Looking for bonds between nonhomogeneous formal contexts

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Abstract. Recently, the concept lattices working with the heterogeneous structures have been fruitfully applied in a fuzzy formal concept analysis. We present a situation under nonhomogeneous formal contexts and explore the bonds in a such nonhomogeneous case. This issue requires to formulate the alternative definition of a bond and to investigate the relationships between bonds and the particular formal contexts.

Keywords: bond, heterogeneous formal context, second order formal context

1 Introduction

Formal concept analysis (FCA) [16] as an applied lattice theory allows us to explore the meaningful groupings of objects with respect to common attributes. In general, FCA is an interesting research area that provides theoretical foundations, fruitful methods, algorithms and underlying applications in many areas and has been investigated in relation to various disciplines and integrated approaches [13,15]. The feasible attempts and generalizations are investigated, one can see dual multi-adjoint concept lattices working with adjoint triples [27–29], interval-valued L-fuzzy concept lattices [1], heterogeneous concept lattices [2,3], connectional concept lattices [12,32,33]. Classical bonds and their generalizations acting on residuated lattices were analyzed from a broader perspective in [17,21,24].

In this paper, we deal with an alternative notion of the bonds and with a problem of looking for bonds in a nonhomogeneous formal contexts. In particular, Section 2 recalls the basic notions of a concept lattice, notion of a bond, its equivalent definition and preliminaries of a second order formal context and a heterogeneous formal context. Section 3 describes the idea of a looking for bonds in a nonhomogeneous case. Sections 4 and 5 provide the solution of this issue in terms of a second order formal context and heterogeneous formal context.

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2 Preliminaries

Definition 1. *Let B and A be the nonempty sets, $R \subseteq B \times A$ be an arbitrary binary relation. Triple $\langle B, A, R\rangle$ is said to be a formal context with a set of objects B and a set of their attributes A. Relationships between objects and their attributes are saved in the relation R. Let us define a pair of derivation operators $(\uparrow, \downarrow)$ as the mappings between powersets of B and A such that*

- $\uparrow : \mathcal{P}(B) \to \mathcal{P}(A)$ *and* $\downarrow : \mathcal{P}(A) \to \mathcal{P}(B)$ *where for any* $X \subseteq B$ *and* $Y \subseteq A$ *is*
- $\uparrow(X) = \{a \in A | (\forall b \in X)(b, a) \in R\}$
- $\downarrow(Y) = \{b \in B | (\forall a \in Y)(b, a) \in R\}$.

Such derivation operators can be defined as the mappings between 2-sets (borrowed from fuzzy generalization of FCA that is sometimes easier to use)

- $\uparrow : 2^B \to 2^A$ *and* $\downarrow : 2^A \to 2^B$ *where for any* $X \in 2^B$ *and* $Y \in 2^A$
- $\uparrow(X)(a) = \bigwedge_{b \in B}((b \in X) \Rightarrow ((b, a) \in R)) = \bigwedge_{b \in B}(X(b) \Rightarrow R(b, a))$
- $\downarrow(Y)(b) = \bigwedge_{a \in A}((a \in Y) \Rightarrow ((b, a) \in R)) = \bigwedge_{a \in A}(Y(a) \Rightarrow R(b, a))$.

Pair of such derivation operators forms an antitone Galois connection between complete lattices of all subsets of B and A. Hence, the compositions of the mappings form closure operators on such complete lattices.

Definition 2. *Let $\mathcal{C} = \langle B, A, R\rangle$ be a formal context. Any pair of sets $(X, Y) \in 2^B \times 2^A$ is said to be a* formal concept *iff $X = \downarrow(Y)$ and $Y = \uparrow(X)$. Object part of any concept is called* extent *and attribute part is called* intent*. Set of all extents of formal context $\mathcal{C}$ will be denoted by* $\mathrm{Ext}(\mathcal{C})$*. The notation* $\mathrm{Int}(\mathcal{C})$ *stands for the set of all intents of $\mathcal{C}$.*

All concepts ordered by set inclusion of extents (or equivalently by dual of intent inclusion) form a complete lattice structure.

2.1 Notion of bond and its equivalent definition

Definition 3. *Let $\mathcal{C}_i = \langle B_i, A_i, R_i\rangle$ for $i \in \{1, 2\}$ be two formal contexts. Relation $\beta \subseteq B_1 \times A_2$ is said to be a* bond *iff any row of the table is an intent of $\mathcal{C}_2$ and any of its column is an extent of $\mathcal{C}_1$. Set of all bonds between $\mathcal{C}_1$ and $\mathcal{C}_2$ will be denoted by* 2-Bonds$(\mathcal{C}_1, \mathcal{C}_2)$.

Lemma 1. *Let $\mathcal{C}_i = \langle B_i, A_i, R_i\rangle$ for $i \in \{1, 2\}$ be two formal contexts. Then $\beta \subseteq B_1 \times A_2$ is a bond between $\mathcal{C}_1$ and $\mathcal{C}_2$ if and only if* $\mathrm{Ext}(\langle B_1, A_2, \beta\rangle) \subseteq \mathrm{Ext}(\mathcal{C}_1)$ *and* $\mathrm{Int}(\langle B_1, A_2, \beta\rangle) \subseteq \mathrm{Int}(\mathcal{C}_2)$.

Proof. $\Rightarrow$: Let $X \in \mathrm{Ext}(\langle B_1, A_2, \beta\rangle)$ be an arbitrary extent of any bond between formal contexts $\mathcal{C}_1$ and $\mathcal{C}_2$. Derivation operators of $\mathcal{C}_i$ will be denoted by $(\uparrow_i, \downarrow_i)$

for $i \in \{1,2\}$. Derivation operators of the bond will be denoted by $(\uparrow_\beta, \downarrow_\beta)$. Then there exists a set of attributes $Y \subseteq A_2$ such that

$$\begin{aligned}
\downarrow_\beta (Y)(b_1) = & \bigwedge_{a_2 \in A_2} (Y(a_2) \Rightarrow \beta(b_1, a_2)) \\
& \beta(-, a_2) \text{ is an extent of } \mathrm{Ext}(\mathcal{C}_1) \text{ hence there exists } Z \subseteq A_1 \\
= & \bigwedge_{a_2 \in A_2} (Y(a_2) \Rightarrow \downarrow_1 (Z)(b_1)) \\
= & \bigwedge_{a_2 \in A_2} \Big(Y(a_2) \Rightarrow \bigwedge_{a_1 \in A_1} (Z(a_1) \Rightarrow R_1(b_1, a_1)) \Big) \\
= & \bigwedge_{a_2 \in A_2} \bigwedge_{a_1 \in A_1} (Y(a_2) \Rightarrow (Z(a_1) \Rightarrow R_1(b_1, a_1))) \\
= & \bigwedge_{a_2 \in A_2} \bigwedge_{a_1 \in A_1} ((Y(a_2) \wedge Z(a_1)) \Rightarrow R_1(b_1, a_1)) \\
= & \bigwedge_{a_1 \in A_1} \Big(\bigvee_{a_2 \in A_2} \big(Y(a_2) \wedge Z(a_1)\big) \Rightarrow R_1(b_1, a_1) \Big) \\
= & \bigwedge_{a_1 \in A_1} (Z_Y(a_1) \Rightarrow R_1(b_1, a_1)) \\
= & \downarrow_1 (Z_Y)(b_1) \text{ where } Z_Y(a_1) = \textstyle\bigvee_{a_2 \in A_2} (Y(a_2) \wedge Z(a_1))
\end{aligned}$$

Hence, $\mathrm{Ext}(\langle B_1, A_2, \beta \rangle) \subseteq \mathrm{Ext}(\mathcal{C}_1)$. Similarly for intents.

$\Leftarrow$: Assume a formal context $\langle B_1, A_2, \beta \rangle$ such that it holds $\mathrm{Ext}(\langle B_1, A_2, \beta \rangle) \subseteq \mathrm{Ext}(\mathcal{C}_1)$ and $\mathrm{Int}(\langle B_1, A_2, \beta \rangle) \subseteq \mathrm{Int}(\mathcal{C}_2)$. From the simple fact that any row of any context is its intent and any column is its extent and from the previous inclusions, we obtain that β is a bond between $\mathcal{C}_1$ and $\mathcal{C}_2$. $\square$

Hence, the notion of bond can be defined equivalently as follows.

Definition 4. *Let $\mathcal{C}_i = \langle B_i, A_i, R_i \rangle$ for $i \in \{1,2\}$ be two formal contexts. Formal context $\mathcal{B} = \langle B_1, A_2, \beta \rangle$ is said to be a* bond *between $\mathcal{C}_1$ and $\mathcal{C}_2$ if $\mathrm{Ext}(\mathcal{B}) \subseteq \mathrm{Ext}(\mathcal{C}_1)$ and $\mathrm{Int}(\mathcal{B}) \subseteq \mathrm{Int}(\mathcal{C}_2)$.*

More about the equivalent definition of bond could be found in [17–19].

2.2 Direct product of two formal contexts and bonds

Let us recall the definition and important property of direct product of two formal contexts. More details about such topic can be found in [21, 26].

Definition 5. *Let $\mathcal{C}_i = \langle B_i, A_i, R_i \rangle$ be two formal contexts. Formal context $\mathcal{C}_1 \Delta \mathcal{C}_2 = \langle B_1 \times A_2, B_2 \times A_1, R_1 \Delta R_2 \rangle$ where*

$$\begin{aligned}
(R_1 \Delta R_2)((b_1, a_2), (b_2, a_1)) &= R_1(b_1, a_1) \vee R_2(b_2, a_2) \\
&= \neg R_1(b_1, a_1) \Rightarrow R_2(b_2, a_2) \\
&= \neg R_2(b_2, a_2) \Rightarrow R_1(b_1, a_1)
\end{aligned}$$

for any $(b_i, a_i) \in B_i \times A_i$ *for all* $i \in \{1, 2\}$ *is said to be a* direct product *of formal contexts* $\mathcal{C}_1$ *and* $\mathcal{C}_2$.

Lemma 2. *Let* $\mathcal{C}_i = \langle B_i, A_i, R_i\rangle$ *be two formal contexts. Every extent of* $\mathcal{C}_1 \Delta \mathcal{C}_2$ *is a bond between* $\mathcal{C}_1$ *and* $\mathcal{C}_2$.

2.3 Second order formal contexts

In this subsection, we remind a notion of a second order formal concept [24].

Definition 6. *Consider two non-empty index sets* I *and* J *and a formal context* $\langle \bigcup_{i\in I} B_i, \bigcup_{j\in J} A_j, r\rangle$, *whereby*

- $B_{i_1} \cap B_{i_2} = \emptyset$ *for any* $i_1, i_2 \in I$, $i_1 \neq i_2$,
- $A_{j_1} \cap A_{j_2} = \emptyset$ *for any* $j_1, j_2 \in J$, $j_1 \neq j_2$,
- $r : \bigcup_{i\in I} B_i \times \bigcup_{j\in J} A_j \to 2$.

Moreover, consider two non-empty sets of 2*-contexts notated*

- $\{\mathcal{C}_i = \langle B_i, T_i, p_i\rangle : i \in I\}$
- $\{\mathcal{D}_j = \langle O_j, A_j, q_j\rangle : j \in J\}$.

Formal context of second order is a tuple

$$\Big\langle \bigcup_{i\in I} B_i,\ \{\mathcal{C}_i; i \in I\},\ \bigcup_{j\in J} A_j,\ \{\mathcal{D}_j; j \in J\},\ \bigcup_{(i,j)\in I\times J} r_{i,j}\Big\rangle,$$

where $r_{i,j} : B_i \times A_j \to 2$ *defined as* $r_{i,j}(b, a) = r(b, a)$ *for any* $b \in B_i$ *and* $a \in A_j$.

In what follows, consider the below described notation. Let us have an L-set $f : X \to 2$ for a non-empty universe set $X = \bigcup_{i\in I} X_i$, where $X_{i_1} \cap X_{i_2} = \emptyset$ for any $i_1, i_2 \in I$. Then $f^i : X_i \to 2$ is defined as $f^i(x) = f(x)$ for an arbitrary $x \in X_i$ and $i \in I$.

We define the mappings between direct products of two sets of concept lattices (that correspond to the two sets of 2-contexts given above) in the following form:

Definition 7. *Let us define the mappings* $\langle \Uparrow, \Downarrow\rangle$ *as follows*

$$\Uparrow: \prod_{i\in I} \mathrm{Ext}(\mathcal{C}_i) \to \prod_{j\in J} \mathrm{Int}(\mathcal{D}_j) \ \textit{and}\ \Downarrow: \prod_{j\in J} \mathrm{Int}(\mathcal{D}_j) \to \prod_{i\in I} \mathrm{Ext}(\mathcal{C}_i)$$

$$\Uparrow(\Phi)^j = \bigwedge_{i\in I} \uparrow_{ij} (\Phi^i),\ \textit{for any}\ \Phi \in \prod_{i\in I} \mathrm{Ext}(\mathcal{C}_i)$$

$$\Downarrow(\Psi)^i = \bigwedge_{j\in J} \downarrow_{ij} (\Psi^j),\ \textit{for any}\ \Psi \in \prod_{j\in J} \mathrm{Int}(\mathcal{D}_j)$$

such that $(\uparrow_{ij}, \downarrow_{ij})$ *is a pair of derivation operators defined on* $\langle B_i, A_j, \rho_{ij}\rangle$ *where*

$$\rho_{ij} = \bigwedge\{\beta \in 2\text{-Bonds}(\mathcal{C}_i, \mathcal{D}_j) : (\forall (b_i, a_j) \in B_i \times A_j)\beta(b_i, a_j) \geq r_{ij}(b_i, a_j)\}.$$

2.4 Heterogeneous formal contexts

A heterogeneous extension in FCA based on the totally diversification of objects, attributes and table fields has been introduced in [3]. In the following, we remind the definition of a heterogeneous formal context and its derivation operators.

Definition 8. *Heterogeneous formal context is a tuple* $\mathcal{C} = \langle B, A, \mathcal{P}, R, \mathcal{U}, \mathcal{V}, \odot \rangle$, *where*

- B *and* A *are non-empty sets,*
- $\mathcal{P} = \{\langle P_{b,a}, \leq_{P_{b,a}} \rangle : (b,a) \in B \times A\}$ *is a system of posets,*
- R *is a mapping from* $B \times A$ *such that* $R(b,a) \in P_{b,a}$ *for any* $b \in B$ *and* $a \in A$,
- $\mathcal{U} = \{\langle U_b, \leq_{U_b} \rangle : b \in B\}$ *and* $\mathcal{V} = \{\langle V_a, \leq_{V_a} \rangle : a \in A\}$ *are systems of complete latices,*
- $\odot = \{\circ_{b,a} : (b,a) \in B \times A\}$ *is a system of isotone and left-continuous mappings* $\circ_{b,a} : U_b \times V_a \longrightarrow P_{b,a}$.

Let us define the derivation operators of a heterogeneous formal context as a pair of mappings $(\nearrow, \swarrow)$, *whereby* $\nearrow : \prod_{b \in B} U_b \to \prod_{a \in A} V_a$ *and* $\swarrow : \prod_{a \in A} V_a \to \prod_{b \in B} U_b$ *such that*

- $\swarrow (f)(a) = \bigvee\{v \in V_a | f(b) \circ_{b,a} v \leq R(b,a)\}$ *for any* $f \in \prod_{b \in B} U_b$
- $\nearrow (g)(b) = \bigvee\{u \in U_b | u \circ_{b,a} g(a) \leq R(b,a)\}$ *for any* $g \in \prod_{a \in A} V_a$.

3 Problem description and sketch of solution

In this section we discussed why we have proposed an equivalent definition of bond. First, consider the classical definition of bond. It is a binary relation (table) between objects and attributes from different contexts such that its rows are intents and columns are extents of different input contexts. The issue of looking for bonds in a classical or homogeneous fuzzy case can be solved successfully [17,21].

The solution of this issue requires the alternative definition of a bond. Hence, new definition of a bond focuses not only on a relation with some special properties, but also on a bond as a formal context, whereby its concept lattice is connected to concept lattices of input contexts in some sense. As a consequence, a generalization for heterogeneous bonds is possible. One can find the methods in effort to equivalently modify the input heterogeneous formal contexts and to extract bonds as the extents of a direct product.

The proposed modification runs as follows. Each individual pair that includes a "conjunction" $\circ_{b,a}$ and a value of the poset $P_{b,a}$ is replaced by a bond from 2-Bonds$(\langle U_b, U_b, \leq \rangle, \langle V_a, V_a, \geq \rangle)$. This completely covers the Galois connection between the complete lattices of any object–attribute pair from $B \times A$.

At the beginning, we will show how this modification looks in terms of second order formal contexts. Then we define new modified heterogeneous formal context such that its concept lattice is identical to the original.

4 Second order form of scaled heterogeneous formal context

In effort to formalize the second order form of scaled heterogeneous formal context and its derivation operators, the definition of the following mappings is required:

Definition 9. *Let $(L, \leq)$ be a complete lattice. Let us define mappings $\overline{(-)}^L$ and $\underline{(-)}_L$ where*

- $\overline{(-)}^L : L \to 2^L$ *such that* $\overline{k}^L(m) = (m \leq k)$ *for any* $k, m \in L$
- $\underline{(-)}_L : 2^L \to L$ *such that* $\underline{X}_L = \bigvee X$ *for any* $X \subseteq L$.

Let us have an arbitrary $f \in \prod_{b \in B} U_b$. *Let us denote* $\overline{f}$ *as a subset of* $\bigcup_{b \in B} U_b$ *defined as* $\overline{f} = \bigcup_{b \in B}\{u \in U_b | u \leq f(b)\}$. *Similarly for any* $g \in \prod_{a \in A} V_a$.

More information about Cartesian representation of fuzzy sets could be found in [10].

Now, consider a heterogeneous formal context $\mathcal{C} = \langle B, A, \mathcal{P}, R, \mathcal{U}, \mathcal{V}, \odot \rangle$. A second order form of scaled heterogeneous formal context is defined as

$$\overline{\mathcal{C}} = \left\langle \bigcup_{b \in B} U_b, \{\langle U_b, U_b, \leq \rangle | b \in B\}, \bigcup_{a \in A} V_a, \{\langle V_a, V_a, \geq \rangle | a \in A\}, \overline{R} \right\rangle,$$

whereby all external contexts are classical crisp contexts and $\overline{R}$ is a classical crisp binary relation defined as $\overline{R}(u, v) = ((u \circ_{b,a} v) \leq R(b, a))$ for any $(u, v) \in U_b \times V_a$ and any $(b, a) \in B \times A$.

In the following, we define the derivation operators of such special second order formal context. First, we state some appropriate remarks and facts. Note that a relation $\overline{R}$ constrained to $U_b \times V_a$ for any pair $(b, a) \in B \times A$ is monotone in both arguments due to its definition. Similarly, consider the fact that any extent of $\langle U_b, U_b, \leq \rangle$ and any intent of $\langle V_a, V_a, \geq \rangle$ is a principal down-set of a corresponding complete lattice (i.e. there exists an element in this complete lattice such that all lower or equal elements are in the extent or in the intent). Hence, a relation $\overline{R}$ constrained to $U_b \times V_a$ for some $(b, a) \in B \times A$ is a 2-bond between $\langle U_b, U_b, \leq \rangle$ and $\langle V_a, V_a, \geq \rangle$ which will be denoted by $\rho_{b,a}$. Note that any $\Phi \in \prod_{b \in B} \mathrm{Ext}(\langle U_b, U_b, \leq \rangle)$ has the form $\overline{f}$ for some $f \in \prod_{b \in B} U_b$. Consider an arbitrary $f \in \prod_{b \in B} U_b$ and $g \in \prod_{a \in A} V_a$. Hence, the derivation operators are defined as follows:

- $\overline{\nearrow}(\overline{f})(v) = \bigwedge_{b \in B} \uparrow_{b,a} (\overline{f(b)}^b)(v)$ for any $v \in V_a$ and $a \in A$
- $\overline{\swarrow}(\overline{g})(u) = \bigwedge_{a \in A} \downarrow_{b,a} (\overline{g(a)}^a)(u)$ for any $u \in U_b$ and $b \in B$.

In a previous definition, the pair of mappings $(\uparrow_{b,a}, \downarrow_{b,a})$ are derivation operators of a formal context $\langle U_b, V_a, \rho_{b,a} \rangle$ for any $(b, a) \in B \times A$. For the sake of brevity, we use the shortened notation $\overline{(-)}^b$ instead of $\overline{(-)}^{U_b}$ and similarly $\overline{(-)}^a$ instead of $\overline{(-)}^{V_a}$.

Lemma 3. *The concept lattices of $\mathcal{C}$ and $\overline{\mathcal{C}}$ are isomorphic.*

Proof. Consider an arbitrary $f \in \prod_{b\in B} U_b$. We will show that $\overline{\nearrow}(\overline{f}) = \overline{\nearrow(f)}$.

Firstly consider the fact of left-continuity of both arguments of $\circ_{b,a}$ for any $(b,a) \in B\times A$. Due to this property, one can define two residuums in the following way. Let $(b,a) \in B \times A$ be an arbitrary object-attribute pair and consider the arbitrary values $u \in U_b$, $v \in V_a$ and $p \in P_{b,a}$. Then define

- $\rightarrow_{b,a}\colon U_b \times P_{b,a} \rightarrow V_a$, such that $u \rightarrow_{b,a} p = \bigvee\{v \in V_a | u \circ_{b,a} v \leq p\}$
- $\rightarrow_{a,b}\colon V_a \times P_{b,a} \rightarrow U_b$, such that $v \rightarrow_{a,b} p = \bigvee\{u \in U_b | u \circ_{b,a} v \leq p\}$.

$$
\begin{aligned}
\overline{\nearrow}\left(\overline{f}\right)(v) &= \bigwedge_{b\in B} \uparrow_{b,a}\left(\overline{f(b)}^{b}\right)(v)\\
&= \bigwedge_{b\in B}\bigwedge_{u\in U_b}\left(\overline{f(b)}^{b}(u) \Rightarrow \rho_{b,a}(u,v)\right)\\
&= \bigwedge_{b\in B}\bigwedge_{u\in U_b}((u \leq f(b)) \Rightarrow (u \circ_{b,a} v \leq R(b,a)))\\
&= \bigwedge_{b\in B}\left(\bigwedge_{u\in U_b; u\not\leq f(b)} 1 \wedge \bigwedge_{u\in U_b; u\leq f(b)} ((u \leq f(b)) \Rightarrow (u \circ_{b,a} v \leq R(b,a)))\right)\\
&= \bigwedge_{b\in B}\bigwedge_{u\in U_b; u\leq f(b)} (u \circ_{b,a} v \leq R(b,a))\\
&= \bigwedge_{b\in B} (f(b) \circ_{b,a} v \leq R(b,a))\\
&= \bigwedge_{b\in B} (v \leq f(b) \rightarrow_{b,a} R(b,a))\\
&= \left(v \leq \bigwedge_{b\in B} (f(b) \rightarrow_{b,a} R(b,a))\right)\\
&= \left(v \leq \bigwedge_{b\in B} \bigvee\{w \in V_a | (f(b) \circ_{b,a} w \leq R(b,a))\}\right)\\
&= \left(v \leq \bigvee\{w \in V_a | (\forall b \in B)(f(b) \circ_{b,a} w \leq R(b,a))\}\right)\\
&= (v \leq \nearrow(f)(a)) = \overline{\nearrow(f)(a)}^{a}(v).
\end{aligned}
$$

Analogously one can obtain $\overline{\swarrow}\left(\overline{g}\right)(u) = \overline{\swarrow(g)(b)}^{b}(u)$. $\square$

4.1 Back to heterogeneous formal contexts

Now, we look at heterogeneous formal context introduced in Subsection 2.3. A second order formal context $\overline{\mathcal{C}}$ can be seen as a special heterogeneous formal context $\widehat{\mathcal{C}}$, whereby the family of posets $\{\langle P_{b,a}, \leq\rangle | (b,a) \in B \times A\}$ is replaced by a set of 2-bonds $\{\rho_{b,a} \in \text{2-Bonds}(\langle U_b, U_b, \leq\rangle, \langle V_a, V_a, \leq\rangle) | (b,a) \in B \times A\}$. Hence, the final form of such heterogeneous formal context is

$$\widehat{\mathcal{C}} = \left\langle B, A, \rho, \widehat{R}, \mathcal{U}, \mathcal{V}, \{\overline{\times}_{b,a} | (b,a) \in B \times A\} \right\rangle$$

where

- $\rho = \{\rho_{b,a} \in \text{2-Bonds}(\langle U_b, U_b, \leq\rangle, \langle V_a, V_a, \leq\rangle) | (b,a) \in B \times A\}$
- $\rho_{b,a}(u,v) = (u \circ_{b,a} v \leq R(b,a))$
- $\widehat{R}(b,a) = \rho_{b,a} \in \text{2-Bonds}(\langle U_b, U_b, \leq\rangle, \langle V_a, V_a, \leq\rangle)$ for any $(b,a) \in B \times A$
- $\overline{\times}_{b,a} : U_b \times V_a \to 2^{U_b \times V_a}$ defined as a Cartesian product $u \overline{\times} v = \overline{u} \times \overline{v}$.

The derivation operators of $\widehat{\mathcal{C}}$ are defined as

- $\uparrow(f)(a) = \bigvee\{v \in V_a | (\forall b \in B) f(b) \overline{\times}_{b,a} v \subseteq \rho_{b,a}\}$ for any $f \in \prod_{b \in B} U_b$
- $\downarrow(g)(b) = \bigvee\{u \in U_b | (\forall a \in A) u \overline{\times}_{b,a} g(a) \subseteq \rho_{b,a}\}$ for any $g \in \prod_{a \in A} V_a$.

Lemma 4. *The concept lattices of $\mathcal{C}$ and $\widehat{\mathcal{C}}$ are identical.*

Proof. Firstly consider that for any $(u,v) \in U_b \times V_a$ for any $(b,a) \in B \times A$ the following holds:

$$\begin{aligned} u \overline{\times} v \subseteq \rho_{b,a} &= \overline{u} \times \overline{v} \subseteq \rho_{b,a} \\ &= \rho_{b,a}(u,v) \\ &= (u \circ_{b,a} v \leq R(b,a)). \end{aligned}$$

Let $f \in \prod_{b \in B} U_b$ be arbitrary. Then

$$\begin{aligned} \uparrow(f)(a) &= \bigvee\{v \in V_a | (\forall b \in B) f(b) \overline{\times}_{b,a} v \subseteq \rho_{b,a}\} \\ &= \bigvee\{v \in V_a | (\forall b \in B) f(b) \circ_{b,a} v \leq R(b,a)\} \\ &= \nearrow(f)(a). \end{aligned}$$

Analogously for $\downarrow(g)(b) = \swarrow(g)(b)$ for any $g \in \prod_{a \in A} V_a$. □

5 Bonds between heterogeneous formal contexts

We present a definition of a bond between two heterogeneous formal contexts which can be formulated as follows.

Definition 10. *Let $\mathcal{C}_i = \langle B_i, A_i, \mathcal{P}_i, R_i, \mathcal{U}_i, \mathcal{V}_i, \odot_i\rangle$ for $i \in \{1,2\}$ be two heterogeneous formal contexts. The heterogeneous formal context $\mathcal{B} = \langle B_1, A_2, \mathcal{P}, R, \mathcal{U}_1, \mathcal{V}_2, \odot\rangle$ such that $\mathrm{Ext}(\mathcal{B}) \subseteq \mathrm{Ext}(\mathcal{C}_1)$ and $\mathrm{Int}(\mathcal{B}) \subseteq \mathrm{Int}(\mathcal{C}_2)$ is said to be a* bond *between two heterogeneous formal contexts $\mathcal{C}_1$ and $\mathcal{C}_2$.*

5.1 Direct product of two heterogeneous formal contexts

In this subsection, we define a direct product of two heterogeneous formal contexts. Further, we give an answer on how to find a bond between two heterogeneous formal contexts.

Definition 11. *Let $\mathcal{C}_i = \langle B_i, A_i, \mathcal{P}_i, R_i, \mathcal{U}_i, \mathcal{V}_i, \odot_i\rangle$ for $i \in \{1,2\}$ be two heterogeneous formal contexts. The heterogeneous formal context*

$$\mathcal{C}_1 \varDelta \mathcal{C}_2 = \langle B_1 \times A_2, B_2 \times A_1, \mathcal{P}_\Delta, R_\Delta, \mathcal{U}_\Delta, \mathcal{V}_\Delta, \times\rangle$$

such that

- $\mathcal{P}_\Delta = \{\rho_{b_1,a_1} \varDelta \rho_{b_2,a_2} | ((b_1,a_2),(b_2,a_1)) \in (B_1 \times A_2) \times (B_2 \times A_1)\}$
- *where* $\rho_{b_i,a_i}(u,v) = (u \circ_{b_i,a_i} v \leq R_i(b_i,a_i))$ *for any* $(u,v) \in U_{b_i} \times V_{a_i}$ *for any* $(b_i,a_i) \in B_i \times A_i$ *for any* $i \in \{1,2\}$
- $R_\Delta((b_1,a_2),(b_2,a_1)) = \rho_{b_1,a_1} \varDelta \rho_{b_2,a_2}$ *for any* $b_i \in B_i$ *and* $a_i \in A_i$ *for all* $i \in \{1,2\}$
- $\mathcal{U}_\Delta = \{\gamma_{1,2} \in \text{2-Bonds}(\langle U_{b_1}, U_{b_1}, \leq\rangle, \langle V_{a_2}, V_{a_2}, \geq\rangle) | (b_1,a_2) \in B_1 \times A_2\}$
- $\mathcal{V}_\Delta = \{\gamma_{2,1} \in \text{2-Bonds}(\langle U_{b_2}, U_{b_2}, \leq\rangle, \langle V_{a_1}, V_{a_1}, \geq\rangle) | (b_2,a_1) \in B_2 \times A_1\}$

is said to be a direct product of two heterogeneous formal contexts*.*

Lemma 5. *Let $\mathcal{C}_i = \langle B_i, A_i, \mathcal{P}_i, R_i, \mathcal{U}_i, \mathcal{V}_i, \odot_i\rangle$ for $i \in \{1,2\}$ be two heterogeneous formal contexts. Let*

$$R \in \prod_{(b_1,a_2) \in B_1 \times A_2} \text{2-Bonds}(\langle U_{b_1}, U_{b_1}, \leq\rangle, \langle V_{a_2}, V_{a_2}, \geq\rangle)$$

be an extent of the direct product $\mathcal{C}_1 \varDelta \mathcal{C}_2$. Then a heterogeneous formal context $\mathcal{B} = \langle B_1, A_2, \rho, R, \mathcal{U}_1, \mathcal{V}_2, \overline{\times}\rangle$ where

$$\rho = \{\text{2-Bonds}(\langle U_{b_1}, U_{b_1}, \leq\rangle, \langle V_{a_2}, V_{a_2}, \geq\rangle) | (b_1,a_2) \in B_1 \times A_2\}$$

is a bond between $\mathcal{C}_1$ and $\mathcal{C}_2$.

Proof. Let us have any intent of $\mathcal{B}$. Then there exists $f \in \prod_{b_1 \in B_1} U_{b_1}$ such that

$$\begin{aligned}
&\overline{\nearrow_{\mathcal{B}} (f)(a_2)}^{a_2}(v_2) = \overline{\nearrow}_{\overline{\mathcal{B}}}(\overline{f})(v_2) \\
&= \bigwedge_{b_1 \in B_1} \uparrow_{R(b_1,a_2)} (\overline{f(b_1)}^{b_1})(v_2) \\
&= \bigwedge_{b_1 \in B_1} \bigwedge_{u_1 \in U_{b_1}} (\overline{f(b_1)}^{b_1}(u_1) \Rightarrow R(b_1,a_2)(u_1,v_2)) \\
&\quad R = \swarrow_\Delta (Q) \text{ for some } Q \in \textstyle\prod_{(b_2,a_1) \in B_2 \times A_1} \text{2-Bonds}(\langle U_{b_2}, U_{b_2}, \leq\rangle, \langle V_{a_1}, V_{a_1}, \geq\rangle) \\
&= \bigwedge_{b_1 \in B_1} \bigwedge_{u_1 \in U_{b_1}} (\overline{f(b_1)}^{b_1}(u_1) \Rightarrow \swarrow_\Delta (Q)(b_1,a_2)(u_1,v_2)) \\
&= \bigwedge_{b_1 \in B_1} \bigwedge_{u_1 \in U_{b_1}} \Big(\overline{f(b_1)}^{b_1}(u_1) \Rightarrow \bigwedge_{(b_2,a_1) \in B_2 \times A_1} \downarrow_{\rho_{b_1,a_1} \varDelta \rho_{b_2,a_2}} (Q(b_2,a_1))(u_1,v_2)\Big)
\end{aligned}$$

$$= \bigwedge_{b_1 \in B_1} \bigwedge_{u_1 \in U_{b_1}} \Big(\overline{f(b_1)}^{b_1}(u_1) \Rightarrow$$

$$\bigwedge_{b_2 \in B_2} \bigwedge_{a_1 \in A_1} \bigwedge_{(u_2,v_1) \in U_{b_2} \times V_{a_1}} \big(Q(b_2,a_1)(u_2,v_1) \Rightarrow (\rho_{b_1,a_1} \Delta \rho_{b_2,a_2})((u_1,v_2),(u_2,v_1))\big)\Big)$$

$$= \bigwedge_{b_1 \in B_1} \bigwedge_{u_1 \in U_{b_1}} \Big(\overline{f(b_1)}^{b_1}(u_1) \Rightarrow$$

$$\bigwedge_{b_2 \in B_2} \bigwedge_{a_1 \in A_1} \bigwedge_{u_2 \in U_{b_2}} \bigwedge_{v_1 \in V_{a_1}} \big(Q(b_2,a_1)(u_2,v_1) \Rightarrow (\neg\rho_{b_1,a_1}(u_1,v_1) \Rightarrow \rho_{b_2,a_2}(u_2,v_2))\big)\Big)$$

$$= \bigwedge_{b_1 \in B_1} \bigwedge_{b_2 \in B_2} \bigwedge_{a_1 \in A_1} \bigwedge_{u_2 \in U_{b_2}} \bigwedge_{v_1 \in V_{a_1}} \bigwedge_{u_1 \in U_{b_1}} \Big(\overline{f(b_1)}^{b_1}(u_1) \Rightarrow$$

$$\big(Q(b_2,a_1)(u_2,v_1) \Rightarrow (\neg\rho_{b_1,a_1}(u_1,v_1) \Rightarrow \rho_{b_2,a_2}(u_2,v_2))\big)\Big)$$

$$= \bigwedge_{b_2 \in B_2} \bigwedge_{u_2 \in U_{b_2}}$$

$$\Big(\bigvee_{b_1 \in B_1} \bigvee_{u_1 \in U_{b_1}} \bigvee_{a_1 \in A_1} \bigvee_{v_1 \in V_{a_1}} \big(\overline{f(b_1)}^{b_1}(u_1) \wedge Q(b_2,a_1)(u_2,v_1) \wedge \neg\rho_{b_1,a_1}(u_1,v_1)\big)$$

$$\Rightarrow \rho_{b_2,a_2}(u_2,v_2)\Big)$$

$$= \bigwedge_{b_2 \in B_2} \bigwedge_{u_2 \in U_{b_2}} \big(\overline{q(b_2)}^{b_2}(u_2) \Rightarrow \rho_{b_2,a_2}(u_2,v_2)\big)$$

$$= \overline{\nearrow_{\overline{\mathcal{C}_2}}}(\overline{q})(v_2) = \overline{\nearrow_{\mathcal{C}_2}(q)(a_2)}(v_2)$$

where

$$q(b_2)(u_2) = \bigvee_{b_1 \in B_1} \bigvee_{u_1 \in U_{b_1}} \bigvee_{a_1 \in A_1} \bigvee_{v_1 \in V_{a_1}} \overline{f(b_1)}(u_1) \wedge Q(b_2,a_1)(u_2,v_1) \wedge \neg\rho_{b_1,a_1}(u_1,v_1)$$

Hence, $\nearrow_{\mathcal{B}}(f) = \nearrow_{\mathcal{C}_2}(q)$. So any intent of $\mathcal{B}$ is an intent of $\mathcal{C}_2$.

By using the following equality

$$(\neg\rho_{b_1,a_1}(u_1,v_1) \Rightarrow \rho_{b_2,a_2}(u_2,v_2)) = (\neg\rho_{b_2,a_2}(u_2,v_2) \Rightarrow \rho_{b_1,a_1}(u_1,v_1))$$

analogously we obtain that any extent of $\mathcal{B}$ is an extent of $\mathcal{C}_1$. Hence, $\mathcal{B}$ is a bond between $\mathcal{C}_1$ and $\mathcal{C}_2$. □

6 Conclusion

Bonds and their L-fuzzy generalizations represent a feasible way to explore the relationships between formal contexts. In this paper we have investigated the notion of a bond with respect to the heterogeneous formal contexts. In conclusion, an alternative definition of a bond provides an efficient tool to work with the nonhomogeneous data and one can further explore this uncharted territory in formal concept analysis.

Categorical properties of heterogeneous formal contexts and bonds as morphisms between such objects and categorical relationship to homogeneous FCA categorical description will be studied in the near future.

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