The Role of Shape in Problem-Solving Activities in Mathematics

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Abstract. In our paper we will rely on research by Grosholz (2007) considering her thesis of the irreducibility of iconic representation in mathematics. Against this background, our aim will be to discuss the epistemic value of "shape" or iconicity in diagrammatic representations in geometry. We show that iconic aspects of diagrams reveal structural relations underlying the method to solve quadrature problems developed by Leibniz (1675/76). As a concluding remark, we shall argue that in retrieving the information embedded in a diagram the reader must establish a meaningful relationship between the information supplied by the diagram and the relevant background knowledge which often remains implicit.

Keywords. Iconic Representation, Diagrams, Visualization, Leibniz, General Method, Background Knowledge.

Introduction

In our paper we rely on research by Grosholz (2007) considering, in particular, her thesis of the irreducibility of iconic representation in mathematics. Against this background, our aim is to discuss the epistemic value of "shape" or iconicity in the representations of diagrams in the case of geometry. In order to illustrate our point, we bring in a case-study selected from Leibniz´s work with diagrams in problem-solving activities in connection with a "master problem, the Squaring of the Circle – or the precise determination of the area of the circle", a problem which remains insoluble by ruler and compass construction within Euclidean geometry.[3] Our main reason to focus on Leibniz is as follows. On the one hand, throughout his work as a mathematician, Leibniz relies on a variety of tools which display rich iconic aspects in the implementation of problem-solving activities. On the other hand, it is precisely in the case of geometry where Leibniz makes important contributions. Reasoning with diagrams plays a central role in this particular case. In order to solve certain geometrical problems which could not be solved within the framework of Euclidean geometry, Leibniz devises a method that proceeds by transforming a certain mathematically intractable curve into a more tractable curve which is amenable to

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[3] See [3, p. 36].

calculation. This method is sometimes called the method of "transmutation" as it is based upon the transformation of one curvilinear figure into another.

For Leibniz depending upon the context of research some methodological tools are more fruitful than others, moreover, simplicity and economy is also amongst the epistemic virtues guiding the design of methods for problem-solving activities. In our case-study, we show how Leibniz devises a method which allows him to re-conceive a given curve by "transforming" it into a more tractable curve as part of his strategy to calculate the area of curves that may contain irrational numbers (the real number π in the case of the circle). In particular, we aim to show that iconic aspects of diagrams reveal structural relations underlying the process of "transformation" developed by Leibniz in *Quadrature arithmetique du circle, de la ellipse et de l' hyperbole* (1675/76).[4]

1. The Idea of "Shape" As Iconic Representation

Let us start by focusing on the idea of "shape" in the sense of "iconic representation". Representations may be iconic, symbolic and indexical depending upon their role in reasoning with signs in specific contexts of work.[5] According to the traditional view representations are iconic when they resemble the things they represent. In many cases this characterization appears as doubtful because of its appeal to a vague idea of similarity which would seem untenable when representations of numbers are involved. But Grosholz argues that in mathematics iconicity is often an irreducible ingredient, as she writes,

> In many cases, the iconic representation is indispensable. This is often, though not always, because shape is irreducible; in many important cases, the canonical representation of a mathematical entity is or involves a basic geometrical figure. At the same time, representations that are 'faithful to' the things they represent may often be quite symbolic, and the likenesses they manifest may not be inherently visual or spatial, though the representations are, and articulate likeness by visual or spatial means [3, p. 262].

In order to determine whether a representation is iconic or symbolic, the context of research with its fundamental background knowledge needs to be taken into account in each particular case, in other words, iconicity cannot simply be read off the representation in isolation of the context of use. We find here a more subtle understanding of "iconicity" than the traditional view. Let us focus on the idea that representations "articulate likeness by visual or spatial means". Grosholz suggests that even highly abstract symbolic reasoning goes hand in hand with certain forms of visualizations.

Giardino (2010) offers a useful characterization of the cognitive activity of "visualizing" in the formal sciences. In visualizing, she explains, we are decoding articulated information which is embedded in a representation, such articulation is a

[4] In this paper, we shall be refereeing to the French translation (Parmentier 2004) of Leibniz original text De Quadratura Arithmetica (1675/76).

[5] The distinction goes back to Charles Peirce´s theory of signs. For a brief discussion of this distinction, see [3, p. 25].

specific kind of spatial organization that lends unicity to a representation turning it intelligible. In other words, spatial organization is not just a matter of physical display on the surface (paper or table) but "intelligible spatiality" which may require substantial background knowledge:

> (...) to give a visualization is to give a contribution to the organization of the available information (...) in visualizing, we are referring also to background knowledge with the aim of getting to a global and synoptic representation of the problem [1, p. 37].

According to this perspective, the ability to read off what is referred to in a representation depends on some background knowledge and expertise of the reader. Such cognitive act is successful only if the user is able to decode the encrypted information of a representation while establishing a meaningful relationship between the representation and the relevant background knowledge which often remains implicit. The starting point of this process is brought about by representations that are iconic in a rudimentary way, namely, they have spatial isolation and organize information by spatial and visual means; and they are indivisible things. In the next section we turn to our case study taken from Leibniz's work in geometry which we hope will help to illustrate some of the above considerations.

2. Our Case Study - Leibniz's *De Quadratura Arithmetica* (1675/76)

In *Quadrature arithmétique du cercle, de l'ellipse et de l'hyperbole* [8] Leibniz provides a general method whereby "quadrature" problems for curvilinear figures can be solved. The first seven propositions of this work form a unity and as Leibniz himself emphasizes, Proposition 7 is the "fruit" of all that has gone before [8, p. 35]. In this context of work, Leibniz presents the reader a diagram (Fig. 1).

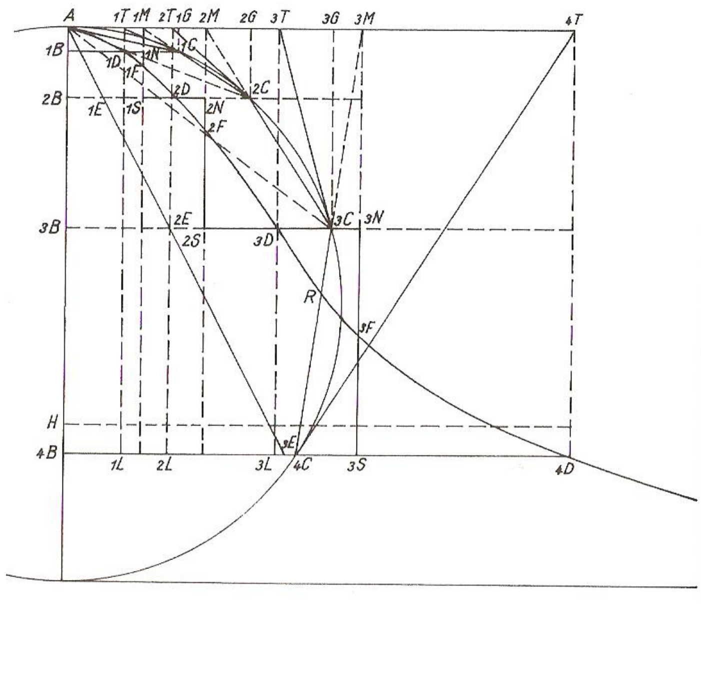

Figure 1. Leibniz *Quadrature arithmetique du circle, de l'ellipse et de l'huperbole* 1675/76, [8, p. 65].

While for the untrained eye this diagram appears as a set of highly entangled shapes, for Leibniz, the diagram should offer the reader an overall assessment of the way his proposed method works. In order to show the most salient aspects of Leibniz's method we shall try to make more explicit some of the features displayed in Figure 1. We proceed to put the original diagram "under the microscope" dissecting it into four diagrams (Figures 2-5). This will allow us to see some of the most relevant steps involved in the resolution of the problem under consideration. These visualizations together with the indications as to how to "read" Figures 2-5 may then be seen as offering a brief outline of Leibniz´s method.

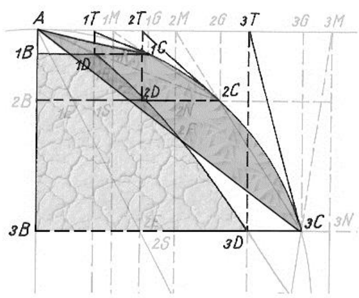

Figure 2

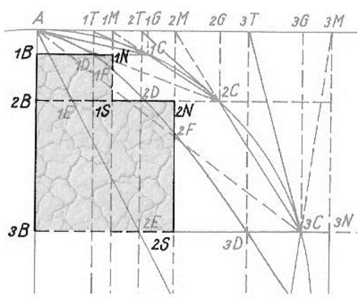

Figure 3

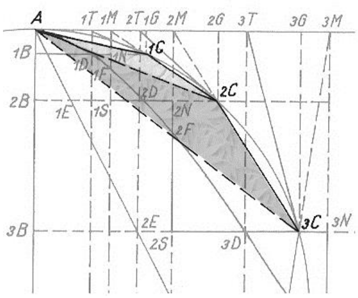

Figure 4

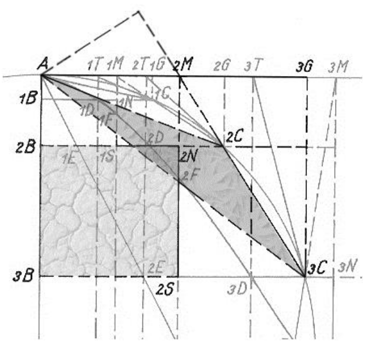

Figure 5

Leibniz aims to show that the area of a curvilinear figure C – which cannot be calculated - may be determined by constructing a second figure D, whose area can be calculated. A crucial step in Leibniz reasoning relies upon certain geometrical results known since Euclid which allow us to assume that the ratio between C and D is known to us. This step in the reasoning is represented in the diagram by two different shapes that we have highlighted in Figure 2. On the one hand, we see an enclosed area delimited by segments A_1C, A_3C and the arc ${}_1C_2C_3C$ – which represents the area C, unknown to us. On the other hand, we see another enclosed area delimited by segments ${}_1B_1D$, ${}_1B_3B$, ${}_3B_3D$ and the curve ${}_1D_2D_3D$ which represents the area of the second figure D. Finally, we can also see some specific lines that represent geometrical relations between both figures according to Euclidean geometry.[6]

With a view to determine the area of curvilinear figure C we first need to find the area of figure D. Leibniz proceeds to decompose D into a finite number of elemental parts - the rectangles ${}_1N_1B_2B_1S$ and ${}_2N_2B_3B_2S$ - which are then added up. We have highlighted this procedure in Figure 3. As we can also see the sum of rectangles makes up a new shape or figure which Leibniz calls *"espace gradiforme"*.[7] At this stage of the reasoning, the construction of such "space" is crucial for Leibniz's problem-solving strategy. Instead of an exact calculation of the area of D, Leibniz approximates the area of D by calculating the area of such *"espace gradiforme"*, so that the difference between both figures will be less than any assignable number.

Next, the newly constructed "*espace gradiforme"* is transposed upon figure C (See Figures 4 and 5). This procedure can be described in two steps.

The first step consists in decomposing the curvilinear figure C into "triangles" which we highlighted in Figure 4. Note that the number of triangles will be greater than any arbitrarily assignable number as it is possible to decompose the figure into arbitrarily many triangles where the whole set of triangles has the single vertex A. Here Leibniz takes distance from other techniques used at the time. While Cavalieri, for instance, often decomposed curvilinear figures into parallelograms, Leibniz proceeds to resolve the problem by decomposing curvilinear figures into triangles (for an illustration of this difference see Figure 6). Accordingly, instead of rectangles or parallelograms, the elemental parts in this case will be triangles, as Leibniz points out in Scholium 1 of the treatise:

> (...) on peut en effet également décomposer en triangles des figures curvilignes qu'à l'exemple d'autres grands savants Cavalieri ne décomposait souvent qu'en parallélogrammes, sans utiliser, à ma connaissance, une résolution générale en triangles [8, p. 39].

[6] Leibniz relies upon a generalization of Euclid's Elements (Proposition 1, Book I) to justify his reasoning when assuming that the "triangle" $\mathbf{A_2C_3C}$ equals one half of the "rectangle" $\mathbf{{}_2B_2N.{}_2B_3B}$ ($\mathbf{A_2C_3C}$ = 1/2 $\mathbf{{}_2B_2N.{}_2B_3B}$). See [9].

[7] The expression Leibniz used in the original Latin is "spatium gradiforme" [8, p. 69].

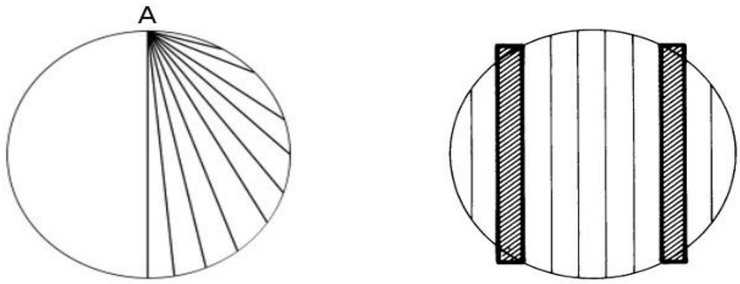

Figure 6. Leibniz's method as opposed to Cavalieri's method.

The second step consists in the construction of the *"espace gradiforme"* upon C (See Figure 5). To this end, Leibniz uses the rectangle with sides $_2B_2N$ and $_2N_2S$ which can be constructed from a given triangle A_2C_3C relying on certain well-established geometrical relations which hold so that the ratio between the areas of figures C and D is ½. It is precisely in this context where Leibniz relies upon results already established by Euclid.[8]

Let us now return to Leibniz's original diagram corresponding to Proposition 7 (See Figure 1). With Euclid's results concerning structural relations between two types of shapes - triangles and rectangles – in mind, we are justified to establish a correlation between triangles A_1C_2C, A_2C_3C,… and corresponding rectangles $_1B_1N_2B_1S$, $_2B_2N_3B_2S$,.... For instance, the triangle A_2C_3C corresponds to the rectangle $_2D_2B_3B_2S$. Next, we recall that the area of figure D can be approximated by the sum of the (finite number of) elemental parts – rectangles – the original figure D was decomposed into.

Finally, the area of the curvilinear figure C can be calculated by applying the ratio of ½ upon the area of figure D. According to Leibniz, the calculation obtained by this method is not exact but one may consider it is a precise determination of the area of the curvilinear figure C. To sum up, it is by recognizing certain geometrical relations holding between triangles and rectangles that one can see that the precise determination of the area of the curvilinear shape will depend upon the value of the approximation of the area of D. The latter, in turn, can be calculated on the basis of the "espace gradiforme", the new shape designed by Leibniz which is required to approximate the value of D.

3. Concluding Remarks

In this section we finally consider some of the requirements which are imposed upon the reader in order to be able to perform the relevant "cognitive act" of successfully decoding a visualization that includes "shapes" in the context of problem solving activities in mathematics. Again, we shall focus on the diagram of our case-study (Figure 1).

Diagrams are shapes that represent by spatial and visual means. Their intelligibility partly depends on their integrity and shape, features that make a diagram intrinsically iconic. But diagrams also often combine, as Grosholz argues, iconic aspects with

[8] See footnote 6 (above).

symbolic ingredients. If diagrams were just iconic, they would be but a copy – a more or less faithful picture - of what we intent to refer to. However, diagrams are inherently general, a drawing of, say, curvilinear shape without being just a drawing of a particular curve on this particular page of a text. On the one hand, diagrams resemble a particular shape, on the other hand, they represent a whole set of (instances of) a certain shape and are in this sense general. To clarify this feature of diagrams we distinguish following M. Giaquinto between "discrete" and "indiscrete" representations,

> (...) diagrams very frequently do represent their objects as having properties that, though not ruled out by the specification, are not demanded by it. In fact this is often unavoidable. Verbal descriptions can be discrete, in that they supply no more information than is needed. But visual representations are typically indiscrete, because for many properties or kinds F, a visual representation cannot represent something as being F without representing it as being F *in a particular way* [1, p. 28].

"Indiscrete representations" as opposed to "discrete representations" are representations that represent by spatial and visual means including the combination of iconic aspects as well as symbolic ingredients. As a consequence of this important feature of diagrams, it follows that both particular instances and generality go hand in hand. Returning to our case-study and Leibniz's diagram, we may offer the following three observations in this regard:

- The diagram that goes with proposition 7 (Figure 1) exhibits a circular shape. We may consider that Leibniz's method to calculate the area for this curvilinear shape works only for this particular curve. But Leibniz intends to use his method as a general method so as to include any curvilinear shape, as he writes in the Schollium to proposition XI:

  La proposition 7 m'a fourni le moyen de construire une infinité de figures de longueur infinie égales au double d'un segment ou d'un secteur (…) d'une courbe donnée quelconque, et ceci d'une infinité de manières (Leibniz 1676, p. 97).[9]

- In the diagram (Figure 1) the curvilinear shape C is actually divided into only four points, namely, ${}_1C_2C_3C_4C$. However, it is possible to divide the arc C into as many points as we want.
- If the number of points is large enough, the diagram will be less faithful to the particular instance that it pretends to represent and when the magnitude of segment A_1C is less than any assignable number, we have the limit-case. At this point, the space ${}_1CA_3C_2C$ (called "triligne" by Leibniz) can be assumed as a space composed by curve ${}_1C_2C_3C$ and the straight line A_3C (called "secteur" by Leibniz)[10].

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[9] Leibniz specifies the class of curves which fall under the domain of application of his method in Proposition 6 of *Quadrature arithmetique*.

[10] See [8, p. 97].

Note that in our case-study, the reader has to select only part of the information furnished by the diagram; he/she has to be able to discern the relevant information contained in the diagram in the light of the problem under consideration. In particular, it is necessary to distinguish in the diagram between iconic ingredients and symbolic ingredients. What exactly is required of the reader to be able to decode the relevant information encrypted in the diagram? To answer this question we return here to Giardino's observation that "to give a visualization is to give a contribution to the organization of the available information". First the reader needs to consider the context in which the diagram is inserted. As already noted, part of the context is made explicit by remarks written in natural language as it is the case in the written text accompanying the diagram [8, pp. 65, 67]. In the written text, Leibniz explains how to construct the diagram he shows together with Proposition 7 (our Figure 1). But such description is hardly enough, as the reader still needs to rely on substantial information – background knowledge concerning relevant chapters of the history of geometry – in order to get "a global and synoptic representation of the problem". However the relevant background knowledge cannot be made fully explicit, at least not "all at once". The expertise of the community of mathematicians which includes different traditions of research and, in a broad sense, the history of mathematics, provides different tools and techniques which need to be acquired by teaching and learning. For instance, in our case-study Leibniz's diagram relies heavily on procedures and techniques whose origin goes back to Euclid and Archimedes but also recalls the work of some of his contemporaries such as Cavalieri's "theorem of indivisibles" and Pascal's "characteristic triangle", which is used by Leibniz in order to "transform" triangles into rectangles. Finally, as we may divide the arc C into "as many points as we want", sometimes the diagram is meant to be read as an infinitesimal configuration and at this point the symbolic dimension of the diagram comes into play so that in each case the trained eye of the reader will be required to be able to recognize the roles of these different dimensions.

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