Isotone L-bonds

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Abstract. L-bonds represent relationships between formal contexts. We study properties of these intercontextual structures w.r.t. isotone concept-forming operators in fuzzy setting. We also focus on the direct product of two formal fuzzy contexts and show conditions under which a bond can be obtained as an intent of the product. In addition, we show that the previously studied properties of their antitone counterparts can be easily derived from the present results.

1 Introduction

Formal Concept Analysis (FCA) has become a very active research topic, both theoretical and practical, for formally describing structural and hierarchical properties of data with "object-attribute" character. It is this wide applicability which justifies the need of a deeper knowledge of its underlying mechanisms: and one important way to obtain this extra knowledge turns out to be via generalization and abstraction.

A number of different approaches have presented towards a generalization of the framework and scope of FCA and, nowadays, one can find papers which extend the theory by using ideas from rough set theory [21, 15, 14], possibility theory [8], fuzzy set theory [1, 2], the multi-adjoint framework [16, 17, 19] or heterogeneous approaches [6, 18].

Goguen argued in [10] that *concepts* should be studied transversally, transcending the natural boundaries between sciences and humanities, and proposed category theory as a unifying language capable of merging different apparently disparate approaches. Krotzsch [13] suggested a categorical treatment of morphisms, understood as fundamental structuring blocks, in order to model, among other applications, data translation, communication, and distributed computing.

In this paper, we deal with an extremely general form of L-fuzzy Formal Concept Analysis, based on categorical constructs and L-fuzzy sets. Particularly, our approach originated in relation to a previous work [12] on the notion of Chu correspondences between formal contexts, which led to obtaining information about the structure of L-bonds in such a generalized framework.

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In this paper, we study properties of L-bonds w.r.t. isotone concept-forming operators in a fuzzy setting. We also focus on the direct product of two formal fuzzy contexts and show conditions under which a bond can be obtained as an extent of the product. In addition, we show that the previously studied properties of their antitone counterparts can be easily derived from the present results.

2 Preliminaries

We use complete residuated lattices as basic structures of truth-degrees. A complete residuated lattice is a structure $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1\rangle$ such that

(i) $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice, i.e. a partially ordered set in which arbitrary infima and suprema exist;
(ii) $\langle L, \otimes, 1\rangle$ is a commutative monoid, i.e. $\otimes$ is a binary operation which is commutative, associative, and $a \otimes 1 = a$ for each $a \in L$;
(iii) $\otimes$ and $\rightarrow$ satisfy adjointness, i.e. $a \otimes b \leqslant c$ iff $a \leqslant b \rightarrow c$.

0 and 1 denote the least and greatest elements. The partial order of $\mathbf{L}$ is denoted by $\leqslant$. Throughout this work, $\mathbf{L}$ denotes an arbitrary complete residuated lattice.

Elements a of L are called truth degrees. Operations $\otimes$ (multiplication) and $\rightarrow$ (residuum) play the role of (truth functions of) "fuzzy conjunction" and "fuzzy implication". Furthermore, we define the complement of $a \in L$ as

$$\neg a = a \rightarrow 0 \tag{1}$$

An $\mathbf{L}$-set (or fuzzy set) A in a universe set X is a mapping assigning to each $x \in X$ some truth degree $A(x) \in L$ where L is a support of a complete residuated lattice. The set of all $\mathbf{L}$-sets in a universe X is denoted L^X, or $\mathbf{L}^X$ if the structure of $\mathbf{L}$ is to be emphasized.

The operations with $\mathbf{L}$-sets are defined componentwise. For instance, the intersection of $\mathbf{L}$-sets $A, B \in L^X$ is an $\mathbf{L}$-set $A \cap B$ in X such that $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in X$, etc. An $\mathbf{L}$-set $A \in L^X$ is also denoted $\{{}^{A(x)}\!/x \mid x \in X\}$. If for all $y \in X$ distinct from $x_1, x_2, \ldots, x_n$ we have $A(y) = 0$, we also write

$$\{{}^{A(x_1)}\!/x_1, {}^{A(x_2)}\!/x_1, \ldots, {}^{A(x_n)}\!/x_n\}.$$

An $\mathbf{L}$-set $A \in L^X$ is called crisp if $A(x) \in \{0, 1\}$ for each $x \in X$. Crisp $\mathbf{L}$-sets can be identified with ordinary sets. For a crisp A, we also write $x \in A$ for $A(x) = 1$ and $x \notin A$ for $A(x) = 0$. An $\mathbf{L}$-set $A \in L^X$ is called empty (denoted by $\varnothing$) if $A(x) = 0$ for each $x \in X$. For $a \in L$ and $A \in L^X$, the $\mathbf{L}$-sets $a \otimes A \in L^X$, $a \rightarrow A$,$A \rightarrow a$, and $\neg A$ in X are defined by

$$(a \otimes A)(x) = a \otimes A(x), \tag{2}$$
$$(a \rightarrow A)(x) = a \rightarrow A(x), \tag{3}$$
$$(A \rightarrow a)(x) = A(x) \rightarrow a, \tag{4}$$
$$\neg A(x) = A(x) \rightarrow 0. \tag{5}$$

An **L**-set $A \in L^X$ is called an a-complement if $A = a \to B$ for some $B \in L^X$.

Binary **L**-relations (binary fuzzy relations) between X and Y can be thought of as **L**-sets in the universe $X \times Y$. That is, a binary **L**-relation $I \in L^{X \times Y}$ between a set X and a set Y is a mapping assigning to each $x \in X$ and each $y \in Y$ a truth degree $I(x, y) \in L$ (a degree to which x and y are related by I).

The composition operators are defined by

$$(A \circ B)(x, y) = \bigvee_{f \in F} A(x, f) \otimes B(f, y), \tag{6}$$

$$(A \triangleleft B)(x, y) = \bigwedge_{f \in F} A(x, f) \to B(f, y), \tag{7}$$

$$(A \triangleright B)(x, y) = \bigwedge_{f \in F} B(f, y) \to A(x, f). \tag{8}$$

An **L**-context is a triplet $\langle X, Y, I\rangle$ where X and Y are (ordinary) sets and $I \in L^{X \times Y}$ is an **L**-relation between X and Y. Elements of X are called objects, elements of Y are called attributes, I is called an incidence relation. $I(x, y) = a$ is read: "The object x has the attribute y to degree a."

Consider the following pairs of operators induced by an **L**-context $\langle X, Y, I\rangle$. First, the pair $\langle \uparrow, \downarrow\rangle$ of operators ${}^{\uparrow} : L^X \to L^Y$ and ${}^{\downarrow} : L^Y \to L^X$ is defined by

$$C^{\uparrow}(y) = \bigwedge_{x \in X} C(x) \to I(x, y), \quad D^{\downarrow}(x) = \bigwedge_{y \in Y} D(y) \to I(x, y). \tag{9}$$

Second, the pair $\langle \cap, \cup\rangle$ of operators ${}^{\cap} : L^X \to L^Y$ and ${}^{\cup} : L^Y \to L^X$ is defined by

$$C^{\cap}(y) = \bigvee_{x \in X} C(x) \otimes I(x, y), \quad D^{\cup}(x) = \bigwedge_{y \in Y} I(x, y) \to D(y), \tag{10}$$

Third, the pair $\langle \wedge, \vee\rangle$ of operators ${}^{\wedge} : L^X \to L^Y$ and ${}^{\vee} : L^Y \to L^X$ is defined by

$$C^{\wedge}(y) = \bigwedge_{x \in X} I(x, y) \to C(x), \quad D^{\vee}(x) = \bigvee_{y \in Y} D(y) \otimes I(x, y), \tag{11}$$

for $C \in L^X$, $D \in L^Y$.

To emphasize that the operators are induced by I, we also denote the operators by $\langle \uparrow_I, \downarrow_I\rangle$, $\langle \cap_I, \cup_I\rangle$, and $\langle \wedge_I, \vee_I\rangle$. Furthermore, denote the corresponding sets of fixpoints by $\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I)$, $\mathcal{B}(X^{\cap}, Y^{\cup}, I)$, and $\mathcal{B}(X^{\wedge}, Y^{\vee}, I)$, i.e.

$$\begin{aligned}
\mathcal{B}(X^{\uparrow}, Y^{\downarrow}, I) &= \{\langle C, D\rangle \in L^X \times L^Y \mid C^{\uparrow} = D,\, D^{\downarrow} = C\},\\
\mathcal{B}(X^{\cap}, Y^{\cup}, I) &= \{\langle C, D\rangle \in L^X \times L^Y \mid C^{\cap} = D,\, D^{\cup} = C\},\\
\mathcal{B}(X^{\wedge}, Y^{\vee}, I) &= \{\langle C, D\rangle \in L^X \times L^Y \mid C^{\wedge} = D,\, D^{\vee} = C\}.
\end{aligned}$$

The sets of fixpoints are complete lattices, called **L**-concept lattices associated to I, and their elements are called formal concepts.

For a concept lattice $\mathcal{B}(X^{\vartriangle}, Y^{\triangledown}, I)$, where $\langle \vartriangle, \triangledown \rangle$ is either of $\langle \uparrow, \downarrow \rangle$, $\langle \cap, \cup \rangle$, or $\langle \wedge, \vee \rangle$, denote the corresponding sets of extents and intents by $\mathrm{Ext}(X^{\vartriangle}, Y^{\triangledown}, I)$ and $\mathrm{Int}(X^{\vartriangle}, Y^{\triangledown}, I)$. That is,

$$\mathrm{Ext}(X^{\vartriangle}, Y^{\triangledown}, I) = \{C \in L^X \mid \langle C, D \rangle \in \mathcal{B}(X^{\vartriangle}, Y^{\triangledown}, I) \text{ for some } D\},$$
$$\mathrm{Int}(X^{\vartriangle}, Y^{\triangledown}, I) = \{D \in L^Y \mid \langle C, D \rangle \in \mathcal{B}(X^{\vartriangle}, Y^{\triangledown}, I) \text{ for some } C\},$$

A system of **L**-sets $V \subseteq L^X$ is called an **L**-*interior system* if

- V is closed under $\otimes$-multiplication, i.e. for every $a \in L$ and $C \in V$ we have $a \otimes C \in V$;
- V is closed under union, i.e. for $C_j \in V$ $(j \in J)$ we have $\bigcup_{j \in J} C_j \in V$.

$V \subseteq L^X$ is called an **L**-*closure system* if

- V is closed under left $\rightarrow$-multiplication (or $\rightarrow$-shift), i.e. for every $a \in L$ and $C \in V$ we have $a \rightarrow C \in V$ (here, $a \rightarrow C$ is defined by $(a \rightarrow C)(i) = a \rightarrow C(i)$ for $i = 1, \ldots, n$);
- V is closed under intersection, i.e. for $C_j \in V$ $(j \in J)$ we have $\bigcap_{j \in J} C_j \in V$.

3 Weak L-bonds

This section introduces some new notions studied in this work. To begin with, we introduce the notion of weak **L**-bonds as a convenient generalization of bond.

Definition 1. *A weak* **L**-*bond between two* **L**-*contexts* $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle$ *and* $\mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$ *w.r.t.* $\langle \cap, \cup \rangle$ *is an* **L**-*relation* $\beta \in L^{X_1 \times Y_2}$ *s.t.*

$$\mathrm{Ext}(X_1^{\cap}, Y_2^{\cup}, \beta) \subseteq \mathrm{Ext}(X_1^{\cap}, Y_1^{\cup}, I_1) \quad \textit{and} \quad \mathrm{Int}(X_1^{\cap}, Y_2^{\cup}, \beta) \subseteq \mathrm{Int}(X_2^{\cap}, Y_2^{\cup}, I_2).$$

This notion can be put in relation with that of i-morphism.

Definition 2. *A mapping* $h : V \rightarrow W$ *from an* **L**-*interior system* $V \subseteq L^X$ *into an* **L**-*interior system* $W \subseteq L^Y$ *is called an* i-morphism *if it is a* $\otimes$- *and* $\bigvee$-*morphism, i.e. if*

- $h(a \otimes C) = a \otimes h(C)$ *for each* $a \in L$ *and* $C \in V$*;*
- $h(\bigvee_{k \in K} C_k) = \bigvee_{k \in K} h(C_k)$ *for every collection of* $C_k \in V$ $(k \in K)$*.*

An i-morphism $h : V \rightarrow W$ *is said to be an* extendable *i*-morphism *if* h *can be extended to an i-morphism of* L^X *to* L^Y*, i.e. if there exists an i-morphism* $h' \colon L^X \rightarrow L^Y$ *such that for every* $C \in V$ *we have* $h'(C) = h(C)$*;*

The following results will be used hereafter.

Lemma 1 ([5]).

1. *For* $V \subseteq L^X$*, if* $h : V \rightarrow L^Y$ *is an extendable i-morphism then there exists an* **L**-*relation* $A \in L^{X \times Y}$ *such that* $h(C) = C \circ A$ *for every* $C \in L^Y$*.*

2. *Let $A \in L^{X \times Y}$, the mapping $h_A : L^X \to L^Y$ defined by $h_A(C) = C \circ A = C^{\cap_A}$ is an extendable i-morphism.*
3. *Consider two contexts $\langle X, Y, I \rangle$ and $\langle F, Y, B \rangle$. Then, we have* $\mathrm{Int}(X^\cap, Y^\cup, I) \subseteq \mathrm{Int}(F^\cap, Y^\cup, B)$ *if and only if there exists $A \in L^{X \times F}$ such that $I = A \circ B$,*
4. *Consider two contexts $\langle X, Y, I \rangle$ and $\langle X, F, A \rangle$. Then, we have* $\mathrm{Ext}(X^\cap, Y^\cup, I) \subseteq \mathrm{Ext}(X^\cap, F^\cup, A)$ *if and only if there exists $B \in L^{F \times Y}$ such that $I = A \circ B$.*

Theorem 1. *The weak* **L***-bonds between $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle$ and $\mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$ are in one-to-one correspondence with extendable i-morphisms* $\mathrm{Int}(X_1^\cap, Y_1^\cup, I_1)$ *to* $\mathrm{Int}(X_2^\cap, Y_2^\cup, I_2)$.

Proof. We show procedures to obtain the i-morphism from a weak **L**-bond and vice versa.

"⇒": Let β be a weak **L**-bond between $\mathbb{K}_1 = \langle X_1, Y_1, I_1 \rangle$ and $\mathbb{K}_2 = \langle X_2, Y_2, I_2 \rangle$. By Definition 1 we have $\mathrm{Int}(X_1^\cap, Y_2^\cup, \beta) \subseteq \mathrm{Int}(X_2^\cap, Y_2^\cup, I_2)$; thus by Lemma 1(3) there exists $S_\mathrm{i} \in L^{Y_1 \times Y_2}$ such that $\beta = I_1 \circ S_\mathrm{i}$. The induced opeator $\cap_{S_\mathrm{i}}$ is an extendable i-morphism $\mathrm{Int}(X_1^\cap, Y_1^\cup, I_1)$ to $\mathrm{Int}(X_2^\cap, Y_2^\cup, I_2)$ by Lemma 1(2).

"⇐": For an extendable i-morphism $f : \mathrm{Int}(X_1^\cap, Y_1^\cup, I_1) \to \mathrm{Int}(X_2^\cap, Y_2^\cup, I_2)$ there is an **L**-relation S_i s.t. $f(B) = B^{\cap_{S_\mathrm{i}}}$ for each $B \in \mathrm{Int}(X_1^\cap, Y_1^\cup, I_1)$ by Lemma 1(1). Then $\beta = I_1 \circ S_\mathrm{i}$ is a weak **L**-bond by Lemma 1(3) and Lemma 1(4).

One can check that these two procedures are mutually inverse. □

Now, consider **L**-bonds w.r.t. $\langle \wedge, \vee \rangle$ defined similarly as in Definition 1, i.e. an **L**-relation $\beta \in L^{X_1 \times X_2}$ s.t.

$$\mathrm{Ext}(X_1^\wedge, Y_2^\vee, \beta) \subseteq \mathrm{Ext}(X_1^\wedge, Y_1^\vee, I_1) \quad \text{and} \quad \mathrm{Int}(X_1^\wedge, Y_2^\vee, \beta) \subseteq \mathrm{Int}(X_2^\wedge, Y_2^\vee, I_2).$$

Note that the weak **L**-bonds w.r.t. $\langle \cap, \cup \rangle$ are different from **L**-bonds w.r.t. $\langle \wedge, \vee \rangle$ as the following example shows.

Example 1. Consider L a finite chain containing $a < b$ with $\otimes$ defined as follows:

$$x \otimes y = \begin{cases} x \wedge y & \text{if } x = 1 \text{ or } y = 1, \\ 0 & \text{otherwise,} \end{cases}$$

for each $x, y \in L$. One can easily see that $x \otimes \bigvee_j y_j = \bigvee_j (x \otimes y_j)$ and thus an adjoint operation $\to$ exists such that $\langle L, \wedge, \vee, \otimes, \to, 0, 1 \rangle$ is a complete residuated lattice. Namely, $\to$ is given as follows:

$$x \to y = \begin{cases} 1 & \text{if } x \leqslant y, \\ y & \text{if } x = 1, \\ b & \text{otherwise,} \end{cases}$$

for each $x, y \in L$. Consider $I_1 = \big(a\big)$ and $I_2 = \big(b\big)$. One can check that, we have $\mathrm{Ext}(\{x\}^\cap, \{y\}^\cup, I_1) = \mathrm{Ext}(\{x\}^\cap, \{y\}^\cup, I_2) = \{\{{}^b\!/x\}, x\}$ and, trivially, $\mathrm{Int}(\{x\}^\cap, \{y\}^\cup, I_2) = \mathrm{Int}(\{x\}^\cap, \{y\}^\cup, I_2)$. Thus I_2 is a weak **L**-bond between I_1 and I_2 w.r.t. $\langle \cap, \cup \rangle$.

On the other hand, I_2 is not a weak **L**-bond between I_1 and I_2 w.r.t. $\langle \wedge, \vee \rangle$ since $\mathrm{Ext}(\{x\}^\wedge, \{y\}^\vee, I_1) = \{\varnothing, \{{}^a\!/x\}\} \nsupseteq \{\varnothing, \{{}^b\!/x\}\} = \mathrm{Ext}(\{x\}^\wedge, \{y\}^\vee, I_2)$.

Theorem 2. *The system of all weak* **L**-*bonds is an* **L**-*interior system.*

Proof. From properties of i-morphism. □

4 Strong L-bonds

We provide a stronger version of the previously studied weak **L**-bond, naturally named strong **L**-bond.

Definition 3. *A strong* **L**-*bond between two* **L**-*contexts* $\mathbb{K}_1 = \langle X_1, Y_1, I_1\rangle$ *and* $\mathbb{K}_2 = \langle X_2, Y_2, I_2\rangle$ *is an* **L**-*relation* $\beta \in L^{X_1 \times Y_2}$ *s.t.* β *is a weak* **L**-*bond w.r.t. both* $\langle \cap, \cup \rangle$ *and* $\langle \wedge, \vee \rangle$.

The following lemma introduces equivalent definitions of strong **L**-bonds.

Lemma 2. *The following propositions are equivalent:*

(a) β *is a strong* **L**-*bond between* $\mathbb{K}_1 = \langle X_1, Y_1, I_1\rangle$ *and* $\mathbb{K}_2 = \langle X_2, Y_2, I_2\rangle$.
(b) β *satisfies both* $\mathrm{Ext}(X_1^{\wedge}, Y_2^{\vee}, \beta) \subseteq \mathrm{Ext}(X_1^{\wedge}, Y_1^{\vee}, I_1)$ *and* $\mathrm{Int}(X_1^{\cap}, Y_2^{\cup}, \beta) \subseteq \mathrm{Int}(X_2^{\cap}, Y_2^{\cup}, I_2)$.
(c) $\beta = S_{\mathrm{e}} \circ I_2 = I_1 \circ S_{\mathrm{i}}$ *for some* $S_{\mathrm{e}} \in L^{X_1 \times X_2}$ *and* $S_{\mathrm{i}} \in L^{Y_1 \times Y_2}$.

Proof. (a) ⇔ (b): By use of the Lemma 1(3) and (4).
(a) ⇔ (c): By definitions. □

In this case, this stronger notion can be related with the c-morphisms, introduced below:

Definition 4. *A mapping* $h : V \to W$ *from a* **L**-*closure system* $V \subseteq L^X$ *into an* **L**-*closure system* $W \subseteq L^Y$ *is called a* c-morphism *if it is a* $\to$*- and* $\bigwedge$*-morphism, i.e. if*

- $h(a \to C) = a \to h(C)$ *for each* $a \in L$ *and* $C \in V$;
- $h(\bigwedge_{k\in K} C_k) = \bigwedge_{k\in K} h(C_k)$ *for every collection of* $C_k \in V$ $(k \in K)$;
- *if* C *is an a-complement then* $h(C)$ *is an a-complement.*

For formally establishing the relationship, the two following results are recalled:

Lemma 3 ([4]).

1. *If* $h : V \to L^Y$ *is an extendable c-morphism then there exists an* **L**-*relation* $A \in L^{X \times Y}$ *such that* $h(C) = C \triangleright A$ *for every* $C \in L^Y$.
2. *Let* $A \in L^{X\times Y}$, *the mapping* $h_A : L^X \to L^Y$ *defined by* $h_A(C) = C \triangleright A \quad (= C^{\wedge_A})$ *is an extendable c-morphism.*

Theorem 3. *The strong* **L**-*bonds between* $\mathbb{K}_1 = \langle X_1, Y_1, I_1\rangle$ *and* $\mathbb{K}_2 = \langle X_2, Y_2, I_2\rangle$ *are in one-to-one correspondence with extendable c-morphisms* $\mathrm{Ext}(X_2^{\cap}, Y_2^{\cup}, I_2)$ *to* $\mathrm{Ext}(X_1^{\cap}, Y_1^{\cup}, I_1)$.

Proof. We show procedures to obtain the c-morphism from a strong **L**-bond and vice versa.

"$\Rightarrow$": Let β be a strong **L**-bond between $\mathbb{K}_1 = \langle X_1, Y_1, I_1\rangle$ and $\mathbb{K}_2 = \langle X_2, Y_2, I_2\rangle$. By Lemma 2 there is $S_\mathrm{e} \in L^{X_1 \times X_2}$ such that $\beta = S_\mathrm{i} \circ I_2$. The induced operator $\cup_{S_\mathrm{i}}$ is an extendable c-morphism $\mathrm{Ext}(X_2^\cap, Y_2^\cup, I_2)$ to $\mathrm{Ext}(X_1^\cap, Y_1^\cup, I_1)$ by Lemma 3(2).

"$\Leftarrow$": For extendable c-morphism $f : \mathrm{Ext}(X_2^\uparrow, Y_2^\downarrow, I_2) \to \mathrm{Ext}(X_1^\cap, Y_1^\cup, I_1)$ there is an **L**-relation S_e s.t. $f(B) = B^{\cup S_\mathrm{i}}$ for each $A \in \mathrm{Ext}(X_2^\cap, Y_2^\cup, I_2)$ by Lemma 3(1). Then $\beta = S_\mathrm{e} \circ I_2$ is a strong **L**-bond by Lemma 2.

One can check that these two procedures are mutually inverse. ◻

Theorem 4. *The system of all strong* **L**-*bonds is an* **L**-*interior system.*

Proof. Using Lemma 2 (b), it is an intersection of the **L**-interior systems from Theorem 2. ◻

Definition 5. *Let* $\mathbb{K}_1 = \langle X_1, Y_1, I_1\rangle, \mathbb{K}_2 = \langle X_2, Y_2, I_2\rangle$ *be* **L**-*contexts. The direct product of* $\mathbb{K}_1$ *and* $\mathbb{K}_2$ *is defined as the* **L**-*context* $\mathbb{K}_1 \boxplus \mathbb{K}_2 = \langle X_2 \times Y_1, X_1 \times Y_2, \Delta\rangle$ *with* $\Delta(\langle x_2, y_1\rangle, \langle x_1, y_2\rangle) = I_1(x_1, y_1) \otimes I_2(x_2, y_2)$.

Theorem 5. *The intents of* $\mathbb{K}_1 \boxplus \mathbb{K}_2$ *are* **L**-*bonds between* $\mathbb{K}_1$ *and* $\mathbb{K}_2$.

Proof. We have

$$\begin{aligned}
\phi^\cap(x_1, y_2) &= \bigvee \phi(x_2, y_1) \otimes \Delta(\langle x_2, y_1\rangle, \langle x_1, y_2\rangle) \\
&= \bigvee_{\langle x_2, y_1\rangle} \phi(x_2, y_1) \otimes I_1(x_1, y_1) \otimes I_2(x_2, y_2) \\
&= \bigvee_{y_1 \in Y_1} \bigvee_{x_2 \in X_2} \phi(x_2, y_1) \otimes I_1(x_1, y_1) \otimes I_2(x_2, y_2) \\
&= \bigvee_{y_1 \in Y_1} I_1(x_1, y_1) \otimes \bigvee_{x_2 \in X_2} \phi(x_2, y_1) \otimes I_2(x_2, y_2) \\
&= \bigvee_{y_1 \in Y_1} I_1(x_1, y_1) \otimes (\phi^T \circ I_2)(y_1, y_2) \\
&= (I_1 \circ \phi^T \circ I_2)(x_1, y_2).
\end{aligned}$$

Now, notice that $(I_1 \circ \phi^T) \circ I_2 = I_1 \circ (\phi^T \circ I_2) = \beta$ is a strong **L**-bond by Lemma 2. ◻

Remark 1. It is worth mentioning that not every strong **L**-bond is included in $\mathrm{Int}((X_1 \times Y_2)^\cap, (X_2 \times Y_1)^\cup, \Delta)$ since there are isotone **L**-bonds which are not of the form of $I_1 \circ \phi^T \circ I_2$. For instance, using the same structure of truth degrees and I_1 as in Example 1, obviously I_1 is **L**-bond on $\mathbb{K}_1$ (i.e. between $\mathbb{K}_1$ and $\mathbb{K}_1$), but $\mathrm{Int}((X_1 \times Y_2)^\cap, (X_2 \times Y_1)^\cup, \Delta)$ contains only $\emptyset$.

The end of proof of the Theorem 5 also explains which **L**-bonds are intents of $\mathbb{K}_1 \boxplus \mathbb{K}_2$:

Corollary 1. *The intents of $\mathbb{K}_1 \boxplus \mathbb{K}_2$ are exactly those* **L***-bonds between $\mathbb{K}_1$ and $\mathbb{K}_2$ which can be decomposed as $I_1 \circ \phi^T \circ I_2$ for some $\phi \in L^{X_2 \times Y_1}$.*

Remark 2 (Relationship to the antitone case in [12]).

Assuming the double negation law, we have the equality $\mathrm{Ext}(X^{\uparrow}, Y^{\downarrow}, I) = \mathrm{Ext}(X^{\cap}, Y^{\cup}, \neg I)$. Thus, for a strong **L**-bond $\beta \in L^{X_1 \times Y_2} = S_{\mathrm{e}} \circ I_2 = I_1 \circ S_{\mathrm{i}}$ between $\mathbb{K}_1$ and $\mathbb{K}_2$ we have $\neg\beta = S_{\mathrm{e}} \triangleleft \neg I_2 = \neg I_1 \triangleright S_{\mathrm{i}}$ being an antitone **L**-bond between $\neg\mathbb{K}_1$ and $\neg\mathbb{K}_2$.

Remark 3. Some papers [9, 12] have considered direct products in the crisp and the fuzzy settings, respectively, for the antitone case. In [12] conditions are specified under which antitone **L**-bonds are present in the concept lattice of the direct product. Corollary 1 and Remark 2 provide a simplification of these conditions. The concept lattice of a direct product $\mathbb{K}_1 \boxtimes \mathbb{K}_2$ defined as in [12] i.e. $\mathbb{K}_1 \boxtimes \mathbb{K}_2 = \langle X_2 \times Y_1, X_1 \times Y_2, \Delta \rangle$ with

$$\Delta(\langle x_2, y_1 \rangle, \langle x_1, y_2 \rangle) = \neg I_1(x_1, y_1) \to I_2(x_2, y_2) \quad (= \neg I_2(x_2, y_2) \to I_1(x_1, y_1))$$

induces concept-forming operator $\phi^{\uparrow_\Delta}$ for which we have

$$\begin{aligned}
\phi^{\uparrow_\Delta}(x_1, y_2) &= \bigwedge_{\langle x_1, y_2 \rangle \in X_1 \times Y_2} \phi(x_2, y_1) \to [\neg I_1(x_1, y_1) \to I_2(x_2, y_2)] \\
&= \bigwedge_{\langle x_2, y_1 \rangle \in X_1 \times Y_2} \neg I_1(x_1, y_1) \to (\phi(x_2, y_1) \to I_2(x_2, y_2)) \\
&= \bigwedge_{x_2 \in X_2} \bigwedge_{y_1 \in Y_1} \neg I_1(x_1, y_1) \to (\phi(x_2, y_1) \to I_2(x_2, y_2)) \\
&= \bigwedge_{y_1 \in Y_1} \neg I_1(x_1, y_1) \to \bigwedge_{x_2 \in X_2} (\phi(x_2, y_1) \to I_2(x_2, y_2)) \\
&= \bigwedge_{y_1 \in Y_1} \neg I_1(x_1, y_1) \to (\phi^T \triangleleft I_2)(y_1, y_2) \\
&= \bigwedge_{y_1 \in Y_1} \neg(\phi^T \triangleleft I_2)(y_1, y_2) \to I_1(x_1, y_1) \\
&= [I_1 \triangleright \neg(\phi^T \triangleleft I_2)](x_1, y_2) \\
&= [\neg I_1 \triangleleft (\phi^T \triangleleft I_2)](x_1, y_2) \\
&= [\neg I_1 \triangleleft (\phi^T \triangleleft I_2)](x_1, y_2) \\
&= [(\neg I_1 \circ \phi^T) \triangleleft I_2)](x_1, y_2) \\
&= [\neg(\neg I_1 \circ \phi^T \circ \neg I_2)](x_1, y_2)
\end{aligned}$$

Whence an antitone **L**-bond is an intent of the concept lattice of $\mathbb{K}_1 \boxtimes \mathbb{K}_2$ iff it is possible to write it as $\neg(\neg I_1 \circ \phi^T \circ \neg I_2)$ i.e. if its complement is an intent of $\neg\mathbb{K}_1 \boxplus \neg\mathbb{K}_2$.

5 Conclusions and future work

We studied **L**-bonds with respect to isotone concept-forming operators, computation of **L**-bonds using dirrect products, and the relationship of these results to the previous results on antitone **L**-bonds.

The present results can be easily generalized to a setting in which extents, intents and the context relation use different structures of truth-degrees. We will bring this generalization in an extended version of the paper.

Our future research in this area includes the the study of yet another type of (extendable) i-morphisms and c-morphisms. In [11], another type of morphism is described: the *a-morphism*. In contrast to the morphisms used in this paper, the a-morphisms are antitone, and their study could shed more light on antitone fuzzy bonds.

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