# Uniform Variable Splitting

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This extended abstract motivates and presents techniques for identifying *variable independence* in free variable calculi for classical logic without equality. Two variables are called *independent* when it is sound to instantiate them differently. The goal of the uniform variable splitting technique, first presented in [14], is to label variables differently (modulo a set of equations) exactly when they are variable independent.

The overall motivation is to have a calculus which simultaneously has: (1) invariance under order of rule application (to enable goal-directed search, since rules then can be applied in any order), (2) introduction of free variables instead of arbitrary terms (to reduce the instantiation problem to a unification problem), and (3) a branchwise restriction of the search space (to allow branchwise termination criteria and early termination in cases of unprovability). Following the notation of Smullyan [11], both formulae and inferences will have type $\alpha$, $\beta$, $\gamma$ or $\delta$. A $\theta$-inference always has a principal formula of type $\theta$; atomic formulae have no type.

$$\dfrac{\dfrac{\mathbf{Pu} \vdash Pa \quad \mathbf{Pu} \vdash Pb}{Pu \vdash Pa \wedge Pb}}{\forall x Px \vdash Pa \wedge Pb}$$

**(a)** $\gamma$ below $\beta$.
$u$ is rigid.

$$\dfrac{\dfrac{\mathbf{Pu} \vdash Pa}{\forall x Px \vdash Pa} \quad \dfrac{\mathbf{Pv} \vdash Pb}{\forall x Px \vdash Pb}}{\forall x Px \vdash Pa \wedge Pb}$$

**(b)** $\beta$ below $\gamma$.
Variable-pure.

$$\dfrac{\dfrac{\mathbf{Pu} \vdash Pa}{\forall x Px \vdash Pa} \quad \dfrac{\mathbf{Pu} \vdash Pb}{\forall x Px \vdash Pb}}{\forall x Px \vdash Pa \wedge Pb}$$

**(c)** $\beta$ below $\gamma$.
Variable-sharing.

$$\dfrac{\dfrac{\mathbf{Pu}^1 \vdash Pa \quad \mathbf{Pu}^2 \vdash Pb}{Pu \vdash Pa \wedge Pb}}{\forall x Px \vdash \underset{1}{Pa} \wedge \underset{2}{Pb}}$$

**(d)** $\gamma$ below $\beta$.
With splitting.

$$\dfrac{\dfrac{\mathbf{Pu}^1 \vdash Pa}{(\forall x Px)^1 \vdash Pa} \quad \dfrac{\mathbf{Pu}^2 \vdash Pb}{(\forall x Px)^2 \vdash Pb}}{\forall x Px \vdash \underset{1}{Pa} \wedge \underset{2}{Pb}}$$

**(e)** $\beta$ below $\gamma$.
With splitting.

**Fig. 1.** Examples over $\forall x Px \vdash Pa \wedge Pb$.

**Examples of variable independence.** In Fig. 1 there are five different derivations over the same sequent. In (a) the variable $u$ occurs in both branches, and it is commonplace to require that both occurrences are instantiated rigidly, i.e. with the same term. However, in this case, it is sound to instantiate the leftmost occurrence with $a$ and the rightmost with $b$. The two occurrences are variable independent. In a calculus with *universal variables* this is easily recognized, but there are cases where a variable is not universal, but still independent of many other occurrences of the same variable. Now, let us reverse the order of rule application such that the $\beta$-inference is below the $\gamma$-inference. If the calculus introduces a new free variable with every $\gamma$-inference, then the derivations are *variable-pure* [13]. This is exemplified in (b), where the above *variable independence is revealed due to the difference in inference order*. In order to have goal-directed search and keep a tight relation to matrix systems [15, 13, 7], it is desirable to have *invariance under order of rule application*, which is not a property enjoyed by variable-pure calculi. (The leaf sequents in (a) differ from those in (b).) To obtain this invariance, one can employ a way of reusing free variables. If the different occurrences of the *same* $\gamma$-formula introduce *the same free variable*, then the derivations are *variable-sharing* [13]. This is exemplified in (c). Again, the occurrences are *variable independent* even though they are syntactically indistinguishable. The derivations in (d) and (e) are the respective variants of (a) and (c) in a calculus with variable splitting. Regardless of inference order, the leaf sequents are the same and variable independent occurrences are labeled differently. The derivations in (d) and (e) are called *permutation variants*, since the leaf sequents are identical.

**Free variable sequent calculus with variable splitting.** For most definitions we refer to [14]. Essential to the variable labeling system is that every formula occurrence is associated with an index. To achieve this we adopt an indexing system for formulae similar to that used by Wallen [15]. When a set of formulae is given, typically in the root sequent of a derivation, every possible formula obtainable from this set, including those created by implicit contraction, will have a *unique* index. Technically, all formulae from the set will be associated with an indexed formula tree, containing all possible indices. (Note that an indexed formula tree is infinite whenever it contains a $\gamma$-formula.) In a derivation, two different formula *occurrences* have identical indices exactly when they are instances of the same formula occurring in different branches due to one or more $\beta$-inferences below in the derivation. The notation $\varphi^{\mathsf{x}}$ is used when $\varphi$ has index $\mathsf{x}$.

**Definition 1.** *If $\varphi^{\mathsf{x}}$ and $\psi^{\mathsf{y}}$ are formulae, let $\mathsf{x} \ll \mathsf{y}$ mean that $\mathsf{x}$ is below $\mathsf{y}$ in the indexed formula tree. ($\ll$ is essentially the subformula ordering.) Two different indices, $\mathsf{x}$ and $\mathsf{y}$, are* $\beta$-related *if they are not $\ll$-related and they have a greatest common descendant of principal type $\beta$. Otherwise, they are* $\alpha$-related. *A* splitting set *is a set of indices such that no two indices are $\beta$-related. A formula occurrence is* decorated *when it is labeled with a splitting set. From now on all formulae will be decorated. The notation $\varphi^{\mathsf{x}}\mathsf{S}$ is used when the formula is labeled with a splitting set $\mathsf{S}$.*

The sequent calculus is similar to **G3c** in [12]. The rules are: L$\land$, L$\lor$, L$\rightarrow$, L$\exists$, L$\forall$, L$\neg$, R$\land$, R$\lor$, R$\rightarrow$, R$\exists$, R$\forall$, R$\neg$. One rule of each type is given in Fig. 2. Due to the built-in contraction of the $\gamma$-rules, structural rules are not needed.

$$\frac{\Gamma, \varphi^x S, \psi^y S \vdash \Delta}{\Gamma, (\varphi^x \wedge \psi^y)^z S \vdash \Delta} \text{ L}\wedge \ (\alpha\text{-rule}) \qquad \frac{\Gamma^x \vdash \varphi^x S, \Delta^x \quad \Gamma^y \vdash \psi^y S, \Delta^y}{\Gamma \vdash (\varphi^x \wedge \psi^y)^z S, \Delta} \text{ R}\wedge \ (\beta\text{-rule})$$

$$\frac{\Gamma, \varphi[w/f^z(u_1, \ldots, u_i)] S \vdash \Delta}{\Gamma, (\exists w \varphi)^z S \vdash \Delta} \text{ L}\exists \ (\delta\text{-rule}) \qquad \frac{\Gamma \vdash (\exists w \varphi)^{z'} S, \varphi[w/u^z]^{x'} S, \Delta}{\Gamma \vdash (\exists w \varphi^x)^z S, \Delta} \text{ R}\exists \ (\gamma\text{-rule})$$

**Fig. 2.** $\beta$**-rules**: $\Gamma^x$ denotes $\{\varphi(S \cup x) \mid \varphi S \in \Gamma\}$, the set of formulae in $\Gamma$ where the index $x$ has been added to all the splitting sets. In a premiss of a $\beta$-rule, the index of the active formula is added to all splitting sets in the sequent except for the splitting set of the active formula itself. It is immediate that the property of being a splitting set is preserved in this operation. Thus, it is possible to track which $\beta$-inferences that caused the branching. $\gamma$**-rules:** The $\gamma$-rules introduce *instantiation variables* of the form $u^z$, where $z$ is the index of the principal $\gamma$-formula. Two formulae are created: one instance of the principal formula and one copy by means of implicit contraction. $\delta$**-rules:** The $\delta$-rules introduce *Skolem functions* of the form $f^z$, where $z$ is the index of the principal $\delta$-formula. (Actually, the same Skolem function is introduced for all contracted copies of the same $\delta$-formula, but this is not so important here.) The skolemization works in the following way: If $(\exists w \varphi)^z S$ is the principal formula in which exactly the instantiation variables $u_1, \ldots, u_i$ occur, then the Skolem term $f^z(u_1, \ldots, u_i)$ is introduced and substituted for the variable $w$. (This $\delta$-rule lies somewhere between a $\delta^+$-rule [5] and a $\delta^{++}$-rule [2].)

In order to identify independent variables, instantiation variables are now labeled with splitting sets. A *colored variable* is a pair $\langle u, S \rangle$, where $u$ is an instantiation variable and $S$ is a splitting set, called a *color*. We write $uS$ for $\langle u, S \rangle$. The terms generated from function symbols (including constants and Skolem functions) and colored variables are called *colored terms*. The terms generated from function symbols and instantiation variables are called *instantiation terms*. Instead of performing unification on the level of instantiation terms, we wish to do so on the level of *colored* terms. This provides an extra degree of freedom, since two colored variables based on the same instantiation variable can be instantiated differently. The labeling is done by means of an operator $\oplus$, which propagates a splitting set to all the instantiation variables in a formula occurrence, i.e. $\varphi \oplus S$ is the formula $\varphi$ where all occurrences of an instantiation variable $u$ has been replaced with $uS$. (An instantiation variable is never bound by a quantifier, so we can do this.)

**Definition 2.** *An* L-pair *("Leaf-pair") $a$ is a pair of complementary formulae, written $\varphi^x S \vdash \psi^y T$, from a leaf sequent. The corresponding* colored L-pair*, denoted $\bar{a}$, is $\varphi^x \oplus (S \setminus T) \vdash \psi^y \oplus (T \setminus S)$. If $A$ is a set of L-pairs, then $\bar{A}$ is the set $\{\bar{a} \mid a \in A\}$. A colored variable $u(S \setminus T)$ is called the* pruning *of the* unpruned variable *$uS$. (The purpose of the pruning is to create a strong dependency between a colored variable and the indices below or equal to* the complementary formula. *E.g., if $z \in (S \setminus T)$, then $z \not\ll x$ and $z \ll y$ ($z = y$).) A* connection *is an L-pair* *in which the two formulae are atomic and contain the same predicate symbols with the same arity. A set of L-pairs is* spanning *for a derivation if it contains exactly one L-pair for each leaf sequent. (We assume that every root sequent* $\Gamma \vdash \Delta$ *of a derivation is such that* $\top \in \Gamma$ *and* $\bot \in \Delta$*.) When* $A$ *is a set of L-pairs,* $\mathsf{Var}^A$ *is the set of all colored variables occurring in* $\bar{A}$*. The set* $\mathsf{Term}^A$ *is the set of colored terms generated from* $\mathsf{Var}^A$ *and the function symbols in* $A$*. An* $A$-substitution *is a partial function* $\sigma : \mathsf{Var}^A \to \mathsf{Term}^A$*. Now, let* $c$ *be the connection* $P(s_1, \ldots, t_1)S \vdash P(t_1, \ldots, t_n)T$*. The set of* primary equations *for* $c$*,* $\mathsf{Prim}(c)$*, is the set* $\{s_i \oplus (S \setminus T) \approx t_i \oplus (T \setminus S) \mid 1 \leq i \leq n\}$*. For a set* $C$ *of connections,* $\mathsf{Prim}(C) = \bigcup_{c \in C} \mathsf{Prim}(c)$*. A* $C$*-substitution* $\sigma$ solves an equation $sS \approx tT$ *from* $\mathsf{Prim}(C)$ *if* $(sS)\sigma = (tT)\sigma$*. It* satisfies $\mathsf{Prim}(C)$ *if it simultaneously solves all equations from* $\mathsf{Prim}(C)$*.*

In general, it is *not* sufficient to characterize provability by only requiring the existence of a satisfying substitution for a spanning set of connections. This is *far too liberal* and gives an inconsistent splitting mechanism due to the fact that *dependent* variables can be labeled differently. In order to obtain consistency, a notion of *admissibility* is introduced. A *proof* in the splitting calculus is a derivation, a spanning set of connections and an admissible substitution satisfying all primary equations. There are several candidates for admissibility that seems natural. However, most of them are inconsistent because too few colored variables are required to be equally instantiated. (In particular, the one proposed in [14] is inconsistent.) A full discussion of this, including a consistent definition of admissibility, is beyond the scope of this abstract. Briefly, an admissible substitution can be defined such that for every proof with splitting there is a proof of the same sequent in a variable-pure calculus without splitting.

The difficulty is illustrated in Fig. 3. The root sequent is falsifiable (a falsifying model is one where $Pb$, $Qa$, $Rb$, $Sa$ are true, and $Pa$, $Qb$, $Ra$, $Sb$ are false. A consistent splitting mechanism must *at least* identify $u^3$ with $u^4$ or $v^1$ with $v^2$. Otherwise, a proof can be obtained by the substitution given over the leaf sequents.

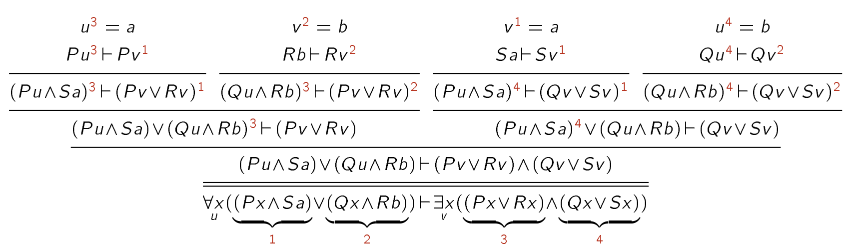


**Fig. 3.** Mutual splitting can give rise to inconsistency

(This extended abstract was rewritten due to the recommendation from one of the reviewers in order to be more suitable as an exercise in technical conference presentation. Several topics for future research were therefore unfortunately left out.)

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