GSR_2 School of Mathematical and Geospatial Sciences, RMIT University
Monday 10 to Wednesday 12 December 2012

TRAVERSE ANALYSIS

R.E. Deakin
School of Mathematical & Geospatial Sciences
RMIT University, Melbourne, Australia.
Email: rod.deakin@rmit.edu.au

Abstract

Traversing is a fundamental operation in surveying and the assessment of the quality of a traverse is a skill that every surveyor develops. Acceptable traverses have angular and linear misclosures that fall within acceptable bounds; which would permit the adjustment of the traverse measurements to remove mathematical inconsistencies. Unfortunately these misclosures tell very little about the precision of the location of the traverse stations – although large misclosures are good indicators of gross errors – and more sophisticated mathematical techniques are required for proper traverse analysis. This paper presents some relatively simple techniques that can be employed to give reliable estimations of the precision of traverse stations that allows a simple assessment of the quality of a traverse.

Keywords: Surveying, traverse, propagation of variances.

Biography of Author Rod Deakin started work in 1968 (age 17) as a surveyor's assistant with John Horne of Frankston, Victoria and fell in the Kananook Creek on his first day. In 1976, he graduated from the RMIT and returned to surveyor Horne's employ until 1980. In 1981, he was appointed as a tutor in surveying and then a lecturer (1983) at RMIT where he has remained. He has lectured in all aspects of surveying and in 2004 was awarded the Francis Ormond medal (RMIT University medal in honour of the founder), and in the Vice Chancellor's address at the presentation it was noted that:

> *"Indeed, Rod even holds the honour of being cited in student evaluations at another university as the 'best lecturer I have had throughout my course at Melbourne University' for a series of guest lectures he provided."*

Rod regards this as his finest achievement.

Introduction

A traverse is the fundamental component of many surveys and consists of a series of sides or lines whose bearings and distances have been determined from Total Station[1] measurements; which for the purposes of this paper will be assumed to be horizontal directions α and horizontal distances l. Traverse bearings θ are horizontal angles measured clockwise from north (0° to 360°); traverse angles β are differences between directions or bearings; and a traverse line has east and north components $\Delta E = l\sin\theta,\ \Delta N = l\cos\theta$ respectively. A <u>closed</u> traverse starts and finishes at the same point and an <u>open</u> traverse starts and finishes at different points.

A closed traverse is a polygon of n sides whose internal angles sum to $(2n-4)\times 90^\circ$ and this rule may be used to determine the <u>angular misclose</u>: the difference between the rule and the sum of the measured angles. A closed traverse has a <u>linear misclose</u> which is the length of an assumed 'misclose vector' (or missing line) whose east and north components are the sums of the east and north components of the n traverse legs. A closed traverse also has a <u>misclose ratio</u> $1:x$ where $x = \text{traverse perimeter}/\text{linear misclose}$; e.g. if the traverse perimeter is 850 m and the linear misclose is 0.050 m then the misclose ratio is 1:17,000. The misclose ratio is often called the traverse accuracy, but this is wrong, since it does not distinguish between random[2] or systematic[3] errors or reveal their effects and random errors do not accumulate in direct proportion to distance (Valentine 1984).

[1] Electronic surveying instrument combining the operation of a theodolite with EDM (Electronic Distance Measurement).

[2] Random errors are the small errors remaining in measurements after mistakes, constant errors and systematic errors have been eliminated. They are due to the imperfection of the equipment; the fallibility of the observer and the changing environmental conditions.

* Review Paper – accepted after double-blind review.
ISBN: 978-0-9872527-1-5
Melbourne: Geospatial Sciences, RMIT 2012

To determine the angular and linear misclosures of an open traverse; the coordinate differences between the start and end points must be known and the bearings of the first and last lines of the traverse must be able to be determined from observations to other known points. If the terminal points of an open traverse do not have a known coordinate relationship then no linear or angular misclosures can be obtained from the traverse observations.

In this paper we will be primarily concerned with estimating precisions of stations in closed traverses although the methods we outline can be just as easily applied to stations in an open traverses; and precision can be taken to mean variances s_E^2, s_N^2 of east and north coordinates respectively; variances $s_\alpha^2, s_\beta^2, s_\theta^2, s_l^2$ of directions, angles, bearings and distances respectively; or their standard deviations $s_E, s_N, s_\alpha, s_\beta,$ etc. noting that standard deviation is defined as the positive square-root of the variance.

It is common practice to assess the quality of traverses by comparing angular/linear misclosures and or misclose ratios with 'practical standards'. For example the Victorian *Surveying (Cadastral Surveys) Regulations 2005* states in part (Regulation 7)

(1) A licensed surveyor must ensure that—

(a) the internal closure of any cadastral survey is such that the length of the misclose vector does not exceed—

(i) 15 millimetres + 100 parts per million of the perimeter for boundaries crossing level or undulating land; and

(ii) ....

These practical standards are not a 'modern' way of assessing traverse quality and are often based on historical survey practices and equipment that may not reflect modern techniques. Instead, a method is proposed that is based on simple assumptions of survey practice; knowledge of precisions of Total Station measurements and *Propagation of Variances* (PoV). This method assesses the quality of a traverse by comparing linear and angular misclosures with statistical estimates that are functions of the actual traverse measurements and the geometry of the traverse. In addition, this method: (i) provides estimates of the precision of individual traverse stations; (ii) is easily programmed on calculators (and computer spread sheets) and (iii) can be enhanced with error ellipse displays.

As a first step the rule for PoV is introduced – with a special case when variables are independent – and then it is shown how this rule can be extended by using matrix algebra and applied to the computation of coordinates of traverse stations. Then, it is shown how the determination of traverse bearings can be broken down into a sequence of simple Total Station measurements with errors than can be plausibly explained and modelled; which in turn enables reasonable estimates of variances of traverse bearings (using PoV). Finally, using the traverse observations (bearings and distances) with estimates of their variances it is shown how they are combined in a sequential application of PoV to give precision estimates of the coordinates of the traverse stations. A rule for assessing the quality of a traverse follows logically from these precision estimates.

Propagation of Variances (PoV)

In surveying, propagation involves obtaining information about a function (or process, or computation) involving variables (measurements or functions of measurements) that are subject to systematic or random variation. Systematic errors are most often due to poor measurement technique or perhaps equipment that is not properly calibrated or used incorrectly. In this paper, there is an assumption that the surveyor properly understands her traversing equipment (Total Station + tribrachs, tripods, prisms, etc.); it is calibrated; her field technique is adequate and measurements have been corrected for the effects of systematic errors. That leaves the effects of random errors to be dealt with and Propagation of Variances (PoV) is also known as propagation of random errors.

Consider a function w of variables $x, y, z, \ldots, t$ that are affected by small random errors $\varepsilon_x, \varepsilon_y, \varepsilon_z, \ldots, \varepsilon_t$ that are assumed to be random variables of infinite populations having means $\mu_x, \mu_y, \mu_z, \ldots, \mu_t$ and variances $\sigma_x^2, \sigma_y^2, \sigma_z^2, \ldots, \sigma_t^2$ then the *Law of Propagation of Variances* allows us to say:

For a function $w = w(x, y, z, \ldots, t)$ the *variance* σ_w^2 is

[3] Systematic errors follow some fixed law (possibly unknown) dependent on local conditions and/or the equipment being used. Propagation of systematic errors can be modelled by using the Total Increment Theorem (or Total Differential) of mathematics

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$$\sigma_w^2 \approx \left(\frac{\partial w}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial w}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial w}{\partial z}\right)^2 \sigma_z^2 + \cdots + \left(\frac{\partial w}{\partial t}\right)^2 \sigma_t^2$$
$$+ 2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\sigma_{xy} + 2\frac{\partial w}{\partial x}\frac{\partial w}{\partial z}\sigma_{xz} + \cdots + 2\frac{\partial w}{\partial x}\frac{\partial w}{\partial t}\sigma_{xt}$$
$$+ 2\frac{\partial w}{\partial y}\frac{\partial w}{\partial z}\sigma_{xz} + \cdots + 2\frac{\partial w}{\partial y}\frac{\partial w}{\partial t}\sigma_{yt} + \cdots + 2\frac{\partial w}{\partial z}\frac{\partial w}{\partial t}\sigma_{zt} \quad (1)$$

where $\sigma_{xy}, \sigma_{xz}, \sigma_{xt}, \ldots$ etc. are *covariances* that measure of how much two variables change together.

If the variables in the function w are independent of each other then their covariance is zero and the *Special Law of Propagation of Variances* follows as

$$\sigma_w^2 \approx \left(\frac{\partial w}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial w}{\partial y}\right)^2 \sigma_y^2 + \left(\frac{\partial w}{\partial z}\right)^2 \sigma_z^2 + \cdots + \left(\frac{\partial w}{\partial t}\right)^2 \sigma_t^2 \quad (2)$$

Suppose that p and q are both functions of variables x, y and z that are affected by small random errors $\varepsilon_x, \varepsilon_y, \varepsilon_z$ assumed to be drawn from populations having variances $\sigma_x^2, \sigma_y^2, \sigma_z^2$. The functions p and q may be combined in a vector $\mathbf{y} = [p \quad q]^T$, and the variables x,y,z in a vector $\mathbf{x} = [x \quad y \quad z]^T$ where $[\ ]^T$ represents the vector (or matrix) transpose; then we may write:

$$\text{If } \mathbf{y} = f(\mathbf{x}) \quad (3)$$

and the Law of Propagation of Variances is expressed as (Mikhail & Gracie 1981)

$$\mathbf{\Sigma}_{yy} = \mathbf{J}_{yx}\mathbf{\Sigma}_{xx}\mathbf{J}_{yx}^T \quad (4)$$

where the variance matrices are $\mathbf{\Sigma}_{yy} = \begin{bmatrix} \sigma_p^2 & \sigma_{pq} \\ \sigma_{qp} & \sigma_q^2 \end{bmatrix}$ $\mathbf{\Sigma}_{xx} = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_y^2 & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_z^2 \end{bmatrix}$

and the matrix of partial derivatives is $\mathbf{J}_{yx} = \begin{bmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} & \frac{\partial p}{\partial z} \\ \frac{\partial q}{\partial x} & \frac{\partial q}{\partial y} & \frac{\partial q}{\partial z} \end{bmatrix}$

Note: (i) covariances $\sigma_{ab} = \sigma_{ba}$; (ii) variance matrices are square and symmetric; and (iii) if variables are independent then the off-diagonal elements of variance matrices are zero.

In measurement sciences, random errors are assumed to be members of infinite populations having means μ and variances σ^2 but in practice these quantities are unknown and instead we use estimates $\bar{x}$ (mean) and s^2 (variance) calculated from small samples of measurements; or just assumed. So equations (1) and (2) can be expressed with estimates $s_x^2, s_y^2, s_z^2, \ldots$ and $s_{xy}, s_{xz}, \ldots$ replacing the population quantities $\sigma_x^2, \sigma_y^2, \sigma_z^2, \ldots$ and $\sigma_{xy}, \sigma_{xz}, \ldots$. And equation (4) can be expressed with *cofactor matrices* $\mathbf{Q}_{yy}$ and $\mathbf{Q}_{xx}$ containing estimates of variances and covariances replacing variance matrices $\mathbf{\Sigma}_{yy}$ and $\mathbf{\Sigma}_{xx}$.

The matrix approach to PoV [equations (3) and (4)] can be demonstrated by the example of a traverse line having a bearing θ and length l connecting points $k-1$ and k of a traverse. The east and north coordinates of the k^{th} traverse station are

$$\begin{aligned} E_k &= l\sin\theta + E_{k-1} = \Delta E + E_{k-1} \\ N_k &= l\cos\theta + N_{k-1} = \Delta N + N_{k-1} \end{aligned} \quad (5)$$

E_k, N_k are functions of $l, \theta, E_{k-1}, N_{k-1}$ which can be expressed symbolically in the matrix equation

$$\mathbf{y} = f(\mathbf{x}) \quad (6)$$

where $\mathbf{y} = [E_k \quad N_k]^T$ and $\mathbf{x} = [l \quad \theta \quad E_{k-1} \quad N_{k-1}]^T$. Applying PoV to (6) gives

* Review Paper – accepted after double-blind review.
ISBN: 978-0-9872527-1-5
Melbourne: Geospatial Sciences, RMIT 2012

$$\mathbf{Q}_{yy} = \mathbf{J}_{yx}\mathbf{Q}_{xx}\mathbf{J}_{yx}^{T} \qquad (7)$$

$\mathbf{Q}_{yy}$ and $\mathbf{Q}_{xx}$ are cofactor matrices containing estimates of variances and covariances of the elements of **y** and **x** respectively.

$$\mathbf{Q}_{yy} = \begin{bmatrix} s_{E_k}^2 & s_{E_kN_k} \\ s_{E_kN_k} & s_{N_k}^2 \end{bmatrix}, \quad \mathbf{Q}_{xx} = \left[\begin{array}{cc|cc} s_l^2 & 0 & 0 & 0 \\ 0 & s_\theta^2 & 0 & 0 \\ \hline 0 & 0 & s_{E_{k-1}}^2 & s_{E_{k-1}N_{k-1}} \\ 0 & 0 & s_{E_{k-1}N_{k-1}} & s_{N_{k-1}}^2 \end{array}\right] \qquad (8)$$

$\mathbf{J}_{yx}$ is the matrix of partial derivatives

$$\mathbf{J}_{yx} = \begin{bmatrix} \frac{\partial E_k}{\partial l} & \frac{\partial E_k}{\partial \theta} & \frac{\partial E_k}{\partial E_{k-1}} & \frac{\partial E_k}{\partial N_{k-1}} \\ \frac{\partial N_k}{\partial l} & \frac{\partial N_k}{\partial \theta} & \frac{\partial N_k}{\partial E_{k-1}} & \frac{\partial N_k}{\partial N_{k-1}} \end{bmatrix} = \begin{bmatrix} \sin\theta & l\cos\theta & 1 & 0 \\ \cos\theta & -l\sin\theta & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\Delta E}{l} & \Delta N & 1 & 0 \\ \frac{\Delta N}{l} & -\Delta E & 0 & 1 \end{bmatrix} \qquad (9)$$

Carrying out the matrix multiplications of (7) gives the cofactor matrix of the computed coordinates

$$\mathbf{Q}_{yy} = \left[\begin{array}{c|c} \left\{\left(\frac{\Delta E}{l}\right)^2 s_l^2 + (\Delta N)^2 s_\theta^2 + s_{E_{k-1}}^2\right\} & \left\{\left(\frac{\Delta E \Delta N}{l^2}\right) s_l^2 - (\Delta E \Delta N) s_\theta^2 + s_{E_{k-1}N_{k-1}}\right\} \\ \hline \left\{\left(\frac{\Delta E \Delta N}{l^2}\right) s_l^2 - (\Delta E \Delta N) s_\theta^2 + s_{E_{k-1}N_{k-1}}\right\} & \left\{\left(\frac{\Delta N}{l}\right)^2 s_l^2 + (\Delta E)^2 s_\theta^2 + s_{N_{k-1}}^2\right\} \end{array}\right] \qquad (10)$$

The elements of $\mathbf{Q}_{yy}$ are the estimates of variances and covariances of the computed coordinates of point *k*. These elements replace the lower-right block in $\mathbf{Q}_{xx}$ in the computation of the precision estimates of the next point in the traverse. The matrix multiplications of equation (4) can easily be done with spreadsheet computer programs.

To carry out this PoV we require estimates of the precisions of traverse bearings and distances. The determination of these is the subject of the following sections.

Estimating the Precision of Traverse Bearings

Consider the operation of determining a traverse bearing.

1. The Total Station is pointed to the back-sight and the horizontal circle (or direction) read, α_{Back}.
2. The Total Station is turned clockwise to the forward-sight (or for-sight) and the horizontal circle read again, α_{For}.
3. The first direction is subtracted from the second direction to obtain the clockwise angle $\beta = \alpha_{For} - \alpha_{Back}$.
4. β is added to the back-sight bearing to give the bearing of the for-sight $\theta_{For} = \theta_{Back} + \beta$

This might not be the simplest or most common field technique but it suffices for separating the operation into certain parts.

To estimate the variance of an observed traverse bearing $\left(s_\theta^2\right)$, equation (2) can be applied to the equation $\theta_{For} = \theta_{Back} + \beta$, assuming that the measured angle β and the bearing of the back-sight θ_{Back} are independent.

This gives

$$s_{\theta_{For}}^2 = s_{\theta_{Back}}^2 + s_\beta^2 \qquad (11)$$

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In equation (11) an estimate of the variance of the measured angle $\left(s_\beta^2\right)$ is required and this may be considered as consisting of two parts; (i) the precision of pointing and reading and (ii) the precision of centring at the observing and target stations. Now since any errors in pointing and reading are independent of direction we may apply equation (2) and write

$$s_\beta^2 = s_{PR}^2 + s_{CENT}^2 \tag{12}$$

s_{PR}^2 and s_{CENT}^2 are estimates of the variances of pointing and reading error and centring error respectively and how these are obtained is discussed in the following sections.

Estimating the Precision of Pointing and Reading Errors

Figure 1 shows two lines, *AB* and *BC*, of a traverse. The Total Station is at *B*, the back-sight target is at *A* and the for-sight target is at *C*. θ_1, l_1 are the bearing and distance respectively of leg 1 and θ_2, l_2 are the bearing and distance of leg 2. α_1, α_2 are the horizontal directions to *A* and *C* respectively and β is the horizontal angle between the two traverse lines.

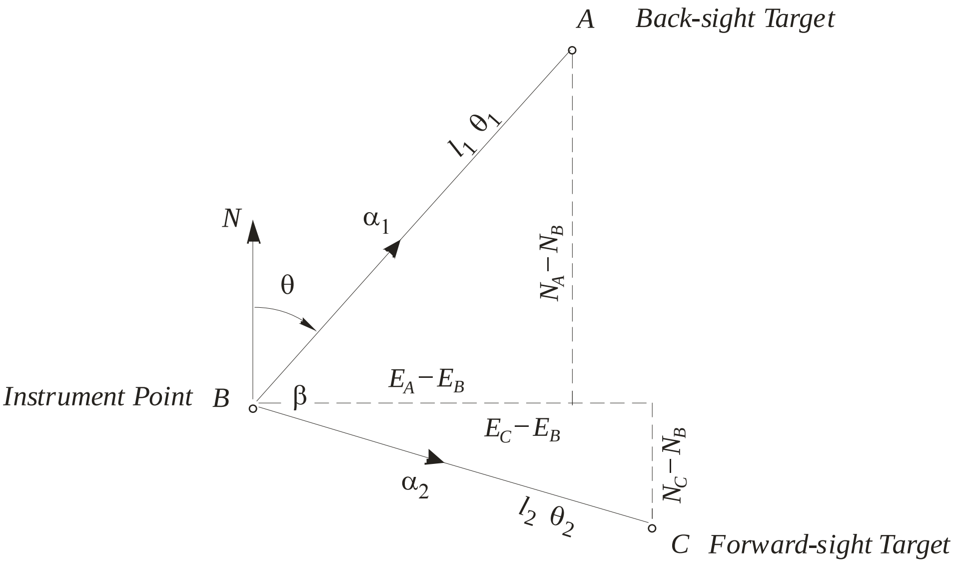

Figure 1

Good survey practice dictates that directions are read on Face Left (FL) and Face Right (FR) of the Total Station and averaged to eliminate the effects of collimation and so we may write

$$\beta_{FL} = \alpha_{2_{FL}} - \alpha_{1_{FL}} \quad \text{and} \quad \beta_{FR} = \alpha_{2_{FR}} - \alpha_{1_{FR}} \tag{13}$$

β_{FL}, β_{FR} are FL/FR angles, $\alpha_{1_{FL}}, \alpha_{1_{FR}}$ are FL/FR directions to the back-sight target and $\alpha_{2_{FL}}, \alpha_{2_{FR}}$ are FL/FR directions to the for-sight station. Applying the Special Law of Propagation of Variances to equation (13) gives the variances of the FL/FR angles as

$$s_{\beta_{FL}}^2 = s_{\alpha 2_{FL}}^2 + s_{\alpha 1_{FL}}^2 \quad \text{and} \quad s_{\beta_{FR}}^2 = s_{\alpha 2_{FR}}^2 + s_{\alpha 1_{FR}}^2$$

Assuming the variances of the FL/FR directions are equal and independent of direction, then we may write $s_{\alpha_{FL}}^2 = s_{\alpha_{FR}}^2 = s_\alpha^2$ and the variances of the FL/FR angles become

$$s_{\beta_{FL}}^2 = 2s_\alpha^2 \quad \text{and} \quad s_{\beta_{FR}}^2 = 2s_\alpha^2$$

The angle β is the average of the FL/FR angles, i.e.,

$$\beta = \tfrac{1}{2}\left(\beta_{FL} + \beta_{FR}\right) \tag{14}$$

and applying the Special Law of Propagation of Variances to equation (14) and using results above gives the variance of a mean angle from a single pair of FR/FL pointings as

$$s_\beta^2 = \left(\tfrac{1}{2}\right)^2 s_{\beta_{FL}}^2 + \left(\tfrac{1}{2}\right)^2 s_{\beta_{FR}}^2 = s_\alpha^2$$

Denoting the variance of the angle as the variance due to pointing and reading errors we say for a single pair of FR/FL pointings

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ISBN: 978-0-9872527-1-5
Melbourne: Geospatial Sciences, RMIT 2012

$$s_{PR}^2 = s_\alpha^2 \qquad (15)$$

s_α^2 is the variance of a single-face direction of the Total Station.

Estimating the Precision of Instrument and Target Centring Errors

We may express the traverse angle β as

$$\beta = \theta_2 - \theta_1 = \arctan\frac{E_C - E_B}{N_C - N_B} - \arctan\frac{E_A - E_B}{N_N - N_B} \qquad (16)$$

so $\beta = \beta\left(E_A, N_A, E_B, N_B, E_C, N_C\right)$ or in symbolic matrix form $\mathbf{y} = f\left(\mathbf{x}\right)$ and applying Propagation of Variances gives

$$\mathbf{Q}_{yy} = \mathbf{J}_{yx}\mathbf{Q}_{xx}\mathbf{J}_{yx}^T \qquad (17)$$

where $\mathbf{Q}_{yy} = \left[s_\beta^2\right]$; $\mathbf{J}_{yx} = \left[\begin{matrix} \dfrac{\partial\beta}{\partial E_A} & \dfrac{\partial\beta}{\partial N_A} & \dfrac{\partial\beta}{\partial E_B} & \dfrac{\partial\beta}{\partial N_B} & \dfrac{\partial\beta}{\partial E_C} & \dfrac{\partial\beta}{\partial N_C} \end{matrix}\right]$ and

$$\mathbf{Q}_{xx} = \left[\begin{array}{cc:cc:cc} s_{E_A}^2 & s_{E_A N_A} & s_{E_A E_B} & s_{E_A N_B} & s_{E_A E_C} & s_{E_A N_C} \\ & s_{N_A}^2 & s_{N_A E_B} & s_{N_A N_B} & s_{N_A E_C} & s_{N_A N_B} \\ \hdashline & & s_{E_B}^2 & s_{E_B N_B} & s_{E_B E_C} & s_{E_B N_C} \\ & & & s_{N_B}^2 & s_{N_B E_C} & s_{N_B N_C} \\ \hdashline & & & & s_{E_C}^2 & s_{E_C N_C} \\ & & & & & s_{N_C}^2 \end{array}\right]$$

For the purposes of developing a working formula for estimating the effects of centring errors, it is assumed that all covariances terms in $\mathbf{Q}_{xx}$ equal zero, i.e., $E_A, N_A, E_B, N_B, E_C, N_C$ are independent random variables. This means that equation (17) can be expressed as

$$s_\beta^2 = \left(\frac{\partial\beta}{\partial E_A}\right)^2 s_{E_A}^2 + \left(\frac{\partial\beta}{\partial N_A}\right)^2 s_{N_A}^2 + \left(\frac{\partial\beta}{\partial E_B}\right)^2 s_{E_B}^2 + \left(\frac{\partial\beta}{\partial N_B}\right)^2 s_{N_B}^2 + \left(\frac{\partial\beta}{\partial E_C}\right)^2 s_{E_C}^2 + \left(\frac{\partial\beta}{\partial N_C}\right)^2 s_{N_C}^2 \qquad (18)$$

Furthermore let $s_{E_A}^2 = s_{N_A}^2 = s_1^2$; $s_{E_B}^2 = s_{N_B}^2 = s_2^2$; $s_{E_C}^2 = s_{N_C}^2 = s_3^2$ and denote the variance of the angle as the variance of the centring errors, i.e., $s_\beta^2 = s_{CENT}^2$ then equation (18) becomes an expression for the variance of the centring errors written as

$$s_{CENT}^2 = \left\{\left(\frac{\partial\beta}{\partial E_A}\right)^2 + \left(\frac{\partial\beta}{\partial N_A}\right)^2\right\} s_1^2 + \left\{\left(\frac{\partial\beta}{\partial E_B}\right)^2 + \left(\frac{\partial\beta}{\partial N_B}\right)^2\right\} s_2^2 + \left\{\left(\frac{\partial\beta}{\partial E_C}\right)^2 + \left(\frac{\partial\beta}{\partial N_C}\right)^2\right\} s_3^2 \qquad (19)$$

Substituting the partial derivatives of equation (16) into equation (19) and simplifying gives

$$s_{CENT}^2 = \frac{l_2^2 s_1^2 + l_1^2 s_3^2 + \left(l_1^2 + l_2^2 - 2 l_1 l_2 \cos\beta\right) s_2^2}{l_1^2 l_2^2} \qquad (20)$$

Equation (20) is the same as equation (16.34) given in Richardus (1966, p. 290). The estimates of variances s_1^2, s_2^2, s_3^2 at the traverse stations *A*, *B*, *C* respectively, are the same in any direction at those points, and can be considered as estimates of the precision of centring. If the back-sight and for-sight target centring errors are considered equal and the traverse lengths equal; i.e., $s_1^2 = s_3^2$ and $l_1 = l_2 = l$ then equation (20) becomes $s_{CENT}^2 = \left\{2s_1^2 + 2\left(1 - \cos\beta\right)s_2^2\right\}/l^2$ which is a maximum when $\beta = 180^\circ$ and $\cos\beta = -1$, in which case $s_{CENT}^2\left(\max\right) = \left\{2s_1^2 + 4s_2^2\right\}/l^2$. The conclusion from this equation is that Total Station centring errors have twice the effect as target centring errors. This makes it clear that greater care should be taken in centring the Total Station (Richardus, 1966, p. 291).

In traversing operations it is plausible to consider that Total Station and target centring errors will be similar and $s_1^2 = s_2^2 = s_3^2 = s_c^2$ and equation (20) can be modified to give an expression of standard deviation

$$s_{CENT} = s_c\sqrt{2}\sqrt{\frac{1}{l_1^2} + \frac{1}{l_2^2} - \frac{\cos\beta}{l_1 l_2}} \qquad (21)$$

* Review Paper – accepted after double-blind review.
ISBN: 978-0-9872527-1-5
Melbourne: Geospatial Sciences, RMIT 2012

But this simplification does not take into account the 'fact' established above that Total Station centring errors have more effect than target centring errors. We may test equation (21) using Matlab and a Monte Carlo simulation[4].

Consider Figure 1 and imagine that the instrument point *B* moves a distance *c* in a random direction and that *c* is a random variable drawn from a normal distribution having a mean of zero and standard deviation s_c and that the back-sight and for-sight targets also move in a similar random manner. The angle at the instrument point is β_1. If this randomized location of the instrument point and target points is repeated then a sample of angles $\beta_1 \beta_2 \beta_3 \ldots \beta_n$ is obtained which will have a sample standard deviation $s_\beta = s'_{CENT}$ that we call the simulated standard deviation. If *n* is large then the sample standard deviation will approach the population standard deviation $\sigma_\beta = \sigma_{CENT}$.

A Matlab function *angletest.m* was used to test the value of s_{CENT} from equation (21) against the simulated value s'_{CENT} for a range of back-sight and for-sight distances and angles and Table 1 below shows the output from this function.

```
>> angletest(0.005,10000,20)

 d1    d2   beta   sx      rule1  (rule1/sx)    rule1a  (rule1a-sx)
-----------------------------------------------------------------
 123    34   104  32.32    45.87   (1.42)        32.43   ( 0.12)
  59   119   267  19.57    27.88   (1.42)        19.71   ( 0.14)
 138    71   132  18.25    26.06   (1.43)        18.43   ( 0.17)
 126   166   175  12.41    17.68   (1.42)        12.50   ( 0.09)
 128   164   260  10.69    15.05   (1.41)        10.64   (-0.04)
  87   102   211  18.71    26.29   (1.41)        18.59   (-0.12)
 115   103    75  12.63    17.74   (1.41)        12.55   (-0.08)
  93   170    72  11.68    16.67   (1.43)        11.79   ( 0.10)
  97    21    65  48.25    67.89   (1.41)        48.00   (-0.25)
  81    48   187  30.13    42.32   (1.40)        29.92   (-0.21)
  74   198   303  13.53    19.07   (1.41)        13.48   (-0.04)
  95   200   337   9.64    13.63   (1.41)         9.64   (-0.00)
 100    88   183  19.15    27.00   (1.41)        19.09   (-0.06)
  85   180   163  15.69    22.20   (1.41)        15.70   ( 0.01)
  69   196   106  16.59    23.36   (1.41)        16.52   (-0.07)
  82   178   221  15.73    22.21   (1.41)        15.71   (-0.02)
 124   186   126  11.22    15.94   (1.42)        11.27   ( 0.05)
 172    43    12  21.87    30.68   (1.40)        21.69   (-0.18)
 120   169   345   7.83    10.99   (1.40)         7.77   (-0.06)
  92    46   305  21.97    31.12   (1.42)        22.00   ( 0.03)
```

Table 1 Output from Matlab function *angletest.m*

In Table 1 `d1` and `d2` are the back-sight and for-sight distances l_1, l_2; `sx` is the simulated standard deviation that we denote s'_{CENT}; and `rule1` is s_{CENT} computed from equation (21). `rule1a = rule1/sqrt(2).` *angletest.m* has the input parameters $s_c = 0.005$ m, $n = 10000$ simulations and 20 combinations of traverse distances and angles. The distances `d1` and `d2` are drawn from a uniform distribution of random integers between 20 and 200 metres. The angle `beta` is drawn from a uniform distribution of random integers between 10 and 350 degrees. The first line of Table 1 has traverse lines $l_1 = 124$ m, $l_2 = 34$ m and traverse angle $\beta = 104^\circ$; then $s'_{CENT} = 32.32''$ from 10000 simulations and $s_{CENT} = 45.87''$ from equation (21). The number in parentheses is $s_{CENT}/s'_{CENT} = 45.87/32.32 = 1.42$ that is the ratio between the computed and simulated standard deviation and is approximately equal to $\sqrt{2}$. The next number is $s_{CENT}/\sqrt{2}$ and the last number in parentheses is the difference $s_{CENT}/\sqrt{2} - s'_{CENT}$. Inspection of the values in Table 1 shows that the standard deviation s_{CENT} computed from equation (21) is consistently larger, by a factor of $\sqrt{2}$, than the simulated value s'_{CENT}.

This leads to a better rule for estimating the standard deviation of a centring error as

[4] A method of repeated sampling to determine the properties of a particular function or phenomenon. The method employs a pseudo-random number generator to simulate small random changes in function variables that can be used to assess their combined effect on the function.

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ISBN: 978-0-9872527-1-5
Melbourne: Geospatial Sciences, RMIT 2012

$$s_{CENT} = s_c \sqrt{\frac{1}{l_1^2} + \frac{1}{l_2^2} - \frac{\cos\beta}{l_1 l_2}} \qquad (22)$$

The rule (22) – developed from Richardus (1966, eq. 16.34, p. 290) – has some similarity with two other rules developed by Briggs (1912, eq. 64, p.80) and Miller (1936, eq. 5, p.29).

$$\text{average angular error due to imperfect centring } = \pm\frac{2r}{\pi}\sqrt{\frac{1}{l_1^2} + \frac{1}{l_2^2} - \frac{2\cos\beta}{l_1 l_2}} \qquad \text{Briggs (1912)} \qquad (23)$$

$$\text{probable error due to imperfect centring } = \pm p\sqrt{2\pi}\sqrt{\frac{1}{l_1^2} + \frac{1}{l_2^2} + \frac{\cos\beta}{2 l_1 l_2}} \qquad \text{Miller (1936)} \qquad (24)$$

Rainsford (1957, pp.30-31) defines the *probable error* γ as that error that has a probability of occurrence of 0.5; and the *average error* η as the mean of all errors taken with the same sign and gives the relationships $\gamma = 0.6745\sigma$ and $\eta = 0.7979\sigma$ where σ is the standard deviation of a Normal distribution with mean zero.

In Briggs' formula *r* is the average centring displacement of the instrument (targets considered error free) and in Miller's formula *p* is the probable error of plumbing over a station (instrument and targets). Miller's derivation was motivated by Briggs only considering the centring of the instrument. Both formula; modified by relationships connecting average and probable error with standard deviation, were compared with the rule (22) but showed no real consistency.

Finally, the precision of a traverse bearing can be obtained from the following sequence given the precisions of a single face pointing of a Total Station s_α; centring error s_c and back-sight bearing $s_{\theta_{Back}}$:

(i) compute the estimate of the precision of the centring error s_{CENT} [equation (22)]

(ii) set the estimate of precision of pointing and reading $s_{PR} = s_\alpha$ [equation (15)]

(iii) compute the estimate of the precision of a traverse angle s_β [equation (12)]

(iv) compute the precision of the forward bearing of the traverse line $s_{\theta_{For}}$ [equation (11)]

Estimating the Precision of Traverse Distances

Most manufacturers of Total Stations state the accuracy of their instrument's EDM in the following form (Rüeger 1990)

$$s = \pm\left(A + B\frac{d}{1000}\right) \qquad (25)$$

where *A* is in mm; *B* in ppm[5]; *d* is distance in metres. For example, if *A* = 5 mm, *B* = 3 ppm and the measured distance was 1355.310 m then *s* = ± 9.1 mm.

s (in mm) is considered to be an estimate of standard deviation. The term *A* includes the electronic resolution of the EDM, compatibility of reflectors, accuracy of pre-set additive constants, maximum effects of short periodic errors and non-linear distance dependent errors. The *B* term is a scale error determined by calibration over known distances. An estimate of the variance (in mm^2) is obtained from

$$s_l^2 = \left(A + B\frac{d}{1000}\right)^2 \qquad (26)$$

Estimating the Precision of the Last Line of a Traverse

Suppose that estimates of precisions of the traverse bearings and distances are obtained using the method and equations set out above; then applying Propagation of Variances [equations (7), (8), (9) and (10)] we can obtain variances and covariances

[5] ppm is parts-per-million and since there are 1 million mm in a km then ppm is also mm per km

ISBN: 978-0-9872527-1-5
Melbourne: Geospatial Sciences, RMIT 2012

s_E^2, s_N^2, s_{EN} of the coordinates of the traverse stations. These quantities define the size, shape and orientation of error ellipses[6] at a traverse station but in pairs (end points of traverse lines) they can be used to estimate the precision of the bearing and distance of a traverse line. This, of course, is perfectly reasonable since these estimates have been used to compute s_E^2, s_N^2 and s_{EN} but general equations can be developed that will be useful for traverse analysis.

The bearing and distance between points P_i and P_k are functions of the east and north coordinates of the points $\theta_{ik} = \tan^{-1}\left[(E_k - E_i)/(N_k - N_i)\right] = f_1(E_i, N_i, E_k, N_k)$ and $l_{ik} = \sqrt{(E_k - E_i)^2 + (N_k - N_i)^2} = f_2(E_i, N_i, E_k, N_k)$ or symbolically; $\mathbf{y} = f(\mathbf{x})$ where $\mathbf{y} = \begin{bmatrix}\theta_{ik} & l_{ik}\end{bmatrix}^T$ and $\mathbf{x} = \begin{bmatrix}E_i & N_i & E_k & N_k\end{bmatrix}^T$; and PoV can be written as

$$\mathbf{Q}_{yy} = \mathbf{J}_{yx}\mathbf{Q}_{xx}\mathbf{J}_{yx}^T \tag{27}$$

where

$$\mathbf{Q}_{yy} = \begin{bmatrix} s_{\theta_{ik}}^2 & s_{\theta l} \\ s_{\theta l} & s_{l_{ik}}^2 \end{bmatrix}, \quad \mathbf{Q}_{xx} = \left[\begin{array}{cc|cc} s_{E_i}^2 & s_{E_iN_i} & s_{E_iE_k} & s_{E_iN_k} \\ s_{E_iN_i} & s_{N_i}^2 & s_{N_iE_k} & s_{N_iN_k} \\ \hline s_{E_iE_k} & s_{N_iE_k} & s_{E_k}^2 & s_{E_kN_k} \\ s_{E_iN_k} & s_{N_iN_k} & s_{E_kN_k} & s_{N_k}^2 \end{array}\right] \tag{28}$$

and the matrix of partial derivatives is

$$\mathbf{J}_{yx} = \begin{bmatrix} \dfrac{\partial\theta_{ik}}{\partial E_i} & \dfrac{\partial\theta_{ik}}{\partial N_i} & \dfrac{\partial\theta_{ik}}{\partial E_k} & \dfrac{\partial\theta_{ik}}{\partial N_k} \\ \dfrac{\partial s}{\partial E_i} & \dfrac{\partial s}{\partial N_i} & \dfrac{\partial s}{\partial E_k} & \dfrac{\partial s}{\partial N_k} \end{bmatrix} = \begin{bmatrix} \dfrac{-\cos\theta_{ik}}{l_{ik}} & \dfrac{\sin\theta_{ik}}{l_{ik}} & \dfrac{\cos\theta_{ik}}{l_{ik}} & \dfrac{-\sin\theta_{ik}}{l_{ik}} \\ -\sin\theta_{ik} & -\cos\theta_{ik} & \sin\theta_{ik} & \cos\theta_{ik} \end{bmatrix} = \begin{bmatrix} -b_{ik} & -a_{ik} & b_{ik} & a_{ik} \\ -d_{ik} & -c_{ik} & d_{ik} & c_{ik} \end{bmatrix} \tag{29}$$

a_{ik}, b_{ik} are *direction coefficients* and c_{ik}, d_{ik} are *distance coefficients.* Performing the matrix multiplications of (27) gives the variances of the bearing and distance as

$$\begin{aligned} s_{\theta_{ik}}^2 &= b_{ik}^2\left(s_{E_i}^2 + s_{E_k}^2 - 2s_{E_iE_k}\right) + a_{ik}^2\left(s_{N_i}^2 + s_{N_k}^2 - 2s_{N_iN_k}\right) \\ &\quad + 2a_{ik}b_{ik}\left(s_{E_iN_i} + s_{E_kN_k} - s_{E_iN_k} - s_{E_kN_i}\right) \end{aligned} \tag{30}$$

$$\begin{aligned} s_{l_{ik}}^2 &= d_{ik}^2\left(s_{E_i}^2 + s_{E_k}^2 - 2s_{E_iE_k}\right) + c_{ik}^2\left(s_{N_i}^2 + s_{N_k}^2 - 2s_{N_iN_k}\right) \\ &\quad + 2c_{ik}d_{ik}\left(s_{E_iN_i} + s_{E_kN_k} - s_{E_iN_k} - s_{E_kN_i}\right) \end{aligned} \tag{31}$$

In this limited analysis: (i) the covariances between points [the upper-right and lower-left blocks of $\mathbf{Q}_{xx}$ in equation (28)] are assumed to be zero; (ii) P_i is the start point of a traverse and assumed to be fixed and 'error free' (variances and covariances = zero) and (iii) P_k is the last point of the traverse. This means that the equations for the estimating standard deviations are simplified to

$$s_{\theta_{ik}} = \sqrt{b_{ik}^2 s_{E_k}^2 + a_{ik}^2 s_{N_k}^2 + 2a_{ik}b_{ik}s_{E_kN_k}} \tag{32}$$

$$s_{l_{ik}} = \sqrt{d_{ik}^2 s_{E_k}^2 + c_{ik}^2 s_{N_k}^2 + 2c_{ik}d_{ik}s_{E_kN_k}} \tag{33}$$

Conclusion

Equations (32) and (33) are proposed as measures to assess the quality of a traverse. And a rule is proposed:

reject a traverse if the angular or linear misclose is greater than two standard deviations.

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[6] Variance in any direction about a point is a function of variances s_E^2, s_N^2 and the covariance s_{EN} and defines the pedal curve of the standard error ellipse

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ISBN: 978-0-9872527-1-5
Melbourne: Geospatial Sciences, RMIT 2012

This is a better approach than using misclose ratios or other 'practical standards' and modern computer/calculator software could make this seemingly complex analysis a simple field task. The example below may assist in understanding the analysis.

Example

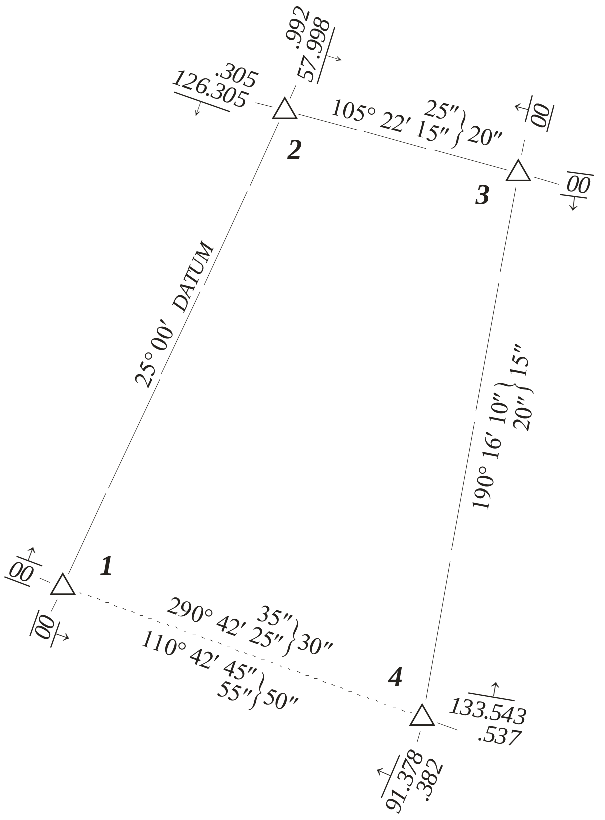

Figure 2 Traverse

Assume:

centring errors $s_c = 0.002$ m

st.dev. of direction $s_\alpha = 5''$

st.dev. of distance $s_l = 5 \text{ mm} + 5 \text{ ppm}$

Use:

$$s_\beta^2 = s_{PR}^2 + s_{CENT}^2$$

$$s_{PR}^2 = s_\alpha^2$$

$$s_{CENT}^2 = \left(\frac{1}{l_1^2} + \frac{1}{l_2^2} - \frac{\cos\beta}{l_1 l_2}\right) s_c^2$$

$$s_{\theta_{For}}^2 = s_{\theta_{Back}}^2 + s_\beta^2$$

Traverse operations:

1. Set up at 1; set bearing 25° 00′ along line 1-2; read bearing line 1-4
2. Set up at 2; set bearing 205° 00′ along line 2-1; read bearing line 2-3
3. Set up at 3; set bearing 285° 22′ 20″ along line 3-2; read bearing line 3-4
4. Set up at 4; set bearing 10° 16′ 15″ along line 4-3; read bearing line 4-1

Computation steps:

1. Tabulate the traverse bearings and distances (last line 4-1 not included) then calculate standard deviation of centring errors and traverse angles.

Line	Bearing	Distance
1-2	25° 00′ 00″	126.305
2-3	105° 22′ 20″	57.995
3-4	190° 16′ 15″	133.545

Point	l_1	l_2	β	s_{PR}	s_{CENT}	s_β
2	126.305	57.995	260° 22′ 20″	5″	8.07″	9.5″
3	57.995	133.545	264° 53′ 55″	5″	7.88″	9.3″

Table 2 Traverse Bearings & Distances and standard deviations of traverse angles

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2. Calculate the standard deviations of the traverse bearings and distances.

Line	s_{Back}	s_β	s_{For}
1-2			$0''$
2-3	$0''$	$9.5''$	$10''$
3-4	$10''$	$9.3''$	$14''$

Line	s_l
1-2	0.006
2-3	0.005
3-4	0.006

Table 3 Standard deviations of traverse bearings and distances

3. Calculate estimates of precision at points 2, 3 and 4 using equation (7)

Point	s_E	s_N	$s_{E,N}$
1	0.0000	0.0000	0
2	0.0025	0.0054	$1.3789E-04$
3	0.0055	0.0062	$9.4194E-06$
4	0.0105	0.0087	$1.3208E-06$

4. Calculate estimates of standard deviations of the bearing and distance of the last line 4-1 using equations (32) and (33)

$$s_\theta = 20.3''$$
$$s_l = 0.010 \text{ m}$$

5. Determine if traverse is acceptable.

Angular misclose: From the traverse shown in Figure 2, the angular misclose is 20″ (the difference between the two observed bearings on the last line 4-1)

Linear misclose: Using the mean bearing of line 4-1 (290° 42′ 40″) a closure of the traverse gives the linear misclose as 0.016 m.

The traverse is acceptable since the angular and linear misclosures are both less than two standard deviations of the relevant estimates of the last line.

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Acknowledgements

The author wishes to thank the reviewers for their thoughtful comments.

* Review Paper – accepted after double-blind review.
ISBN: 978-0-9872527-1-5
Melbourne: Geospatial Sciences, RMIT 2012