of A, $x \in A^{'} \in \Lambda$ and $A^{'} \setminus C \neq \emptyset$. Let $y \in A \setminus C$. By construction one has $\{B \in A : x \in B, B \subset A\} \subset \{B \in \Lambda : x \in B, y \notin B\}$ since C belong to the latter but not to the former set. Since assumption, $\lambda_C > 0$, this implies $\lambda(\{B : x \in B \subset A\}) - \lambda(\{B : x \in B, y \notin B\}) < 0$.
Therefore,

$$\begin{aligned}
&d(x, A^c) - \max_{z \in A} d(x,z) \\
&\quad = v(\{x\} \cup A^c) - v(A^c) - \max_{z \in A} d(x,z) \\
&\quad \leq v(\{x\} \cup A^c) - v(A^c) - d(x,y) \\
&\quad = \lambda(\{B : x \in B \subseteq A\}) - \lambda(\{B : x \in B, y \notin B\}) \\
&\quad = \lambda_A + \lambda(\{B : x \in B \subset A\}) - \lambda(\{B : x \in B, y \notin B\}) \\
&\quad < \lambda_A.
\end{aligned}$$

According to next definition, we can divide the concept lattice into hierarchies to compute weights of concepts by CMI.

Definition 8. *A* **lattice hierarchy** *H in lattice L is join-sublattice where $a \cap b \neq \emptyset \Rightarrow [a \leq b \vee b \leq a]$, $a, b \in H$. $\mathfrak{H}(L, \subseteq)$ denotes the poset of all lattice hierarchies of lattice L.*

Theorem 3. *Let H be the lattice hierarchy. Then Hasse diagram $H \setminus 0$ is rooted tree.*

Proof. Because H is finite join-sublattice then exists join r for all element of H. It is easy to show that r is root. Hence Hasse diagram $H \setminus 0$ is connected. Let $H \setminus 0$ contain a cycle i.e. suppose there exists $a, b \in H \setminus 0$ such that $a \| b$ and $a \wedge b = c$, $c \neq 0$. We obtain a contradiction with presumption.

4 Conclusion

Concept lattice is ordered but we can not use a simple method to compute weights of concepts by Conjugate Möebius Inverse. We try to delegate this problem to pairwise of elements of the hierarchical model. We get values of weights or diversities but this method ensures right results if and only if, concept lattice or a part of lattice satisfy the condition of hierarchical structure (9).

We see another way to solve this problem. In future, we want to prove, that we can "supply" any concepts lattice by finite set of trees, that are ordered and they satisfy our condition of hierarchical structure. We want to find minimal count of hierarchies, that can cover concept lattice.

References

1. B. Ganter and R. Wille. *Formal Concept Analysis.* Springer-Verlag Berlin Heidelberg, 1999.
2. K. Nehring. A theory of diversity. *Ecometrica 70*, pages 1155–1198, 2002.
3. K. Nehring and C. Puppe. Modelling phylogenetic diversity. *Resource and Energy Economics*, 2002.
4. K. Nehring and C. Puppe. Diversity and dissimilarity in lines and hierarchies. *Mathematical Social Sciences 45*, pages 167–183, 2003.
5. I. Vondrak and V. Snasel. Using concept lattices for organizational structure analysis. *In Proceedings of European Concurrent Engineering Conference ECEC '02 (Modena, Italy)*, 2002.