Comparing the Expressiveness of Description Logics

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Abstract. This work studies the expressiveness of the description logics that extend $\mathcal{ALC}_{reg}$ (a variant of PDL) with any combination of the features: inverse roles, nominals, quantified number restrictions, the universal role, the concept constructor for expressing the local reflexivity of a role. We compare the expressiveness of these description logics w.r.t. concepts, positive concepts, TBoxes and ABoxes. Our results about separating the expressiveness of description logics are based on bisimulations and bisimulation-based comparisons. They are naturally extended to the case when instead of $\mathcal{ALC}_{reg}$ we have any sublogic of $\mathcal{ALC}_{reg}$ that extends $\mathcal{ALC}$.

1 Introduction

Expressiveness (expressive power) is a topic studied in the fields of formal languages, databases and logics. The Chomsky hierarchy provides fundamental results on the expressiveness of formal languages. In the field of databases, the works by Fagin [11, 12], Immerman [15, 16], Abiteboul and Vianu [1] provide important results on the expressiveness of query languages. Many results on the expressiveness of logics have also been obtained, e.g. in [13, 15, 3, 17, 22, 18, 23].

The expressiveness of description logics (DLs) has been studied in a number of works [2, 5, 6, 19, 20]. In [2] Baader proposed a formal definition of the expressive power of DLs. His definition is liberal in that it allows the compared logics to have different vocabularies. His work provides separation results for some early DLs. In [5] Borgida showed that certain DLs have the same expressiveness as the two or three variable fragment of first-order logic. The class of DLs considered in [5] is large, but the results only concern DLs without the reflexive and transitive closure of roles. In [6] Cadoli et al. considered the expressiveness of hybrid knowledge bases that combine a DL knowledge base with Horn rules. The used DL is $\mathcal{ALCNR}$. The work [19] by Kurtonina and de Rijke is a comprehensive work on the expressiveness of DLs that are sublogics of $\mathcal{ALCNR}$. It is based on bisimulation and provides many interesting results. In [20] Lutz et al. characterized the expressiveness and rewritability of DL TBoxes for the DLs that are sublogics of $\mathcal{ALCQIO}$. They used semantic notions such as bisimulation, equisimulation, disjoint union and direct product.

This work studies the expressiveness of the DLs that extend $\mathcal{ALC}_{reg}$ (a variant of PDL) with any combination of the features: inverse roles, nominals, quantified number restrictions, the universal role, the concept constructor for expressing the local reflexivity of a role. We compare the expressiveness of these DLs w.r.t. concepts, positive concepts, TBoxes and ABoxes. Our results about separating the expressiveness of DLs are based on bisimulations and bisimulation-based comparisons studied in our joint works [8–10]. They are naturally extended to the case when instead of $\mathcal{ALC}_{reg}$ we have any sublogic of $\mathcal{ALC}_{reg}$ that extends $\mathcal{ALC}$.

Our work differs significantly from all of [2, 5, 6, 19, 20], as the class of considered DLs is much larger than the ones considered in those works (we allow PDL-like role constructors as well as the universal role and the concept constructor $\exists r.\mathtt{Self}$) and our results about separating the expressiveness of DLs are obtained not only w.r.t. concepts and TBoxes but also w.r.t. positive concepts and ABoxes.

The rest of this paper is structured as follows. In Section 2 we recall notation and semantics of DLs, including the notion of positive concepts [10]. In Section 3 we recall the definitions of bisimulations and bisimulation-based comparisons as well as some invariance or preservation results of [8–10]. In Section 4 we present our results on the expressiveness of DLs. Section 5 concludes this work.

2 Notation and Semantics of Description Logics

Our languages use a finite set Σ_C of *concept names* (atomic concepts), a finite set Σ_R of *role names* (atomic roles), and a finite set Σ_I of *individual names*. Let $\Sigma = \Sigma_C \cup \Sigma_R \cup \Sigma_I$. We denote concept names by letters like A and B, role names by letters like r and s, and individual names by letters like a and b.

We consider some *DL-features* denoted by I (*inverse*), O (*nominal*), Q (*quantified number restriction*), U (*universal role*), $\mathtt{Self}$ (the local reflexivity of a role). A *set of DL-features* is a set consisting of some or zero of these names. We sometimes abbreviate sets of DL-features, writing, e.g., IOQ instead of $\{I, O, Q\}$. From now on, if not stated otherwise, let Φ be any set of DL-features and let $\mathcal{L}$ stand for $\mathcal{ALC}_{reg}$.

Definition 2.1. The DL language $\mathcal{L}_\Phi$ allows *roles* and *concepts* defined inductively as follows:

- if $r \in \Sigma_R$ then r is a role of $\mathcal{L}_\Phi$
- if $A \in \Sigma_C$ then A is a concept of $\mathcal{L}_\Phi$
- if R and S are roles of $\mathcal{L}_\Phi$ and C is a concept of $\mathcal{L}_\Phi$ then
  - ε, $R \circ S$, $R \sqcup S$, R^* and $C?$ are roles of $\mathcal{L}_\Phi$
  - $\top$, $\bot$, $\neg C$, $C \sqcup D$, $C \sqcap D$, $\exists R.C$ and $\forall R.C$ are concepts of $\mathcal{L}_\Phi$
  - if $I \in \Phi$ then R^- is a role of $\mathcal{L}_\Phi$
  - if $O \in \Phi$ and $a \in \Sigma_I$ then $\{a\}$ is a concept of $\mathcal{L}_\Phi$
  - if $Q \in \Phi$, $r \in \Sigma_R$ and n is a natural number
    then $\geq n\, r.C$ and $\leq n\, r.C$ are concepts of $\mathcal{L}_\Phi$

- if $\{Q, I\} \subseteq \Phi$, $r \in \Sigma_R$ and n is a natural number then $\geq n\, r^-.C$ and $\leq n\, r^-.C$ are concepts of $\mathcal{L}_\Phi$
- if $U \in \Phi$ then U is a role of $\mathcal{L}_\Phi$
- if $\mathtt{Self} \in \Phi$ and $r \in \Sigma_R$ then $\exists r.\mathtt{Self}$ is a concept of $\mathcal{L}_\Phi$. □

The following definition introduces positive concepts of $\mathcal{L}_\Phi$.

Definition 2.2. Let $\mathcal{L}_\Phi^{pos}$ be the smallest set of concepts and $\mathcal{L}_{\Phi,\exists}^{pos}$, $\mathcal{L}_{\Phi,\forall}^{pos}$ be the smallest sets of roles defined recursively as follows:

– if $r \in \Sigma_R$ then r is a role of $\mathcal{L}_{\Phi,\exists}^{pos}$ and $\mathcal{L}_{\Phi,\forall}^{pos}$,
– if $I \in \Phi$ and $r \in \Sigma_R$ then r^- is a role of $\mathcal{L}_{\Phi,\exists}^{pos}$ and $\mathcal{L}_{\Phi,\forall}^{pos}$,
– if R and S are roles of $\mathcal{L}_{\Phi,\exists}^{pos}$ and C is a concept of $\mathcal{L}_\Phi^{pos}$ then ε, $R \circ S$, $R \sqcup S$, R^* and $C?$ are roles of $\mathcal{L}_{\Phi,\exists}^{pos}$,
– if R and S are roles of $\mathcal{L}_{\Phi,\forall}^{pos}$ and C is a concept of $\mathcal{L}_\Phi^{pos}$ then ε, $R \circ S$, $R \sqcup S$, R^* and $(\neg C)?$ are roles of $\mathcal{L}_{\Phi,\forall}^{pos}$,
– if $A \in \Sigma_C$ then A is a concept of $\mathcal{L}_\Phi^{pos}$,
– if $O \in \Phi$ and $a \in \Sigma_I$ then $\{a\}$ is a concept of $\mathcal{L}_\Phi^{pos}$,
– if $\mathtt{Self} \in \Phi$ and $r \in \Sigma_R$ then $\exists r.\mathtt{Self}$ is a concept of $\mathcal{L}_\Phi^{pos}$,
– if C is a concept of $\mathcal{L}_\Phi^{pos}$, R is a role of $\mathcal{L}_{\Phi,\exists}^{pos}$ and S is a role of $\mathcal{L}_{\Phi,\forall}^{pos}$ then
  - $\top$, $C \sqcup D$, $C \sqcap D$, $\exists R.C$ and $\forall S.C$ are concepts of $\mathcal{L}_\Phi^{pos}$,
  - if $Q \in \Phi$, $r \in \Sigma_R$ and n is a natural number then $\geq n\, r.C$ and $\leq n\, r.(\neg C)$ are concepts of $\mathcal{L}_\Phi^{pos}$,
  - if $\{Q, I\} \subseteq \Phi$, $r \in \Sigma_R$ and n is a natural number then $\geq n\, r^-.C$ and $\leq n\, r^-.(\neg C)$ are concepts of $\mathcal{L}_\Phi^{pos}$,
  - if $U \in \Phi$ then $\forall U.C$ and $\exists U.C$ are concepts of $\mathcal{L}_\Phi^{pos}$.

A concept of $\mathcal{L}_\Phi^{pos}$ is called a *positive concept* of $\mathcal{L}_\Phi$. □

We introduce both $\mathcal{L}_{\Phi,\forall}^{pos}$ and $\mathcal{L}_{\Phi,\exists}^{pos}$ due to the test constructor of roles. The concepts $\exists(A?).B$ and $\forall((\neg A)?).B$ are positive concepts. As we will see, they are equivalent to $A \sqcap B$ and $A \sqcup B$, respectively.[1]

If Φ is empty then we abbreviate $\mathcal{L}_\Phi$ by $\mathcal{L}$. We use letters like R and S to denote arbitrary roles, and use letters like C and D to denote arbitrary concepts. We refer to elements of Σ_R also as *atomic roles*. Let $\Sigma_R^\pm = \Sigma_R \cup \{r^- \mid r \in \Sigma_R\}$. From now on, by *basic roles* we refer to elements of $\Sigma_R^\pm$ if the considered language allows inverse roles, and refer to elements of Σ_R otherwise. In general, the language decides whether inverse roles are allowed in the considered context.

An *interpretation* $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$ consists of a non-empty set $\Delta^\mathcal{I}$, called the *domain* of $\mathcal{I}$, and a function $\cdot^\mathcal{I}$, called the *interpretation function* of $\mathcal{I}$, which maps every concept name A to a subset $A^\mathcal{I}$ of $\Delta^\mathcal{I}$, maps every role name r to a binary relation $r^\mathcal{I}$ on $\Delta^\mathcal{I}$, and maps every individual name a to an element $a^\mathcal{I}$ of $\Delta^\mathcal{I}$. The interpretation function $\cdot^\mathcal{I}$ is extended to complex roles and complex concepts as shown in Figure 1, where $\#\Gamma$ stands for the cardinality of the set Γ. We write $C^\mathcal{I}(x)$ to denote $x \in C^\mathcal{I}$, and write $R^\mathcal{I}(x, y)$ to denote $\langle x, y \rangle \in R^\mathcal{I}$.

[1] That the concept $\leq n\, R.(\neg A)$ is positive should not be a surprise, as $\forall R.A$ is equivalent to $\leq 0\, R.(\neg A)$.

$$(R \circ S)^{\mathcal{I}} = R^{\mathcal{I}} \circ S^{\mathcal{I}} \qquad \top^{\mathcal{I}} = \Delta^{\mathcal{I}}$$
$$(R \sqcup S)^{\mathcal{I}} = R^{\mathcal{I}} \cup S^{\mathcal{I}} \qquad \bot^{\mathcal{I}} = \emptyset$$
$$(R^*)^{\mathcal{I}} = (R^{\mathcal{I}})^* \qquad (\neg C)^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$$
$$(C?)^{\mathcal{I}} = \{\langle x, x\rangle \mid C^{\mathcal{I}}(x)\} \qquad (C \sqcup D)^{\mathcal{I}} = C^{\mathcal{I}} \cup D^{\mathcal{I}}$$
$$\varepsilon^{\mathcal{I}} = \{\langle x, x\rangle \mid x \in \Delta^{\mathcal{I}}\} \qquad (C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$$
$$U^{\mathcal{I}} = \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \qquad \{a\}^{\mathcal{I}} = \{a^{\mathcal{I}}\}$$
$$(R^-)^{\mathcal{I}} = (R^{\mathcal{I}})^{-1} \qquad (\exists r.\mathtt{Self})^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid r^{\mathcal{I}}(x, x)\}$$
$$(\exists R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \exists y\,[R^{\mathcal{I}}(x, y) \text{ and } C^{\mathcal{I}}(y)]$$
$$(\forall R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \forall y\,[R^{\mathcal{I}}(x, y) \text{ implies } C^{\mathcal{I}}(y)]\}$$
$$(\geq n\, R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \#\{y \mid R^{\mathcal{I}}(x, y) \text{ and } C^{\mathcal{I}}(y)\} \geq n\}$$
$$(\leq n\, R.C)^{\mathcal{I}} = \{x \in \Delta^{\mathcal{I}} \mid \#\{y \mid R^{\mathcal{I}}(x, y) \text{ and } C^{\mathcal{I}}(y)\} \leq n\}$$

Fig. 1. Interpretation of complex roles and complex concepts.

A *terminological axiom* in $\mathcal{L}_\Phi$, also called a *general concept inclusion* (GCI) in $\mathcal{L}_\Phi$, is an expression of the form $C \sqsubseteq D$, where C and D are concepts of $\mathcal{L}_\Phi$. An interpretation $\mathcal{I}$ *validates* an axiom $C \sqsubseteq D$, denoted by $\mathcal{I} \models C \sqsubseteq D$, if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$.

A *TBox* in $\mathcal{L}_\Phi$ is a finite set of terminological axioms in $\mathcal{L}_\Phi$. An interpretation $\mathcal{I}$ is a *model* of a TBox $\mathcal{T}$, denoted by $\mathcal{I} \models \mathcal{T}$, if it validates all the axioms of $\mathcal{T}$.

An *individual assertion* in $\mathcal{L}_\Phi$ is an expression of one of the forms $C(a)$ (*concept assertion*), $R(a, b)$ (*positive role assertion*), $\neg R(a, b)$ (*negative role assertion*), $a = b$, and $a \neq b$, where C is a concept and R is a role in $\mathcal{L}_\Phi$.

Given an interpretation $\mathcal{I}$, define that:

$$\begin{array}{ll} \mathcal{I} \models a = b & \text{if } a^{\mathcal{I}} = b^{\mathcal{I}}, \\ \mathcal{I} \models a \neq b & \text{if } a^{\mathcal{I}} \neq b^{\mathcal{I}}, \\ \mathcal{I} \models C(a) & \text{if } C^{\mathcal{I}}(a^{\mathcal{I}}) \text{ holds}, \\ \mathcal{I} \models R(a, b) & \text{if } R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \text{ holds}, \\ \mathcal{I} \models \neg R(a, b) & \text{if } R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \text{ does not hold.} \end{array}$$

We say that $\mathcal{I}$ *satisfies* an individual assertion φ if $\mathcal{I} \models \varphi$.

An *ABox* in $\mathcal{L}_\Phi$ is a finite set of individual assertions in $\mathcal{L}_\Phi$. An interpretation $\mathcal{I}$ is a *model* of an ABox $\mathcal{A}$, denoted by $\mathcal{I} \models \mathcal{A}$, if it satisfies all the assertions of $\mathcal{A}$.

3 Bisimulations and Bisimulation-Based Comparisons

Bisimulation is a very useful notion for DLs. It can be used for analyzing expressiveness of DLs (as investigated in [19] and the current paper), minimizing interpretations [8–10] and concept learning in DLs [21, 25, 14, 7, 24, 26]. The following definition comes from our joint works [8–10].

Definition 3.1. Let $\mathcal{I}$ and $\mathcal{I}'$ be interpretations. A binary relation $Z \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}'}$ is called an $\mathcal{L}_\Phi$-*bisimulation* between $\mathcal{I}$ and $\mathcal{I}'$ if the following conditions hold for every $a \in \Sigma_I$, $A \in \Sigma_C$, $r \in \Sigma_R$, $x, y \in \Delta^{\mathcal{I}}$, $x', y' \in \Delta^{\mathcal{I}'}$:

$$Z(a^{\mathcal{I}}, a^{\mathcal{I}'}) \tag{1}$$

$$Z(x, x') \Rightarrow [A^{\mathcal{I}}(x) \Leftrightarrow A^{\mathcal{I}'}(x')] \tag{2}$$

$$[Z(x, x') \land r^{\mathcal{I}}(x, y)] \Rightarrow \exists y' \in \Delta^{\mathcal{I}'}[Z(y, y') \land r^{\mathcal{I}'}(x', y')] \tag{3}$$

$$[Z(x, x') \land r^{\mathcal{I}'}(x', y')] \Rightarrow \exists y \in \Delta^{\mathcal{I}}[Z(y, y') \land r^{\mathcal{I}}(x, y)], \tag{4}$$

if $I \in \Phi$ then

$$[Z(x, x') \land r^{\mathcal{I}}(y, x)] \Rightarrow \exists y' \in \Delta^{\mathcal{I}'}[Z(y, y') \land r^{\mathcal{I}'}(y', x')] \tag{5}$$

$$[Z(x, x') \land r^{\mathcal{I}'}(y', x')] \Rightarrow \exists y \in \Delta^{\mathcal{I}}[Z(y, y') \land r^{\mathcal{I}}(y, x)], \tag{6}$$

if $O \in \Phi$ then

$$Z(x, x') \Rightarrow [x = a^{\mathcal{I}} \Leftrightarrow x' = a^{\mathcal{I}'}], \tag{7}$$

if $Q \in \Phi$ then

if $Z(x, x')$ holds then, for every role name r, there exists a bijection $h : \{y \mid r^{\mathcal{I}}(x, y)\} \to \{y' \mid r^{\mathcal{I}'}(x', y')\}$ such that $h \subseteq Z$, (8)

if $\{Q, I\} \subseteq \Phi$ then (additionally)

if $Z(x, x')$ holds then, for every role name r, there exists a bijection $h : \{y \mid r^{\mathcal{I}}(y, x)\} \to \{y' \mid r^{\mathcal{I}'}(y', x')\}$ such that $h \subseteq Z$, (9)

if $U \in \Phi$ then

$$\forall x \in \Delta^{\mathcal{I}}\, \exists x' \in \Delta^{\mathcal{I}'}\, Z(x, x') \tag{10}$$

$$\forall x' \in \Delta^{\mathcal{I}'}\, \exists x \in \Delta^{\mathcal{I}}\, Z(x, x'), \tag{11}$$

if $\mathtt{Self} \in \Phi$ then

$$Z(x, x') \Rightarrow [r^{\mathcal{I}}(x, x) \Leftrightarrow r^{\mathcal{I}'}(x', x')]. \tag{12}$$

By (2′), (7′) and (12′) we denote the conditions obtained respectively from (2), (7) and (12) by replacing equivalence ($\Leftrightarrow$) by implication ($\Rightarrow$). If the conditions (2), (7) and (12) are replaced by (2′), (7′) and (12′), respectively, then the relation Z is called an $\mathcal{L}_\Phi$-*comparison* between $\mathcal{I}$ and $\mathcal{I}'$ [10]. □

As shown in [4], the PDL-like role constructors are "safe" for the conditions (3)-(6). That is, we need to specify these conditions only for atomic roles, and as a consequence, they also hold for complex roles.

Definition 3.2. A concept C in $\mathcal{L}_\Phi$ is *invariant* for $\mathcal{L}_\Phi$-bisimulation if, for any interpretation $\mathcal{I}, \mathcal{I}'$ and any $\mathcal{L}_\Phi$-bisimulation Z between $\mathcal{I}$ and $\mathcal{I}'$, if $Z(x, x')$ holds then $x \in C^{\mathcal{I}}$ iff $x' \in C^{\mathcal{I}'}$. A TBox $\mathcal{T}$ in $\mathcal{L}_\Phi$ is *invariant* for $\mathcal{L}_\Phi$-bisimulation if, for every interpretations $\mathcal{I}$ and $\mathcal{I}'$, if there exists an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$ then $\mathcal{I}$ is a model of $\mathcal{T}$ iff $\mathcal{I}'$ is a model of $\mathcal{T}$. The notion of whether an ABox in $\mathcal{L}_\Phi$ is *invariant* for $\mathcal{L}_\Phi$-bisimulation is defined similarly. □

Definition 3.3. An interpretation $\mathcal{I}$ is said to be *unreachable-objects-free* (w.r.t. the considered language) if every element of $\Delta^\mathcal{I}$ is reachable from some $a^\mathcal{I}$, where $a \in \Sigma_I$, via a path consisting of edges being instances of basic roles. □

The following theorem comes from our joint work [8].

Theorem 3.4.

1. *All concepts in $\mathcal{L}_\Phi$ are invariant for $\mathcal{L}_\Phi$-bisimulation.*
2. *If $U \in \Phi$ then all TBoxes in $\mathcal{L}_\Phi$ are invariant for $\mathcal{L}_\Phi$-bisimulation.*
3. *Let $\mathcal{T}$ be a TBox in $\mathcal{L}_\Phi$ and $\mathcal{I}$, $\mathcal{I}'$ be unreachable-objects-free interpretations (w.r.t. $\mathcal{L}_\Phi$) such that there exists an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$. Then $\mathcal{I}$ is a model of $\mathcal{T}$ iff $\mathcal{I}'$ is a model of $\mathcal{T}$.*
4. *Let $\mathcal{A}$ be an ABox in $\mathcal{L}_\Phi$. If $O \in \Phi$ or $\mathcal{A}$ contains only assertions of the form $C(a)$ then $\mathcal{A}$ is invariant for $\mathcal{L}_\Phi$-bisimulation.*

Definition 3.5. A concept C of $\mathcal{L}_\Phi$ is *preserved by $\mathcal{L}_\Phi$-comparisons* if, for any interpretations $\mathcal{I}$, $\mathcal{I}'$ and any $\mathcal{L}_\Phi$-comparison Z between $\mathcal{I}$ and $\mathcal{I}'$, if $Z(x, x')$ holds and $x \in C^\mathcal{I}$ then $x' \in C^{\mathcal{I}'}$. □

The following theorem comes from our joint work [10].

Theorem 3.6. *All concepts of $\mathcal{L}_\Phi^{pos}$ are preserved by $\mathcal{L}_\Phi$-comparisons.*

4 Comparing the Expressiveness of Description Logics

Definition 4.1. Two concepts C and D are *equivalent* if, for every interpretation $\mathcal{I}$, $C^\mathcal{I} = D^\mathcal{I}$. Two TBoxes $\mathcal{T}_1$ and $\mathcal{T}_2$ are *equivalent* if, for every interpretation $\mathcal{I}$, $\mathcal{I}$ is a model of $\mathcal{T}_1$ iff $\mathcal{I}$ is a model of $\mathcal{T}_2$. Two ABoxes $\mathcal{A}_1$ and $\mathcal{A}_2$ are *equivalent* if, for every interpretation $\mathcal{I}$, $\mathcal{I}$ is a model of $\mathcal{A}_1$ iff $\mathcal{I}$ is a model of $\mathcal{A}_2$.

Definition 4.2. We say that a logic $\mathcal{L}_1$ is *at most as expressive as* a logic $\mathcal{L}_2$ w.r.t. concepts (resp. positive concepts, TBoxes, ABoxes), denoted by $\mathcal{L}_1 \leq_C \mathcal{L}_2$ (resp. $\mathcal{L}_1 \leq_{PC} \mathcal{L}_2$, $\mathcal{L}_1 \leq_T \mathcal{L}_2$, $\mathcal{L}_1 \leq_A \mathcal{L}_2$), if every concept (resp. positive concept, TBox, ABox) in $\mathcal{L}_1$ has an equivalent concept (resp. positive concept, TBox, ABox) in $\mathcal{L}_2$.

We say that a logic $\mathcal{L}_2$ is *more expressive* than a logic $\mathcal{L}_1$ (or $\mathcal{L}_1$ is *less expressive* than $\mathcal{L}_2$) w.r.t. concepts (resp. positive concepts, TBoxes, ABoxes), denoted by $\mathcal{L}_1 <_C \mathcal{L}_2$ (resp. $\mathcal{L}_1 <_{PC} \mathcal{L}_2$, $\mathcal{L}_1 <_T \mathcal{L}_2$, $\mathcal{L}_1 <_A \mathcal{L}_2$), if $\mathcal{L}_1 \leq_C \mathcal{L}_2$ (resp. $\mathcal{L}_1 \leq_{PC} \mathcal{L}_2$, $\mathcal{L}_1 \leq_T \mathcal{L}_2$, $\mathcal{L}_1 \leq_A \mathcal{L}_2$) and $\mathcal{L}_2 \not\leq_C \mathcal{L}_1$ (resp. $\mathcal{L}_2 \not\leq_{PC} \mathcal{L}_1$, $\mathcal{L}_2 \not\leq_T \mathcal{L}_1$, $\mathcal{L}_2 \not\leq_A \mathcal{L}_1$). □

The following proposition clearly holds.

Proposition 4.3. *If a logic $\mathcal{L}_1$ is at most as expressive as a logic $\mathcal{L}_2$ w.r.t. concepts (resp. positive concepts, TBoxes, ABoxes) and a logic $\mathcal{L}_2$ is at most as expressive as $\mathcal{L}_3$ w.r.t. concepts (resp. positive concepts, TBoxes, ABoxes) then $\mathcal{L}_1$ is at most as expressive as $\mathcal{L}_3$ w.r.t. concepts (resp. positive concepts, TBoxes, ABoxes).*

Lemma 4.4. *Let Φ_1 and Φ_2 be sets of DL-features such that $\Phi_1 \subseteq \Phi_2$. Denote $\mathcal{L}_1 = \mathcal{L}_{\Phi_1}$ and $\mathcal{L}_2 = \mathcal{L}_{\Phi_2}$. Let $\mathcal{I}$, $\mathcal{I}'$ be interpretations and Z an $\mathcal{L}_1$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$.*

1. *If $\mathcal{L}_1 \leq_C \mathcal{L}_2$, $x \in \Delta^{\mathcal{I}}, x' \in \Delta^{\mathcal{I}'}, Z(x,x')$ holds, and there exists a concept C of $\mathcal{L}_2$ such that $x \in C^{\mathcal{I}}$ but $x' \notin C^{\mathcal{I}'}$, then $\mathcal{L}_1 <_C \mathcal{L}_2$.*
2. *Suppose that $U \in \Phi_1$ or both $\mathcal{I}$ and $\mathcal{I}'$ are unreachable-objects-free. If $\mathcal{L}_1 \leq_T \mathcal{L}_2$ and there exists a TBox $\mathcal{T}$ in $\mathcal{L}_2$ such that $\mathcal{I}$ is a model of $\mathcal{T}$ but $\mathcal{I}'$ is not, then $\mathcal{L}_1 <_T \mathcal{L}_2$.*
3. *Suppose $O \in \Phi_1$. If $\mathcal{L}_1 \leq_A \mathcal{L}_2$ and there exists an ABox $\mathcal{A}$ in $\mathcal{L}_2$ such that $\mathcal{I}$ is a model of $\mathcal{A}$ but $\mathcal{I}'$ is not, then $\mathcal{L}_1 <_A \mathcal{L}_2$.*

Proof. Consider the first assertion. Suppose $\mathcal{L}_1 \leq_C \mathcal{L}_2$, $x \in \Delta^{\mathcal{I}}$, $x' \in \Delta^{\mathcal{I}'}$, $Z(x,x')$ holds and there exists a concept C of $\mathcal{L}_2$ such that $x \in C^{\mathcal{I}}$ but $x' \notin C^{\mathcal{I}'}$. We prove that $\mathcal{L}_2 \not\leq_C \mathcal{L}_1$. For the sake of contradiction, suppose $\mathcal{L}_2 \leq_C \mathcal{L}_1$. It follows that there exists a concept C' of $\mathcal{L}_1$ that is equivalent to C. Thus, $x \in C'^{\mathcal{I}}$ but $x' \notin C'^{\mathcal{I}'}$. Hence, C' is not invariant for Z, which contradicts Theorem 3.4(1). Therefore, $\mathcal{L}_1 <_C \mathcal{L}_2$.

Consider the second assertion. Suppose $\mathcal{L}_1 \leq_T \mathcal{L}_2$ and there exists a TBox $\mathcal{T}$ in $\mathcal{L}_2$ such that $\mathcal{I}$ is a model of $\mathcal{T}$ but $\mathcal{I}'$ is not. We prove that $\mathcal{L}_2 \not\leq_T \mathcal{L}_1$. For the sake of contradiction, suppose $\mathcal{L}_2 \leq_T \mathcal{L}_1$. It follows that there exists a TBox $\mathcal{T}'$ in $\mathcal{L}_1$ that is equivalent to $\mathcal{T}$. Thus, $\mathcal{I}$ is a model of $\mathcal{T}'$ but $\mathcal{I}'$ is not, which contradicts Theorem 3.4(2) or Theorem 3.4(3). Therefore, $\mathcal{L}_1 <_T \mathcal{L}_2$.

Consider the third assertion. Suppose $\mathcal{L}_1 \leq_A \mathcal{L}_2$ and there exists an ABox $\mathcal{A}$ in $\mathcal{L}_2$ such that $\mathcal{I}$ is a model of $\mathcal{A}$ but $\mathcal{I}'$ is not. We prove that $\mathcal{L}_2 \not\leq_A \mathcal{L}_1$. For the sake of contradiction, suppose $\mathcal{L}_2 \leq_A \mathcal{L}_1$. It follows that there exists an ABox $\mathcal{A}'$ in $\mathcal{L}_1$ that is equivalent to $\mathcal{A}$. Thus, $\mathcal{I}$ is a model of $\mathcal{A}'$ but $\mathcal{I}'$ is not, which contradicts Theorem 3.4(4). Therefore, $\mathcal{L}_1 <_A \mathcal{L}_2$. □

Lemma 4.5. *Let Φ_1 and Φ_2 be sets of DL-features such that $\Phi_1 \subseteq \Phi_2$. Denote $\mathcal{L}_1 = \mathcal{L}_{\Phi_1}$ and $\mathcal{L}_2 = \mathcal{L}_{\Phi_2}$. Let $\mathcal{I}$, $\mathcal{I}'$ be interpretations and Z an $\mathcal{L}_1$-comparison between $\mathcal{I}$ and $\mathcal{I}'$. If $\mathcal{L}_1 \leq_{PC} \mathcal{L}_2$, $x \in \Delta^{\mathcal{I}}, x' \in \Delta^{\mathcal{I}'}, Z(x,x')$ holds, and there exists a positive concept C of $\mathcal{L}_2$ such that $x \in C^{\mathcal{I}}$ but $x' \notin C^{\mathcal{I}'}$, then $\mathcal{L}_1 <_{PC} \mathcal{L}_2$.*

Proof. Suppose $\mathcal{L}_1 \leq_{PC} \mathcal{L}_2$, $x \in \Delta^{\mathcal{I}}$, $x' \in \Delta^{\mathcal{I}'}$, $Z(x,x')$ holds and there exists a positive concept C of $\mathcal{L}_2$ such that $x \in C^{\mathcal{I}}$ but $x' \notin C^{\mathcal{I}'}$. We prove that $\mathcal{L}_2 \not\leq_{PC} \mathcal{L}_1$. For the sake of contradiction, suppose $\mathcal{L}_2 \leq_{PC} \mathcal{L}_1$. It follows that there exists a positive concept C' of $\mathcal{L}_1$ that is equivalent to C. Thus, $x \in C'^{\mathcal{I}}$ but $x' \notin C'^{\mathcal{I}'}$. It follows that C' is not preserved by Z, which contradicts Theorem 3.6. Hence, $\mathcal{L}_1 <_{PC} \mathcal{L}_2$. □

From now on, we assume that Σ_C and Σ_R are not empty and Σ_I contains at least two individual names. Let $\{a,b\} \subseteq \Sigma_I$, $A \in \Sigma_C$ and $r \in \Sigma_R$.

Lemma 4.6.

1. *For any pair $\langle \mathcal{L}_1, \mathcal{L}_2 \rangle$ among $\langle \mathcal{L}_I, \mathcal{L}_{OQU\mathtt{Self}} \rangle, \langle \mathcal{L}_Q, \mathcal{L}_{IOU\mathtt{Self}} \rangle, \langle \mathcal{L}_{\mathtt{Self}}, \mathcal{L}_{IOQU} \rangle$, we have that: $\mathcal{L}_1 \not\leq_C \mathcal{L}_2$, $\mathcal{L}_1 \not\leq_{PC} \mathcal{L}_2$, $\mathcal{L}_1 \not\leq_T \mathcal{L}_2$, $\mathcal{L}_1 \not\leq_A \mathcal{L}_2$.*

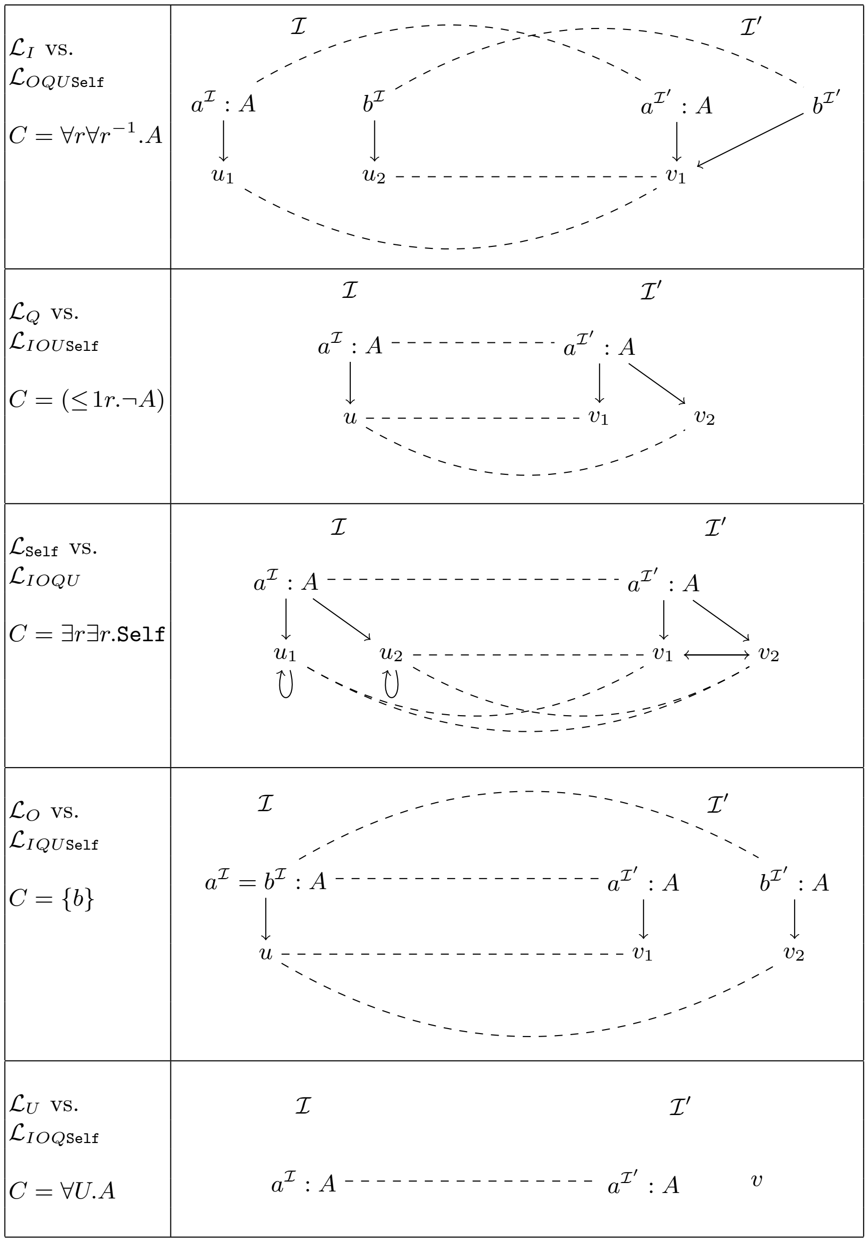

Fig. 2. An illustration for Lemma 4.6.

2. $\mathcal{L}_O \not\leq_C \mathcal{L}_{IQU\texttt{Self}}$, $\mathcal{L}_O \not\leq_{PC} \mathcal{L}_{IQU\texttt{Self}}$, $\mathcal{L}_O \not\leq_T \mathcal{L}_{IQU\texttt{Self}}$.
3. $\mathcal{L}_U \not\leq_C \mathcal{L}_{IOQ\texttt{Self}}$, $\mathcal{L}_U \not\leq_{PC} \mathcal{L}_{IOQ\texttt{Self}}$, $\mathcal{L}_U \not\leq_A \mathcal{L}_{IOQ\texttt{Self}}$.

Proof. Let us compare $\mathcal{L}_I$ with $\mathcal{L}_{OQU\mathtt{Self}}$. Consider the interpretations $\mathcal{I}, \mathcal{I}'$ and the relation Z shown in the first part of Figure 2. The arrows denote the instances of r in $\mathcal{I}$ and $\mathcal{I}'$. The instances of A in $\mathcal{I}$ and $\mathcal{I}'$ are explicitly indicated in the figure. Let $B^{\mathcal{I}} = B^{\mathcal{I}'} = \emptyset$ for all $B \in \Sigma_C \setminus \{A\}$, $s^{\mathcal{I}} = s^{\mathcal{I}'} = \emptyset$ for all $s \in \Sigma_R \setminus \{r\}$, and $c^{\mathcal{I}} = a^{\mathcal{I}}$, $c^{\mathcal{I}'} = a^{\mathcal{I}'}$ for all $c \in \Sigma_I \setminus \{a, b\}$. The dotted lines in the figure indicate the instances of a binary relation $Z \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}'}$. It can be checked that Z is an $\mathcal{L}_{OQU\mathtt{Self}}$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$. Consider the positive concept $C = \forall r \forall r^{-1}.A$ of $\mathcal{L}_I$. Clearly, $a^{\mathcal{I}} \in C^{\mathcal{I}}$ but $a^{\mathcal{I}'} \notin C^{\mathcal{I}'}$. By Theorem 3.4(1), C does not have any equivalent concept in $\mathcal{L}_{OQU\mathtt{Self}}$. Hence, $\mathcal{L}_I \not\leq_C \mathcal{L}_{OQU\mathtt{Self}}$. As Z is also an $\mathcal{L}_{OQU\mathtt{Self}}$-comparison between $\mathcal{I}$ and $\mathcal{I}'$, by Theorem 3.6, C does not have any equivalent positive concept in $\mathcal{L}_{OQU\mathtt{Self}}$ either. Hence, $\mathcal{L}_I \not\leq_{PC} \mathcal{L}_{OQU\mathtt{Self}}$. Consider the TBox $\mathcal{T} = \{A \sqsubseteq C\}$. Since $\mathcal{I} \models \mathcal{T}$ but $\mathcal{I}' \not\models \mathcal{T}$, by Theorem 3.4(3), $\mathcal{T}$ does not have any equivalent TBox in $\mathcal{L}_{OQU\mathtt{Self}}$. Hence $\mathcal{L}_I \not\leq_T \mathcal{L}_{OQU\mathtt{Self}}$. Consider the ABox $\mathcal{A} = \{C(a)\}$. Since $\mathcal{I} \models \mathcal{A}$ but $\mathcal{I}' \not\models \mathcal{A}$, by Theorem 3.4(4), $\mathcal{A}$ does not have any equivalent ABox in $\mathcal{L}_{OQU\mathtt{Self}}$. Hence $\mathcal{L}_I \not\leq_A \mathcal{L}_{OQU\mathtt{Self}}$.

The proofs for the other pairs of logics can be done similarly, using $\mathcal{I}$, $\mathcal{I}'$, C specified in the next parts of Figure 2. For the parts without the presence of b, let $b^{\mathcal{I}} = a^{\mathcal{I}}$ and $b^{\mathcal{I}'} = a^{\mathcal{I}'}$. □

Theorem 4.7. *Let Φ and Φ' be subsets of $\{I, O, Q, U, \mathtt{Self}\}$.*

1. *If $\Phi \subset \Phi'$ then $\mathcal{L}_\Phi <_C \mathcal{L}_{\Phi'}$ and $\mathcal{L}_\Phi <_{PC} \mathcal{L}_{\Phi'}$.*
2. *If $\Phi \not\subseteq \Phi'$ then $\mathcal{L}_\Phi \not\leq_C \mathcal{L}_{\Phi'}$ and $\mathcal{L}_\Phi \not\leq_{PC} \mathcal{L}_{\Phi'}$.*
3. *If $\Phi \subset \Phi'$ and $\Phi' \setminus \Phi \neq \{U\}$ then $\mathcal{L}_\Phi <_T \mathcal{L}_{\Phi'}$.*
4. *If $\Phi \not\subseteq \Phi'$ and $\Phi \setminus \Phi' \neq \{U\}$ then $\mathcal{L}_\Phi \not\leq_T \mathcal{L}_{\Phi'}$.*
5. *If $\Phi \subset \Phi'$ and $\Phi' \setminus \Phi \neq \{O\}$ then $\mathcal{L}_\Phi <_A \mathcal{L}_{\Phi'}$.*
6. *If $\Phi \not\subseteq \Phi'$ and $\Phi \setminus \Phi' \neq \{O\}$ then $\mathcal{L}_\Phi \not\leq_A \mathcal{L}_{\Phi'}$.*

Proof. Consider the first assertion and suppose $\Phi \subset \Phi'$. Since every concept (resp. positive concept) of $\mathcal{L}_\Phi$ is also a concept (resp. positive concept) of $\mathcal{L}_{\Phi'}$, we have that $\mathcal{L}_\Phi \leq_C \mathcal{L}_{\Phi'}$ (resp. $\mathcal{L}_\Phi \leq_{PC} \mathcal{L}_{\Phi'}$). Since $\Phi' \setminus \Phi \neq \emptyset$, at least one feature among I, O, Q, U, $\mathtt{Self}$ belongs to $\Phi' \setminus \Phi$. Consider the case $I \in \Phi' \setminus \Phi$. The cases of other features are similar and omitted. For the sake of contradiction, suppose $\mathcal{L}_{\Phi'} \leq_C \mathcal{L}_\Phi$ (resp. $\mathcal{L}_{\Phi'} \leq_{PC} \mathcal{L}_\Phi$). Since $\mathcal{L}_I \leq_C \mathcal{L}_{\Phi'}$ (resp. $\mathcal{L}_I \leq_{PC} \mathcal{L}_{\Phi'}$) and $\mathcal{L}_\Phi \leq_C \mathcal{L}_{OQU\mathtt{Self}}$ (resp. $\mathcal{L}_\Phi \leq_{PC} \mathcal{L}_{OQU\mathtt{Self}}$), it follows that $\mathcal{L}_I \leq_C \mathcal{L}_{OQU\mathtt{Self}}$ (resp. $\mathcal{L}_I \leq_{PC} \mathcal{L}_{OQU\mathtt{Self}}$), which contradicts Lemma 4.6. Therefore, $\mathcal{L}_\Phi <_C \mathcal{L}_{\Phi'}$ (resp. $\mathcal{L}_\Phi <_{PC} \mathcal{L}_{\Phi'}$).

Consider the second assertion and suppose $\Phi \not\subseteq \Phi'$. Since $\Phi \setminus \Phi' \neq \emptyset$, at least one feature among I, O, Q, U, $\mathtt{Self}$ belongs to $\Phi \setminus \Phi'$. Consider the case $I \in \Phi \setminus \Phi'$. The cases of other features are similar and omitted. For the sake of contradiction, suppose $\mathcal{L}_\Phi \leq_C \mathcal{L}_{\Phi'}$ (resp. $\mathcal{L}_\Phi \leq_{PC} \mathcal{L}_{\Phi'}$). Since $\mathcal{L}_I \leq_C \mathcal{L}_\Phi$ (resp. $\mathcal{L}_I \leq_{PC} \mathcal{L}_\Phi$) and $\mathcal{L}_{\Phi'} \leq_C \mathcal{L}_{OQU\mathtt{Self}}$ (resp. $\mathcal{L}_{\Phi'} \leq_{PC} \mathcal{L}_{OQU\mathtt{Self}}$), it follows that $\mathcal{L}_I \leq_C \mathcal{L}_{OQU\mathtt{Self}}$ (resp. $\mathcal{L}_I \leq_{PC} \mathcal{L}_{OQU\mathtt{Self}}$), which contradicts Lemma 4.6. Therefore, $\mathcal{L}_\Phi \not\leq_C \mathcal{L}_{\Phi'}$ (resp. $\mathcal{L}_\Phi \not\leq_{PC} \mathcal{L}_{\Phi'}$).

Consider the third assertion and suppose $\Phi \subset \Phi'$ and $\Phi' \setminus \Phi \neq \{U\}$. At least one feature among I, O, Q, $\mathtt{Self}$ belongs to $\Phi' \setminus \Phi$. Consider the case $I \in \Phi' \setminus \Phi$. The cases of other features are similar and omitted. Since $\Phi \subset \Phi'$,

$\mathcal{L}_\Phi \leq_T \mathcal{L}_{\Phi'}$. For the sake of contradiction, suppose $\mathcal{L}_{\Phi'} \leq_T \mathcal{L}_\Phi$. Since $\mathcal{L}_I \leq_T \mathcal{L}_{\Phi'}$ and $\mathcal{L}_\Phi \leq_T \mathcal{L}_{OQU\texttt{Self}}$, it follows that $\mathcal{L}_I \leq_T \mathcal{L}_{OQU\texttt{Self}}$, which contradicts Lemma 4.6. Therefore, $\mathcal{L}_\Phi <_T \mathcal{L}_{\Phi'}$.

Consider the fourth assertion and suppose $\Phi \not\subseteq \Phi'$ and $\Phi \setminus \Phi' \neq \{U\}$. At least one feature among I, O, Q, $\texttt{Self}$ belongs to $\Phi \setminus \Phi'$. Consider the case $I \in \Phi \setminus \Phi'$. The cases of other features are similar and omitted. For the sake of contradiction, suppose $\mathcal{L}_\Phi \leq_T \mathcal{L}_{\Phi'}$. Since $\mathcal{L}_I \leq_T \mathcal{L}_\Phi$ and $\mathcal{L}_{\Phi'} \leq_T \mathcal{L}_{OQU\texttt{Self}}$, it follows that $\mathcal{L}_I \leq_T \mathcal{L}_{OQU\texttt{Self}}$, which contradicts Lemma 4.6. Therefore, $\mathcal{L}_\Phi \not\leq_T \mathcal{L}_{\Phi'}$.

Consider the fifth assertion and suppose $\Phi \subset \Phi'$ and $\Phi' \setminus \Phi \neq \{O\}$. At least one feature among I, Q, U, $\texttt{Self}$ belongs to $\Phi' \setminus \Phi$. Consider the case $I \in \Phi' \setminus \Phi$. The cases of other features are similar and omitted. Since $\Phi \subset \Phi'$, $\mathcal{L}_\Phi \leq_A \mathcal{L}_{\Phi'}$. For the sake of contradiction, suppose $\mathcal{L}_{\Phi'} \leq_A \mathcal{L}_\Phi$. Since $\mathcal{L}_I \leq_A \mathcal{L}_{\Phi'}$ and $\mathcal{L}_\Phi \leq_A \mathcal{L}_{OQU\texttt{Self}}$, it follows that $\mathcal{L}_I \leq_A \mathcal{L}_{OQU\texttt{Self}}$, which contradicts Lemma 4.6. Therefore, $\mathcal{L}_\Phi <_A \mathcal{L}_{\Phi'}$.

Consider the last assertion and suppose $\Phi \not\subseteq \Phi'$ and $\Phi \setminus \Phi' \neq \{O\}$. At least one feature among I, Q, U, $\texttt{Self}$ belongs to $\Phi \setminus \Phi'$. Consider the case $I \in \Phi \setminus \Phi'$. The cases of other features are similar and omitted. For the sake of contradiction, suppose $\mathcal{L}_\Phi \leq_A \mathcal{L}_{\Phi'}$. Since $\mathcal{L}_I \leq_A \mathcal{L}_\Phi$ and $\mathcal{L}_{\Phi'} \leq_A \mathcal{L}_{OQU\texttt{Self}}$, it follows that $\mathcal{L}_I \leq_A \mathcal{L}_{OQU\texttt{Self}}$, which contradicts Lemma 4.6. Therefore, $\mathcal{L}_\Phi \not\leq_A \mathcal{L}_{\Phi'}$. □

Definition 4.8. *We define $\mathcal{ALC}$ to be the sublogic of $\mathcal{ALC}_{reg}$ such that the role constructors ε, $R \circ S$, $R \sqcup S$, R^* and $C?$ are disallowed. We say that $\mathcal{L}$ is a sublogic of $\mathcal{ALC}_{reg}$ that extends $\mathcal{ALC}$, denoted $\mathcal{ALC} \leq \mathcal{L} \leq \mathcal{ALC}_{reg}$, if it extends $\mathcal{ALC}$ with some of those role constructors. For $\Phi \subseteq \{I, O, Q, U, \texttt{Self}\}$ and $\mathcal{ALC} \leq \mathcal{L} \leq \mathcal{ALC}_{reg}$, let $\mathcal{L}_\Phi$ and $\mathcal{L}_\Phi^{pos}$ be defined as usual in the spirit of Definitions 2.1 and 2.2.* □

Corollary 4.9. *Let $\mathcal{L}$ be any sublogic of $\mathcal{ALC}_{reg}$ that extends $\mathcal{ALC}$ and let Φ and Φ' be subsets of $\{I, O, Q, U, \texttt{Self}\}$.*

1. *If $\Phi \subset \Phi'$ then $\mathcal{L}_\Phi <_C \mathcal{L}_{\Phi'}$ and $\mathcal{L}_\Phi <_{PC} \mathcal{L}_{\Phi'}$.*
2. *If $\Phi \not\subseteq \Phi'$ then $\mathcal{L}_\Phi \not\leq_C \mathcal{L}_{\Phi'}$ and $\mathcal{L}_\Phi \not\leq_{PC} \mathcal{L}_{\Phi'}$.*
3. *If $\Phi \subset \Phi'$ and $\Phi' \setminus \Phi \neq \{U\}$ then $\mathcal{L}_\Phi <_T \mathcal{L}_{\Phi'}$.*
4. *If $\Phi \not\subseteq \Phi'$ and $\Phi \setminus \Phi' \neq \{U\}$ then $\mathcal{L}_\Phi \not\leq_T \mathcal{L}_{\Phi'}$.*
5. *If $\Phi \subset \Phi'$ and $\Phi' \setminus \Phi \neq \{O\}$ then $\mathcal{L}_\Phi <_A \mathcal{L}_{\Phi'}$.*
6. *If $\Phi \not\subseteq \Phi'$ and $\Phi \setminus \Phi' \neq \{O\}$ then $\mathcal{L}_\Phi \not\leq_A \mathcal{L}_{\Phi'}$.*

Proof. Just observe that the concepts C listed in Figure 2 do not use any of the role constructors ε, $R \circ S$, $R \sqcup S$, R^*, $C?$. All the lemmas and theorems given in this paper hold for the case when $\mathcal{L}_\Phi$ is a sublogic of $\mathcal{ALC}_{reg}$ that extends $\mathcal{ALC}$. Their proofs do not require any change. □

Figure 3 illustrates the relationship between the expressiveness of all the DLs that extend $\mathcal{L}$, where $\mathcal{ALC} \leq \mathcal{L} \leq \mathcal{ALC}_{reg}$, with any non-empty combination of the features I, O, Q, U, $\texttt{Self}$. Note that the problems whether $\mathcal{L}_\Phi <_T \mathcal{L}_{\Phi'}$ when $\Phi' \setminus \Phi = \{U\}$ and whether $\mathcal{L}_\Phi <_A \mathcal{L}_{\Phi'}$ when $\Phi' \setminus \Phi = \{O\}$ remain open.

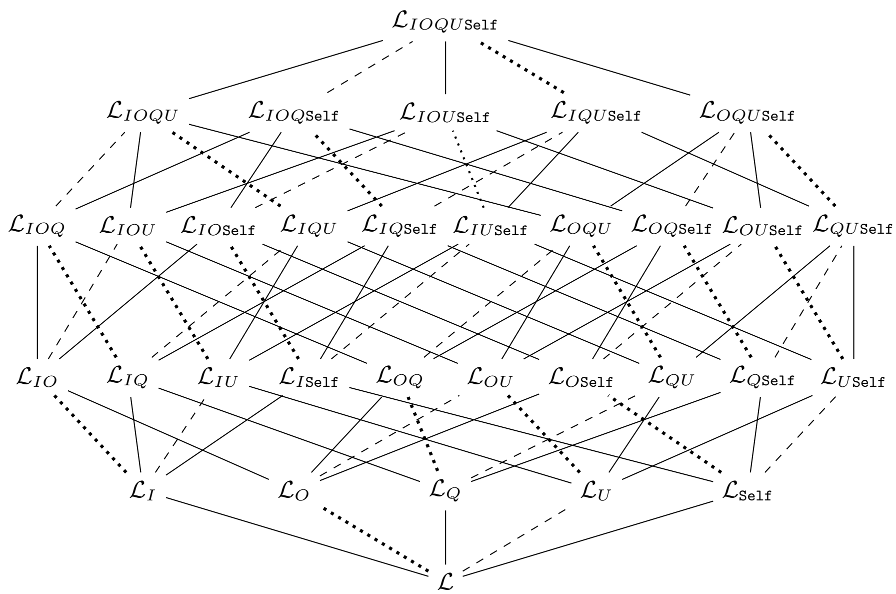

Fig. 3. Comparing the expressiveness of description logics, where $\mathcal{ALC} \leq \mathcal{L} \leq \mathcal{ALC}_{reg}$. If there is a path from a logic $\mathcal{L}_2$ down to a logic $\mathcal{L}_1$ that contains either a normal edge or at least two edges then $\mathcal{L}_2$ is more expressive than $\mathcal{L}_1$ w.r.t. concepts, positive concepts, TBoxes and ABoxes. If the path is a dotted edge then $\mathcal{L}_2$ is more expressive than $\mathcal{L}_1$ w.r.t. concepts, positive concepts and TBoxes. If the path is a dashed edge then $\mathcal{L}_2$ is more expressive than $\mathcal{L}_1$ w.r.t. concepts, positive concepts and ABoxes.

5 Conclusions

Analyzing the expressiveness of logics is a theoretical topic that has attracted a lot of logicians. In this paper we have studied the expressiveness of large classes of DLs. Namely, we have provided results about separating the expressiveness of the DLs that extend $\mathcal{L}$, where $\mathcal{ALC} \leq \mathcal{L} \leq \mathcal{ALC}_{reg}$, with any combination of the features I, O, Q, U, `Self`. Our separation results are w.r.t. concepts, positive concepts, TBoxes and ABoxes. Our work differs significantly from all of [2, 5, 6, 19, 20], as the class of considered DLs is much larger than the ones considered in those works and our results about separating the expressiveness of DLs are obtained not only w.r.t. concepts and TBoxes but also w.r.t. positive concepts and ABoxes.

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