Interval-Timed Petri Nets with Auto-concurrent Semantics and their State Equation

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Abstract. In this paper we consider Interval-Timed Petri nets (ITPN), an extension of Timed Petri nets in which the discrete time delays of transitions are allowed to vary within fixed intervals including possible zero durations. These nets will be analyzed for the first time under some maximal step semantics with auto-concurrency. This matches well with the reality of time critical systems which could be modeled and analyzed with our model. We introduce in particular the notion of global firing step which regroups all what happens inbetween two time ticks. Full algebraic representations of the semantics are proposed. We introduce time-dependent state equations for a sequence of global firing steps of ITPNs which are analogous to the state equation for a firing sequence in standard Petri nets and we prove its correctness using linear algebra. Our result delivers a necessary condition for reachability which is also a sufficient condition for non-reachability of an arbitrary marking in an ITPN.

1 Introduction

Petri nets (PN) as proposed initially by Carl Adam Petri [4] are applied to design models of systems considering only causal relations in it and not temporal ones. Of course there is a huge field of applications in which time does not really matter. In real systems, however, the time is mostly indispensable and therefore it cannot be ignored. Thus a certain number of time-dependent Petri net classes had been proposed in the meanwhile, cf.([3], [9], [5], [2], [11], [1], [6]). Moreover, it is well known that the majority of these classes are more expressive then the classic model: Almost all time-dependent Petri net classes are Turing-powerful, while the power of classic Petri nets is less than that of Turing-machines.

In this paper we are dealing with Interval-Timed Petri nets (ITPN), which are an extension of Timed Petri nets (TPN), introduced by Ramhandani in [9] and extensively studied by Sifakis [10]. TPNs are classic PNs where each transition is associated with a natural number which describes its firing duration. TPNs, as well as their extensions like ITPNs, are Turing-powerful (cf. Popova [6]).

In ITPNs the firing duration of a transition is also given by a natural number but this duration is not fixed. It may vary within an interval which is associated with the transition. The apparition of a transition is thus divided in two events, the startfire and the endfire event. Inbetween them tick events may happen, corresponding to the passing (or elapsing) of one time unit of some global clock [1].

When transitions are enabled they must start firing. This is the reason why we consider as firing modus for ITPNs the firing in maximal steps. Two different step semantics are possible: with or without auto-concurrency. In this article, we consider ITPNs with auto-concurrency. This means that when a transition becomes enabled, irrespective of whether or not an instance of it is firing already, a new instance must immediately start firing. The firing duration of each new instance is choosen in a non-deterministic way and is a natural number, describing how many tick events may occur before the endfiring event. This number belongs to the interval associated with the transition. Contrary to previous work, zero firing durations are allowed in this article.

A configuration in a PN is described by a marking. Because of the explicit presence of time a marking alone cannot completely represent the configuration of a time-dependent Petri net however. For this reason we use the notion of "state" which includes both the marking and the corresponding temporal informations. The first aim of the paper is to introduce the maximal step semantics for the ITPNs formally: a firing step sequence in an ITPN consists of alternating so called *Globalsteps* (multisets of startfire and endfire events) and tick events. And we will prove some semantical properties.

The second aim of this paper is to provide a sufficient condition for non-reachability of states in ITPNs similar to the sufficient condition for non-reachability of markings for classic Petri nets. To illustrate this purpose, let us consider first the problem in a classic Petri net $\mathcal{N}$, starting with a firing sequence σ of $\mathcal{N}$. After the firing of such a sequence a certain marking M of $\mathcal{N}$ is reached. We can compute this marking using the following well known equation:

$$M = M_0 + C \cdot \psi_\sigma \tag{1}$$

where C is the incidence matrix of the Petri net $\mathcal{N}$ and ψ_σ is the Parikh vector of σ(whose i-th component gives the number of appearance of transition t_i in σ). This equation is also called the *state equation of the sequence* σ. Actually, it can be used in many more ways. We can consider each marking suitable for a net as reached after the firing of an unknown sequence. Now, we can consider the state equation of the unknown sequence, where the elements of the Parikh vector are variables. If this equality has no non-negative integer solution then there does not exist a sequence making the considered marking reachable. Therefore, this is a sufficient condition for the non-reachability of the marking. The following simple example illustrates this approach:

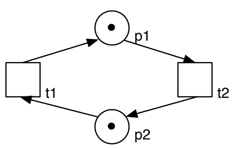

Fig. 1: PN $\mathcal{N}_1$.

Let us consider the PN $\mathcal{N}_1$ with $M_0 = (1,1)^T$ and show that the empty marking $M = (0,0)^T$ is not reachable in this net. The incidence matrix of $\mathcal{N}_1$ is $C_{\mathcal{N}_1} = \left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$. Let us assume that there is a transition sequence σ such that after its firing in $\mathcal{N}_1$ the empty marking is reached. When the transition $t1$ appears x_1 times in σ and $t2$ appears x_2 times then the Parikh vector of σ is $\psi_\sigma = (x_1, x_2)^T$. Subsequently, the equality (1) for this

transition sequence leads to the system of equations $\begin{cases} -1 = x_1 - x_2 \\ -1 = -x_1 + x_2 \end{cases}$. This equation system is obviously not solvable and therefore there is no such firing transition sequence σ in $\mathcal{N}_1$ leading to the empty marking M.
Furthermore, it is evident that the marking $M' = (2,0)^T$ is reachable in $\mathcal{N}_1$.

Let us now consider the Interval-Timed Petri net $\mathcal{D}_1$ arising from the PN $\mathcal{N}_1$ by adding time durations to each transition – the firing of each transition should take exactly one time unit, thus $[1,1]$ is the duration interval associated to $t1$ and $t2$. As both transitions are firable from the initial state, after startfiring both transitions in one step, the empty marking M is reached. After one tick event, both transitions need to endfire in one step, and the initial state is reached again. Thus it is easy to see that in this ITPN $\mathcal{D}_1$ the marking M' is not reachable. This simple example shows that reachability and non-reachability in an Interval-Timed Petri net are essentially unrelated to reachability and non-reachability in its untimed skeleton. Our aim is to prove with the help of a time-dependent state equation that for instance, it is impossible to reach M' in $\mathcal{D}_1$.

Of course, the time-dependent state equations we are establishling in this paper are much more complex than (1) or our previous results in [8], [7] and [2] because of the possibility of zero durations and the auto concurrent maximal step semantics. Nevertheless, our equations of a firing step sequence in an ITPN are consistent extensions of (1).

The paper is organized as follows: First formal definitions of ITPNs and their maximal step semantics are given in Section 2, and some semantical equivalence is proved. Then original algebraic representations and calculus of these semantics are proposed in Section 3. Some of them are adaptations of definitions known for the algebraic presentation of a firing step sequence for TPN [8], or ITPN without zero duration and without auto-concurrency [7], and others are entirely new here. Within this frame intermediate algebraic properties are first established in Section 4, leading then to the state equations. Full proofs of all results are included in the paper.

2 Interval-Timed Petri Nets and their semantics

This section will define the objects treated in this article.

As usual, $\mathbb{N}$ denotes the set of all natural numbers including zero, $\mathbb{N}^+$ is that without zero. A matrix A is a $(m \times n)$ - matrix when A has m rows and n columns. The denotation $A = \Big(a_{ij}\Big)_{\substack{i=1\cdots m\\ j=1\cdots n}}$ for a matrix A means that A is a $(m \times n)$ - matrix and a_{ij} is the element of A in the $(i)-$th row and in the $j-$th column. Furthermore, $A_{.j} = (a_{.j})$ denotes the j-th column of the matrix A and $A_{i.} = (a_{i.})$ denotes the i-th row. The $(d \times d)$ - matrix O_d denotes the $(d \times d)$ zero-matrix (all its elements are zero), the $(d \times d)$ - matrix E_d is the $(d \times d)$ identity matrix.

2.1 Net definitions

A (marked) *Petri net* (PN) is a quadruple $\mathcal{N} = (P, T, v, M_0)$, where P (the set of places) and T (the set of transitions) are finite and disjoint sets and

$v : (P \times T) \cup (T \times P) \longrightarrow \mathbb{N}$ defines the arcs with their weights and $M_0 : P \longrightarrow \mathbb{N}$ fixes the initial p-marking. In general, a p-marking $M : P \longrightarrow \mathbb{N}$ is presented by a vector of dimension $|P|$. As usual, t is called enabled in a p-marking M if for all $p \in P,\ v(p,t) \leq M(p)$.
Let $\mathcal{N}$ be a PN and $D : T \longrightarrow \mathbb{N} \times \mathbb{N}$ be a function. Then, a pair $\mathcal{Z} = (\mathcal{N}, D)$ is called an *Interval-Timed Petri net* (ITPN) where $\mathcal{N}$ is its *skeleton* and D its *duration function* including zero duration. Thus, D defines an interval for each transition. within which its firing duration can vary.
The bounds $\mathit{sfd}(t)$ and $\mathit{lfd}(t)$ with $D(t) = \Big(\mathit{sfd}(t), \mathit{lfd}(t)\Big)$ are called the *shortest firing duration* for t and the *longest firing duration* for t, respectively. Furthermore, each $\delta_i \in \Big(D(t_i) \cap \mathbb{N}\Big)$ can be the actual duration of transition t_i firing. The bounds are allowed to be zero, i.e. the firing can be considered to take no time.
An ITPN behaves similarly to a PN with regards to maximal step semantics. In this article auto-concurrency is not only allowed, but forced. Thus a *maximal step* will be a *multiset* of events which appears at the same moment.
Formally, a *multiset* U of events E is a total function $U : E \longrightarrow \mathbb{N}$, where $U(e_i)$ defines the number of occurrences of the event e_i in the multiset U. We can write U in the extended set notation $U = \{e^{U(e)} \mid e \in E \text{ and } U(e) \neq 0\}$ and we denote by $\uplus$ the operator of multisets union.
Let $\bar{t}$ be a transitions sequence of length n, $\bar{t} = t_1 t_2 \cdots t_n$. The transitions sequence $\bar{t}$ is called an *undesired cycle* if, for all $i \leqslant n$, $sfd(t_i) = 0$ and for all p, $\sum_{1 \leqslant i \leqslant n} \big(v(t_i, p) - v(p, t_i)\big) \geqslant 0$. Thus *undesired cycles* have firing duration zero and could be infinitely repeated without time elapsing.
An ITPN is *well formed* if it has no undesired cycles. In order to avoid infinite steps only *well formed* nets are considered in this paper.
Note that a token will reach the post-set of a transition t_i only after the time corresponding to the actual duration of this transition has elapsed. The exact value of the actual duration δ_i is unknown at the beginning of the firing of t_i. The transition may stop firing after an arbitrary number $\delta_i \in D(t_i)$ of time ticks has elapsed.
As usual in time-dependent PNs, *states* in ITPNs are pairs $S = (M, h)$ of mappings, M being the p -marking and h codes the clocks of the transitions. In [7] h was defined as *clock-vector*, whereas now, in the context of auto-concurrency, h needs to be a matrix of dimension $(|T| \times d)$. Thus the *clock-matrix* h has $|T|$ rows (i.e. the number of transitions in the skeleton $\mathcal{Z}$) and $d = \max_{t_i \in T}(lfd(t_i)) + 1$ columns. The value $h_{i,j+1}$ represents the number of active transitions t_i with age j (i.e. fired since j time ticks), where $j \in D(t_i)$. Please, note that we need to use 'j+1" because the first column of the matrix has number 1 and not number 0. The *initial state* $S^{(0)} = (M^{(0)}, h^{(0)})$ of $\mathcal{Z}$ is given by the initial marking $M^{(0)} = M_0$ of $\mathcal{Z}$ and the *zero-clock-matrix* $h^{(0)}$ where $h^{(0)}_{i,j} = 0$ for all i, j. The ITPN $\mathcal{Z}_o$ which is used as a running example is shown in Fig.2.

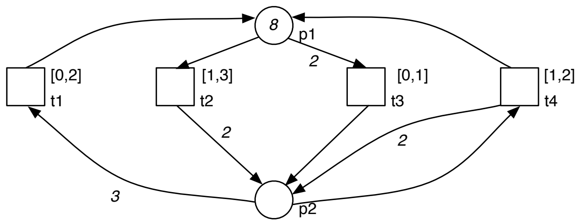

Fig. 2: ITPN $\mathscr{Z}_0$.

2.2 Semantics of Interval-Timed Petri Nets

Now, the behavior of ITPNs will be defined. For the *transition rule* of an ITPN we distinguish three types of events, namely

- *Startfire events:* A startfire event, denoted as $[\mathtt{t_i}$, *must* occur immediately (even n times) if t_i becomes enabled in the skeleton (resp. if n transitions t_i become enabled at the same time). For each occurrence of $[\mathtt{t_i}$ the input tokens of t_i are removed from their preplaces, the clock associated with t_i will count this occurrence by incrementing the number $h_{i,1}$ and t_i will be called *active.*
- *Endfire events:* An endfire event, denoted as $\mathtt{t_i}\rangle$, *must* occur (even n times) if the clock associated with t_i is expiring, i.e. $h_{i,j+1} = n \neq 0$ and $j = lfd(t_i)$. The event $\mathtt{t_i}\rangle$ *may* occur (at most q_i times) if $\sum\limits_{sfd(t_i)\leqslant j<lfd(t_i)} h_{i,j+1} = q_i \geqslant 1$.
  For each of the endfire events $\mathtt{t_i}\rangle$ which occurs the corresponding $h_{i,j+1}$ is decremented and the output tokens are delivered at the postplaces of t_i. There is not only some choice, if some active transitions which need not to endfire may endfire. But once the number of these may endfire events is fixed (for instance $q \leq q_i$ times transition t_i), there is a choice to take these q events totally nondeterministically or to take deterministically those q which are the oldest among the q_i active ones.
- *Tick events:* A tick event, denoted as ✓, is enabled iff there is no firing event which must either start firing or stop firing. Upon occurring, a tick event increments the clocks for all active transitions. Hence the tick events are global. More precisely the incrementation is realised with a right shift of the clock-matrix and by setting the first column to zero.

The initial state is considered to be the first *after-tick state.* The whole set of such states is defined by induction in the sequel. An ITPN can change from one after-tick state into another one by the occurrence of the so-called *Globalstep*, which due to zero duration and auto concurrency extends the definition of firing triple known from [7]. A *Globalstep* consists of several parts, first a multiset of endfire events (called *Endstep*), then an *iterative union* of two multisets *Maxstep* and *EndstepZero*, (called *Iteratedstep*). A *Maxstep* is a maximal step of startfire events and an *EndstepZero* is a multiset of endfire events of transitions with zero firing duration. The iteration stops when no further *Maxstep* is possible. Note that it always stops as only wellformed ITPNs are considered. The *Globalstep* is followed by one tick event for time elapsing.

During the execution of the ITPN *Globalsteps* and single tick events alternate in the following way. Let $S^{(1)} = (M^{(1)}, h^{(1)})$ be an arbitrary after-tick state of $\mathcal{Z}$.

1) An *Endstep* (for *end-firing-step*), denoted by $\mathfrak{G}_{\rangle}^{(1)}$, represents the union of two multisets: That of all active transitions T_1 which must end their firing in this state, and a multiset T_2' that contains several transitions which may end their firing in this state s.

Thus *Endstep* $\mathfrak{G}_{\rangle}^{(1)} = T_1 \uplus T_2'$ where $T_2' \subseteq T_2$,
$T_1 = \{\mathtt{t_i^{n_i}}\rangle \mid i \in [1, |T|], \ \ h_{i,j+1}^{(1)} = n_i \neq 0, \ \ j = lfd(t_i)\}$ and
$T_2 = \{\mathtt{t_i^{q_i}}\rangle \mid i \in [1, |T|], \ \ q_i = \sum_{sfd(t_i) \leqslant j < lfd(t_i)} h_{i,j+1}^{(1)} \ \}.$

Without loss of generality, we can choose for each i to put in T_2' the *oldest* active transitions $t_i \in T_2$, as shown later in Theorem 3.

Its occurrence $S^{(1)} \xrightarrow{\mathfrak{G}_{\rangle}^{(1)}} \tilde{S}^{(1)}$ leads to $\tilde{S}^{(1)} = (\tilde{M}^{(1)}, \tilde{h}^{(1)})$
such that

$$\forall p \in P \ \ \widetilde{M}^{(1)}(p) = M^{(1)}(p) + \sum_{t_i \in \mathfrak{G}_{\rangle}^{(1)}} \mathfrak{G}_{\rangle}^{(1)}(\mathtt{t_i}\rangle) \cdot v(t_i, p) \tag{2}$$

$$\text{and } \widetilde{h}_{i,j}^{(1)} := \begin{cases} 0 & \text{if } \mathfrak{G}_{\rangle}^{(1)}(\mathtt{t_i}\rangle) - \sum_{j' \geqslant j} h_{i,j'}^{(1)} \geqslant 0 \\ h_{i,j}^{(1)} - q & \text{if } \mathfrak{G}_{\rangle}^{(1)}(\mathtt{t_i}\rangle) - \sum_{j' \geqslant j+1} h_{i,j'}^{(1)} = q \textit{ and } 0 < q < h_{i,j}^{(1)} \\ h_{i,j}^{(1)} & \text{otherwise.} \end{cases} \tag{3}$$

The state $\tilde{S}^{(1)}$ is called an *intermediate state*.

2) An *Iteratedstep* is the iterative union of two multisets, the first one being a *Maxstep*. The second one contains only *Endfiring* events of transitions with zero duration, we denote that as *EndstepZero*.

We start by setting $k := 0$ and

$$\widetilde{M}^{(1,k)} = \widetilde{M}^{(1,0)} := \widetilde{M}^{(1)} \text{ and } \ \ \tilde{h}^{(1,k)} = \tilde{h}^{(1,0)} := \ \tilde{h}^{(1)}. \tag{4}$$

a) A *Maxstep* (for maximal start firing step) represents a maximal multiset of concurrently enabled transitions which must start to fire after an *Endstep* or an *EndstepZero*. The multiset of startfire events is denoted by $\mathfrak{G}\mathrm{M}^{(1,k+1)} = \{[\mathtt{t_i^{n_i}} | i \in [1, |T|] \text{ and } \widetilde{M}^{(1,k)} \geqslant \sum_{i=1}^{|T|} n_i \cdot v(t_i, p)\}$.

If there are several enabled Maxsteps, the choice will be arbitrary solved.

The iterative union is stopped if the calculated $k+1$-th *Maxstep* is empty ($\mathfrak{G}\mathrm{M}^{(1,k+1)} = \emptyset$,i.e. a fixpoint is reached). This implies that no further transitions can fire in this step, which always arrives because of the wellformedness of the net. The value of k is stocked in k_{max} ($k_{max} := k$).

b) An *EndstepZero*, denoted by $\mathfrak{G}\mathrm{Z}^{(1,k+1)}$, is a multiset of endfire events of just activated transitions, which *must* or *may* end their firing immediately.

Precisely, *EndstepZero* contains only transitions started in the same step of iteration and whose *shortest firing duration* is equal to zero; all of them whose longest firing duration is equal to zero too must end their firing; among the others an arbitrary number of transitions may end their firing. Thus *EndstepZero* is defined as

$$\mathfrak{G}\mathrm{Z}^{(1,k+1)} = \left\{ \mathtt{t_i^{n_i}}\rangle \;\middle|\; \begin{array}{l} i \in [1,|T|] \text{ and } sfd(t_i)=0 \text{ and}\Big[\big(\, lfd(t_i)=0 \text{ and} \\ n_i = \mathfrak{G}\mathrm{M}^{(1,k+1)}([\mathtt{t_i})\,\big) \text{ or } \big(\, lfd(t_i)\neq 0 \text{ and} \\ n_i \leqslant \mathfrak{G}\mathrm{M}^{(1,k+1)}([\mathtt{t_i})\,\big)\Big] \end{array} \right\}.$$

A state $\tilde{S}^{(1,k+1)}$ is calculated after the k-th iteration such that for each $p \in P$ it holds that:

$$\widetilde{M}^{(1,k+1)}(p) = \widetilde{M}^{(1,k)}(p) - \sum_{t_i\in T} \mathfrak{G}\mathrm{M}^{(1,k+1)}([\mathtt{t_i}) \cdot v(p,t_i) + \sum_{t_i\in T} \mathfrak{G}\mathrm{Z}^{(1,k+1)}(\mathtt{t_i}\rangle) \cdot v(t_i,p) \quad \text{and} \tag{5}$$

$$\tilde{h}_{i,j}^{(1,k+1)} := \begin{cases} \Big(\tilde{h}_{i,j}^{(1,k)} + \mathfrak{G}\mathrm{M}^{(1,k+1)}([\mathtt{t_i}) - \mathfrak{G}\mathrm{Z}^{(1,k+1)}(\mathtt{t_i}\rangle)\Big) & \text{if } j=1 \\ \tilde{h}_{i,j}^{(1,k)} & \text{otherwise} \end{cases}. \tag{6}$$

All newly fired and not ended events obtain age zero, i.e. are counted in column $j=1$ of the clock-matrix.

The *Iteratedstep* is now defined by

$$\mathfrak{G}_I^{(1)} = \biguplus_{1\leqslant k\leqslant k_{max}} (\mathfrak{G}\mathrm{M}^{(1,k)} \uplus \mathfrak{G}\mathrm{Z}^{(1,k)}). \tag{7}$$

The occurrence of an *Iteratedstep* $(\mathfrak{G}_I)$ $\tilde{S}^{(1)} \xrightarrow{\mathfrak{G}_I^{(1)}} S'^{(1)}$ leads to $S'^{(1)} = (M'^{(1)}, h'^{(1)})$ with

$$M'^{(1)} := \widetilde{M}^{(1,k_{max})} \quad \text{and} \quad h'^{(1)} := \tilde{h}^{(1,k_{max})}. \tag{8}$$

$S'^{(1)}$ is called an *intermediate state.*

3) After the *Globalstep* $(\mathfrak{G}_\rangle^{(l)},\ \mathfrak{G}_I{}^{(l)})$, *one tick event* has to occur now in state S', as no further firing event must happen. Its occurrence $S'^{(1)} \xrightarrow{\checkmark} S^{(2)}$ leads to $S^{(2)} = (M^{(2)}, h^{(2)})$. The state $S^{(2)}$ is a new *after-tick state*, with

$$M^{(2)} := M'^{(1)} \quad \text{and} \quad h_{i,j}^{(2)} := \begin{cases} h'^{(1)}_{i,j-1} & \text{if } 1 < j \leqslant d \\ 0 & \text{if } j = 1 \end{cases} \tag{9}$$

4) A *firing step sequence* σ in an ITPN $\mathcal{Z}$ is an alternating sequence of *Globalsteps* and ticks, starting with the initial time state $S^{(0)} = (M^{(0)}, h^{(0)})$

$$\sigma = S^{(0)} \xrightarrow{\mathfrak{G}_\rangle^{(0)}=\emptyset} \tilde{S}^{(0)} \xrightarrow{\mathfrak{G}_I^{(0)}} S'^{(0)} \xrightarrow{\checkmark} S^{(1)} \xrightarrow{\mathfrak{G}_\rangle^{(1)}} \tilde{S}^{(1)} \xrightarrow{\mathfrak{G}_I^{(1)}} S'^{(1)} \xrightarrow{\checkmark} S^{(2)} \xrightarrow{\mathfrak{G}_\rangle^{(2)}} \tilde{S}^{(2)} \dots S^{(n-1)} \xrightarrow{\mathfrak{G}_\rangle^{(n-1)}} \tilde{S}^{(n-1)} \xrightarrow{\mathfrak{G}_I^{(n-1)}} S'^{(n-1)} \xrightarrow{\checkmark} S^{(n)}. \tag{10}$$

where for all $l \geqslant 0$, the *Endstep* $\mathfrak{G}_\rangle^{(l)}$, *Iteratedstep* $\mathfrak{G}_I^{(l)}$ and states $S^{(l)} = (M^{(l)}, h^{(l)})$, $S'^{(l)} = (M'^{(l)}, h'^{(l)})$ and $\tilde{S}^{(l)} = (\widetilde{M}^{(l)}, \tilde{h}^{(l)})$ verify the above conditions. In particular each $S^{(l)}$ has the same marking, i.e. the same first column in the time marking as $S'^{(l-1)}$.

The following lemma states that the definition of $S'^{(1)}$ is well founded

Lemma 1 *Let us consider state $S'^{(l)} = (M'^{(l)}, h'^{(l)})$ as defined in (8). Then this state fulfils*

$$M'^{(l)} = \widetilde{M}^{(l)} - \sum_{i=1}^{|T|} \mathfrak{G}_I^{(l)}([\mathtt{t}_i) \cdot v(p,t_i) + \sum_{i=1}^{|T|} \mathfrak{G}_I^{(l)}(\mathtt{t_i}\rangle) \cdot v(t_i,p) \quad \textit{and}$$

$$h'^{(l)}_{i,j} = \begin{cases} \tilde{h}^{(l)}_{i,j} + [\mathfrak{G}_I^{(l)}([\mathtt{t}_i) - \mathfrak{G}_I^{(l)}(\mathtt{t_i}\rangle)] & \textit{if } j = 1 \\ \tilde{h}^{(l)}_{i,j} & \textit{otherwise.} \end{cases}$$

□

Proof. We start with

$$M'^{(l)} \underset{(8)}{=} \widetilde{M}^{(l,k_{max})}$$

$$\underset{(5)}{=} \widetilde{M}^{(l,k_{max}-1)} - \sum_{i=1}^{|T|} \mathfrak{G}\mathrm{M}^{(l,k_{max})}([\mathtt{t}_i) \cdot v(p,t_i) + \sum_{i=1}^{|T|} \mathfrak{G}\mathrm{Z}^{(l,k_{max})}(\mathtt{t_i}\rangle) \cdot v(t_i,p)$$

and after k_{max} iterations we obtain

$$M'^{(l)} \underset{(5)}{=} \widetilde{M}^{(l,0)} - \sum_{k=1}^{k_{max}} \sum_{i=1}^{|T|} \mathfrak{G}\mathrm{M}^{(l,k)}([\mathtt{t}_i) \cdot v(p,t_i) + \sum_{k=1}^{k_{max}} \sum_{i=1}^{|T|} \mathfrak{G}\mathrm{Z}^{(l,k)}(\mathtt{t_i}\rangle) \cdot v(t_i,p)$$

$$\underset{(4)+(7)}{=} \widetilde{M}^{(l)} - \sum_{i=1}^{|T|} \mathfrak{G}_I^{(l)}([\mathtt{t}_i) \cdot v(p,t_i) + \sum_{i=1}^{|T|} \mathfrak{G}_I^{(l)}(\mathtt{t_i}\rangle) \cdot v(t_i,p).$$

Further, we start with the definition of $h'^{(l)}$.

$$h'^{(l)} \underset{(8)}{=} \tilde{h}^{(l,k_{max})}$$

$$\underset{(6)}{=} \begin{cases} \tilde{h}^{(l,k_{max}-1)}_{i,j} + [\mathfrak{G}_{\mathrm{M}}^{(l,k_{max})}([\mathtt{t}_i) - \mathfrak{G}_{\mathrm{Z}}^{(l,k_{max})}(\mathtt{t_i}\rangle)] & \text{if } j = 1 \\ \tilde{h}^{(l,k_{max}-1)}_{i,j} & \text{otherwise.} \end{cases}$$

and after k_{max} iterations we obtain

$$h'^{(l)} \underset{(6)}{=} \begin{cases} \tilde{h}^{(l,0)}_{i,j} + [\sum_{k=1}^{k_{max}} \mathfrak{G}_{\mathrm{M}}^{(l,k)}([\mathtt{t}_i) - \sum_{k=1}^{k_{max}} \mathfrak{G}_{\mathrm{Z}}^{(l,k)}(\mathtt{t_i}\rangle)] & \text{if } j = 1 \\ \tilde{h}^{(l,0)}_{i,j} & \text{otherwise.} \end{cases}$$

$$\underset{(4)+(7)}{=} \begin{cases} \tilde{h}^{(l)}_{i,j} + [\mathfrak{G}_I^{(l)}([\mathtt{t}_i) - \mathfrak{G}_I^{(l)}(\mathtt{t_i}\rangle)] & \text{if } j = 1 \\ \tilde{h}^{(l)}_{i,j} & \text{otherwise.} \end{cases}.$$

□

The set of all after-tick states and intermediate states forms the set of *reachable states* of $\mathcal{Z}$. The *reachability graph* start with the initial state s_0 and has all these states as nodes and the concerned *Endsteps*, *Iteratedsteps* or ticks ✓ as arc inscriptions. Each after-tick state has as many successor nodes as the number of subsets of the set of endfiring events which may occur in the state. Each of these nodes has as many successor nodes as *Iteratedsteps*. Thus the reachability graph grows very quickly. To avoid the construction of such an enormous reachability graph the consideration of the state equation to decide unreachability will be a good alternative.

2.3 Semantic equivalences

We could have defined firing step sequences of an ITPN as in (10) where for all $l \geqslant 0$, the *Endstep* $\mathfrak{S}_{\rangle}^{(l)}$ may contain transitions to be endfired independently of their age. We would like to define the notion of similar firing step sequences which only differ in the choice of the age of transitions which may and will endfire.

Two firing step sequences σ and σ_0 are called *similar*, denoted by $\sigma_0 \sim \sigma$ if both start at the same state and in all states $S^{(l)}$ and ${S_0}^{(l)}$ the marking (i.e. their first column) is the same, and the *Globalsteps* are the same.
Thus, in similar firing step sequences only the clock matrices may differ, which signifies that transitions of different ages could have endfired.

The following sentence establishes that w.l.o.g., we can always use as may endfire events the oldest active transitions (as chosen in Definition 1 of Subsection 2.2. above).
Note that in both cases, transitions whose actual durations are the upper bound of their respective time interval ($\delta_i = lfd(t_i)$) must endfire. For the others active transitions (i.e. those which may endfire) we have the choice to choose which transitions do so. Choosing to endfire the oldest active transitions make the choice deterministic.

Example 2 *Let be* $S^{(3)} = (M^{(3)}, h^{(3)})$ *the state reached from the initial state of our running example in Fig.2. by the firing steps sequence* $\sigma =$
$\big(\emptyset, \{[\mathtt{t}_2^8\}, \checkmark\big), \big(\{\mathtt{t}_2^2\rangle\}, \{[\mathtt{t}_1, [\mathtt{t}_4, \mathtt{t}_1\rangle, [\mathtt{t}_2\}, \checkmark\big), \big(\{\mathtt{t}_4\rangle, \mathtt{t}_2^2\rangle\}, \{[\mathtt{t}_1^2, [\mathtt{t}_2, \mathtt{t}_1^2\rangle, [\mathtt{t}_2^2\}, \checkmark\big)$ *with* $M^{(3)} = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ *and* $h^{(3)} = \begin{pmatrix} 0&0&0&0 \\ 0&3&1&4 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$.
Note that in this state, there are eight active transitions t_2 *whose time interval is* $[1, 3]$.
From the clock matrix $h^{(3)}$ *we can see that there are four transitions* t_2 *of age 3, one transition of age 2 and three transitions of age 1. Imagine that seven transitions will be endfired.*
(a) If only the oldest active transitions are chosen
The intermediate state $\tilde{S}$ *with* $\widetilde{M}^{(3)} = \left(\begin{smallmatrix} 0 \\ 14 \end{smallmatrix}\right)$ *and* $\tilde{h}^{(3)} = \begin{pmatrix} 0&0&0&0 \\ 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$ *will be reached.*
(b) If transitions of any age may be chosen, then that of age two can be ignored and the following state $\tilde{S}$ *with* $\widetilde{M}^{(3)} = \left(\begin{smallmatrix} 0 \\ 14 \end{smallmatrix}\right)$ *and* $\tilde{h}^{(3)} = \begin{pmatrix} 0&0&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$ *could be reached, too.* □

Theorem 3 *Let* $\mathcal{Z}$ *be an ITPN and* $n \in \mathbb{N}^+$. *For each firing step sequence* σ *of* n *Globalsteps where we choose to may endfire active transitions of any ages, we* *can find a sequence σ_0 where always the oldest active transitions are endfired, and $\sigma_o \sim \sigma$.* □

Proof. Let σ be a sequence of $n \geqslant 1$ global steps where may endfire events are chosen arbitrarily among the active transitions independently of their ages. As defined in (10) it holds that

$$\sigma = S^{(0)} \xrightarrow{\mathfrak{G}_{\rangle}^{(0)}} \tilde{S}^{(0)} \xrightarrow{\mathfrak{G}_{I}^{(0)}} S'^{(0)} \xrightarrow{\checkmark} S^{(1)} \xrightarrow{\mathfrak{G}_{\rangle}^{(1)}} \tilde{S}^{(1)} \xrightarrow{\mathfrak{G}_{I}^{(1)}} S'^{(1)} \xrightarrow{\checkmark} S^{(2)} \xrightarrow{\mathfrak{G}_{\rangle}^{(2)}} \tilde{S}^{(2)} \ldots$$
$$\ldots\ S^{(n-1)} \xrightarrow{\mathfrak{G}_{\rangle}^{(n-1)}} \tilde{S}^{(n-1)} \xrightarrow{\mathfrak{G}_{I}^{(n-1)}} S'^{(n-1)} \xrightarrow{\checkmark} S^{(n)}$$

and $\forall i \leqslant n,\ S^{(i)} = (M^{(i)}, h^{(i)})$, where $M^{(i)}$ is a marking and $h^{(i)}$ its associated clock matrix. We want to prove, by induction on n, that we can obtain another sequence σ_0 which has the same global steps as σ but different states, by endfiring the oldest active transitions first.

Base : $n = 1$. For the first global step $\sigma = S^{(0)} \xrightarrow{\mathfrak{G}_{\rangle}^{(0)}} \tilde{S}^{(0)} \xrightarrow{\mathfrak{G}_{I}^{(0)}} S'^{(0)} \xrightarrow{\checkmark} S^{(1)}$ we want to construct σ_o similar to σ. The initial state is the same in both cases because we begin from the initial marking and no transition is active. Thus ${S_o}^{(0)} = (M^{(0)}, h^{(0)}) = S^{(0)}$.
The first endfiring multi-set is empty and the age does not play any role. Thus $\tilde{S_o}^{(0)} = \tilde{S}^{(0)}$.
The iterated step contains only endfiring events of zero ages, thus we can use the same multiset of firing ${S'_o}^{(0)} = S'^{(0)}$. After the tick event ${S_o}^{(1)} = S^{(1)}$ holds. We conclude that $\sigma_o = {S_o}^{(0)} \xrightarrow{\mathfrak{G}_{\rangle}^{(0)}} \tilde{S_o}^{(0)} \xrightarrow{\mathfrak{G}_{I}^{(0)}} {S'_o}^{(0)} \xrightarrow{\checkmark} {S_o}^{(1)}$ is a valid firing step sequence and $\sigma_o \sim \sigma$.
The base of induction is proved.

Induction hypothesis : For all firing step sequences σ of length $i \leqslant n$, with arbitrarily aged endfiring events, there exists σ_o of length i such that $\sigma_o \sim \sigma$ is supposed to be true and σ_o endfires only the oldest active transitions.

Induction step: Let σ be a firing step sequence of size $(n + 1)$ with arbitrary aged endfiring events.
Thus, the prefix of σ of size n is the following firing step sequence

$$\sigma' = S^{(0)} \xrightarrow{\mathfrak{G}_{\rangle}^{(0)}} \tilde{S}^{(0)} \xrightarrow{\mathfrak{G}_{I}^{(0)}} S'^{(0)} \xrightarrow{\checkmark} S^{(1)} \xrightarrow{\mathfrak{G}_{\rangle}^{(1)}} \tilde{S}^{(1)} \xrightarrow{\mathfrak{G}_{I}^{(1)}} S'^{(1)} \xrightarrow{\checkmark} S^{(2)} \xrightarrow{\mathfrak{G}_{\rangle}^{(2)}}$$
$$\tilde{S}^{(2)} \ldots\ S^{(n-1)} \xrightarrow{\mathfrak{G}_{\rangle}^{(n-1)}} \tilde{S}^{(n-1)} \xrightarrow{\mathfrak{G}_{I}^{(n-1)}} S'^{(n-1)} \xrightarrow{\checkmark} S^{(n)}$$

and its $(n+1)$-th global step is $S^{(n)} = (M^{(n)}, h^{(n)}) \xrightarrow{\mathfrak{G}_{\rangle}^{(n)}} \widetilde{S}^{(n)} = (\widetilde{M}^{(n)}, \widetilde{h}^{(n)}) \xrightarrow{\mathfrak{G}_{I}^{(n)}} S'^{(n)} = (M'^{(n)}, h'^{(n)}) \xrightarrow{\checkmark} S^{(n+1)} = (M^{(n+1)}, h^{(n+1)})$. As the ages of transitions in $\mathfrak{G}_{\rangle}^{(n)}$ are arbitrary, we only know the following about $\tilde{h}^{(n)}$, $h^{(n)}$:

(a) For all i, $x_i := h_{i,lfd(t_i)+1}$ transitions t_i must endfire in this step. Thus, for each i, $t_i^{x_i}$ is in $\mathfrak{G}_{\rangle}^{(n)}$ and $\tilde{h}^{(n)}_{i,lfd(t_i)+1} = 0$ follows.

(b) For all i, $y_i := \sum\limits_{1 \leqslant j \leqslant lfd(t_i)} h^{(n)}_{i,j} - \sum\limits_{1 \leqslant j \leqslant lfd(t_i)} \tilde{h}^{(n)}_{i,j}$ is the number of may endfire transitions in $\mathfrak{G}_{\rangle}^{(n)}$.

(c) It follows that for all i, $z_i := x_i + y_i = \mathfrak{G}_{\rangle}^{(n)}(t_i)$.

Now let us prove that there exist σ_o of size $(n+1)$ with $\sigma_o \sim \sigma$, such that the oldest active transitions endfire.
By hypothesis, we have σ'_o, such $\sigma'_o \sim \sigma'$ and σ'_o ends with state ${S_o}^{(n)}$, such that the states $S^{(n)}$ and ${S_o}^{(n)}$ have the same markings but may have different clock matrices. In σ'_o only the oldest active transitions have endfired.

We need to prolongate σ'_o by the same $(n+1)$-th global step $({\mathfrak{G}_{\rangle}}^{(n)}, {\mathfrak{G}_I}^{(n)}, \checkmark)$. Thus, we have to show the existence of fitting $\tilde{h_o}^{(n)}, {h'_o}^{(n)}$ and ${h_0}^{(n+1)}$ such that ${S_o}^{(n)} = (M^{(n)}, {h_o}^{(n)}) \xrightarrow{\mathfrak{G}_{\rangle}^{(n)}} \tilde{S_o}^{(n)} = (\widetilde{M}^{(n)}, \tilde{h_o}^{(n)}) \xrightarrow{\mathfrak{G}_I^{(n)}} {S'_o}^{(n)} = (M'^{(n)}, {h'_o}^{(n)}) \xrightarrow{\checkmark} {S_o}^{(n+1)} = (M^{(n+1)}, {h_o}^{(n+1)})$.

We have first to show that we can endfire z_i active transitions by choosing the oldest ones. Clearly, as the same global steps appeared in σ' and σ'_o, the same number of active transitions appears in the two states $S^{(n)}$ and ${S_o}^{(n)}$, i.e., for all i, it holds that

$$\sum_{i \geqslant 1^{d+1}} h_{i,j}^{(n)} = \sum_{i \geqslant 1}^{d+1} {h_{o\ i,j}}^{(n)} \text{ and } z_i \leqslant \sum_{i \geqslant 1}^{d+1} h_{i,j}^{(n)}.$$

Because all preceding global steps are the same for the two sequences, we have precisely the same number of transitions too young to be endfired, i.e., for all i,
$\sum\limits_{j \leqslant sfd(t_i)} h_{i,j}^{(n)} = \sum\limits_{j \leqslant sfd(t_i)} h_{o\ i,j}^{(n)}$. Thus, there are also the same number of active transitions which must or may endfire in $S^{(n)}$ and ${S_o}^{(n)}$.
By consequence, we can take exactly the same endfiring multiset $\mathfrak{G}_{\rangle}^{(n)}$ as in σ, by choosing the oldest active instance of transitions.

The state $\tilde{S_o}^{(n)} = (\widetilde{M}^{(n)}, \tilde{h_o}^{(n)})$, as defined in (2) and (3), and $\widetilde{S}^{(n)}$ have clearly the same markings.
Now the same iterated step ${\mathfrak{G}_I}^{(n)}$ can appear in both states leading to ${S'_o}^{(n)} = (M'^{(n)}, {h'_o}^{(n)})$, as defined in (5), (6) and (8), and to $S'^{(n)}$.
Finally, by the tick event we obtain ${S_o}^{(n+1)} = (M^{(n+1)}, {h_o}^{(n+1)})$, as defined in (9).
Thus, the firing step sequence σ_o is successfully completed. We can conclude that $\sigma_o \sim \sigma$. □

3 Algebraic representations

As already quoted, the relationship between a firing step sequence σ and a reachable p-marking M in an ordinary PN with initial p-marking M_0 and a incidence matrix C can be described formally by the following linear equation, where ψ_σ is the Parikh vector of σ: $M = M_0 + C \cdot \psi_\sigma$. A Parikh vector of a word α defined over the finite set, here of transitions $T = \{t_1 \cdots t_n\}$ is a vector of dimension n and the i-th component is the number of appearance of t_i in the word α. Our goal is to obtain a similar result for ITPNs, i.e. to give an algebraic description, precisely, a linear equation, for each firing step sequence, now of

Globalsteps as defined above, in an arbitrary ITPN which takes into account the time, too. Meanwhile state equations had been introduced for TPN with fixed duration [8] and for ITPN without auto-concurrency and without zero duration [7], where the semantics had been formulated in a more algebraic way. We will present in the following the formal definitions of the notions we need later for the different proofs. Some of them are adaptations of definitions known for the algebraic presentation of a firing step sequence for TPN, or ITPN without zero duration and without auto-concurrency [8,7] and others are entirely new here.

3.1 Semantics with time markings

In this subsection we introduce a more detailed view of the p-markings in an arbitrary ITPN with respect to the time. This view makes it possible to obtain a time-dependent state equation for a firing step sequence and it delivers a sufficient condition for the non-reachability of p-markings (timeless) as well as of time markings in such a net.

First, to calculate the effect of *Endstep* $\mathfrak{G}_{\rangle}$ we introduce a new $(|T| \times d)$ matrix, denoted by $G_{\rangle}$ which is the matrix representation of the *Endstep* multiset, fixing which events have to endfire, by taking the oldest ones.

Let $S^{(1)} = (M^{(1)}, h^{(1)}) \xrightarrow{\mathfrak{G}_{\rangle}^{(1)}} \widetilde{S}^{(1)} = (\widetilde{M}^{(1)}, \tilde{h}^{(1)})$

$$G_{\rangle i,j}^{(1)} := \begin{cases} h_{i,j}^{(1)} & \text{if } \mathfrak{G}_{\rangle}^{(1)}(\mathtt{t_i}\rangle) - \sum\limits_{j' \geqslant j} h_{i,j'}^{(1)} \geqslant 0 \\ q & \text{if } \mathfrak{G}_{\rangle}^{(1)}(\mathtt{t_i}\rangle) - \sum\limits_{j' > j} h_{i,j'}^{(1)} = q > 0 \text{ and } q < h_{i,j}^{(1)} \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

The element $G_{\rangle i,j}^{(1)}$ fixes the number of t_i whose age is $(j-1)$ and which is chosen to endfire.

Lemma 4 *Let an Endstep* $\mathfrak{G}_{\rangle}^{(1)}$ *appear in state* $S^{(1)}$*, i.e.,* $S^{(1)} = (M^{(1)}, h^{(1)}) \xrightarrow{\mathfrak{G}_{\rangle}^{(1)}} \widetilde{S}^{(1)} = (\widetilde{M}^{(1)}, \tilde{h}^{(1)})$ *and let* $G_{\rangle}^{(1)}$ *be its associated matrix as defined in (11). Then for all* i, j *it holds that* $\tilde{h}_{i,j}^{(1)} = h_{i,j}^{(1)} - G_{\rangle i,j}^{(1)}$. □

The proof is an immediate consequence of the above definition (11).

Second, in order to describe the relation between tokens and time algebraically, we use a generalization of the p-marking, called *time marking*, cf. [8]. A time marking is a $(|P| \times (d+1))$- matrix. The number of rows is equal to the number of places and the number of columns, $d+1$, equals the maximum of all longest durations in the considered ITPN, plus 2. They are numbered from 1 to $d+1$. Each column can be considered to be a p-marking. The first column represents the number of visible tokens in each place, i.e. the actual p-marking M. The other columns represent tokens which are on their way to the places: column number two for those arriving immediately, column number three for those arriving in one time unit (one tick later), the column number four for those arriving in two time units (after two ticks), and so on. We may observe, that only a finite number of time markings can be associated with a given p-marking M.

This number depends on the time-dimension d of the net and is exponential in $|T|$.

A *time state* s is now defined as a pair (m, h), where m is a time marking and h is a clock-matrix. The *initial time marking* $m^{(0)}$ is defined as

$$m^{(0)}_{\cdot 1} = M^{(0)} \text{ and } m^{(0)}_{i,j} = 0 \text{ for } i = 1 \dots |P| \text{ and } j = 2 \dots d+1. \qquad (12)$$

The *initial time state* $s^{(0)}$ is the pair $(m^{(0)}, h^{(0)})$ considered now to be the *first after-tick time state.*

***E*xample 5** *Consider the ITPN $\mathcal{Z}_o$ with $d = 4$ and $m^{(0)} = \left(\begin{smallmatrix} 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{smallmatrix}\right)$. This initial time marking allows many possible Globalsteps such as, e.g.,*

1. $(\mathfrak{G}^{(0)}_{\rangle} = \emptyset,\ \mathfrak{G}^{(0)}_{I} = \{[t_2^8\});$
2. $(\mathfrak{G}^{(0)}_{\rangle} = \emptyset,\ \mathfrak{G}^{(0)}_{I} = \{[t_2^6, [t_3\} \uplus \{\mathtt{t_3}\rangle\} \uplus \{[t_4\} = \{[t_2^6, [t_3, [t_4, t_3\rangle\});$
3. $(\mathfrak{G}^{(0)}_{\rangle} = \emptyset,\ \mathfrak{G}^{(0)}_{I} = \{[t_2^2, [t_3^3\} \uplus \{t_3\rangle^3\} \uplus \{[t_1\} \uplus \{t_1\rangle\} \uplus \{[t_2\}$
   $= \{[t_2^3, [t_3^3, [t_1, t_3\rangle^3, t_1\rangle\}).$

The choice of one Globalsteps among those above is arbitrary. We will consider later the third one appearing. □

Let $s^{(1)} = (m^{(1)}, h^{(1)})$ be an after-tick time state in some ITPN $\mathcal{Z}$, and $(\mathfrak{G}^{(1)}_{\rangle},\ \mathfrak{G}^{(1)}_{I})$ a *Globalstep* which may appear from state $S^{(1)} = (M^{(1)}, h^{(1)})$ as defined in Subsection 2.1. above. We will adapt the definitions now to show how the execution of this *Globalstep* changes the time state $s^{(1)}$, by using matrix $G^{(1)}_{\rangle}$ for the calculations.

a) By firing the *Endstep* we obtain $s^{(1)} \xrightarrow{\mathfrak{G}^{(1)}_{\rangle}} \tilde{s}^{(1)} = (\ \widetilde{m}^{(1)},\ \widetilde{h}^{(1)})$, with

$$\widetilde{m}^{(1)}_{i,j} := \begin{cases} m^{(1)}_{i,j} + \sum\limits_{k\geqslant 1}^{|T|} (\sum\limits_{r\geqslant 1}^{d} G^{(1)}_{\rangle k,r}) \cdot v(t_k, p_i) & \text{if } j = 1 \\ m^{(1)}_{i,j} - \sum\limits_{k\geqslant 1}^{|T|} G^{(1)}_{\rangle k,j'} \cdot v(t_k, p_i) & \text{if } j > 1 \text{ and} \\ & j' = lfd(t_k) - j + 3 \end{cases} . \qquad (13)$$

and $\tilde{h}^{(1)}_{i,j} := h^{(1)}_{i,j} - G^{(1)}_{\rangle i,j}$ (by Lemma 4). It is clear that $\widetilde{m}^{(1)}_{i,2} = 0$.

b) By firing the *Iteratedstep* we obtain $\tilde{s}^{(1)} \xrightarrow{\mathfrak{G}^{(1)}_{I}} s'^{(1)} = (m'^{(1)},\ h'^{(1)})$. The *Iteratedstep* change the first column of the time marking, $m'^{(1)}_{i,1} = M'^{(1)}$, as shown in Lemma 1. For each transition $t_k \in \mathfrak{G}^{(1)}_{I}$ the j-th column can be modified if $j = lfd(t_k) + 2$, but t_k does not influence the others columns. Hence, it holds that $m'^{(1)}_{i,j} :=$

$$\begin{cases} \widetilde{m}^{(1)}_{i,j} - \sum\limits_{k\geqslant 1}^{|T|} \mathfrak{G}^{(1)}_{I}([\mathtt{t}_k) \cdot v(p_i, t_k) + \sum\limits_{k\geqslant 1}^{|T|} \mathfrak{G}^{(1)}_{I}(\mathtt{t_k}\rangle) \cdot v(t_k, p_i) & \text{if } j = 1 \\ \widetilde{m}^{(1)}_{i,j} + \sum\limits_{\substack{1\leqslant k\leqslant |T| \\ \mathbf{j=lfd(t_k)+2}}} \left[\mathfrak{G}^{(1)}_{I}([\mathtt{t}_k) - \mathfrak{G}^{(1)}_{I}(\mathtt{t_k}\rangle)\right] \cdot v(t_k, p_i) & \text{if } j > 1 \end{cases} . \qquad (14)$$

The clock matrix ${h'}^{(1)}$ does not need to be recalculated: the definitions of (6) and (8) apply.

c) Now one tick has to occur $s'^{(1)} \xrightarrow{\checkmark} s^{(2)} = (m^{(2)}, h^{(2)})$ with

$$m^{(2)}_{i,j} := \begin{cases} m'^{(1)}_{i,j} & \text{if } j = 1 \\ m'^{(1)}_{i,j+1} & \text{if } 2 \leqslant j \leqslant d \\ 0 & \text{if } j = d+1 \end{cases} . \tag{15}$$

The clock matrix $h^{(2)}$ is already defined in (9). The time state $s^{(2)}$ is a new after-tick time state. We can observe, that in the defined time markings the first column is in fact always the usual p-marking of the corresponding state.

***E*xample 6** *Let us reconsider the running example $\mathcal{Z}_o$ and the selected Global-step appearing from the initial state $s^{(0)}$: ($\mathfrak{G}^{(0)}_{\rangle} = \emptyset$, $\mathfrak{G}^{(0)}_{I} = \{[t_2^3, [t_3^3, [t_1, t_3\rangle^3, t_1\rangle\}$). Then the time states reached during its firing and the subsequent tick $s^{(0)} \xrightarrow{\mathfrak{G}^{(0)}_{\rangle}} \tilde{s}^{(0)} \xrightarrow{\mathfrak{G}^{(0)}_{I}} s'^{(0)} \xrightarrow{\checkmark} s^{(1)}$ have the following time markings*

$$\tilde{m}^{(0)} = m^{(0)} = \left(\begin{smallmatrix} 8&0&0&0&0 \\ 0&0&0&0&0 \end{smallmatrix}\right), m'^{(0)} = \left(\begin{smallmatrix} 0&0&0&0&0 \\ 0&0&0&0&6 \end{smallmatrix}\right), m^{(1)} = \left(\begin{smallmatrix} 0&0&0&0&0 \\ 0&0&0&6&0 \end{smallmatrix}\right).$$

As $G^{(0)}_{\rangle} = \mathcal{O}$, it holds that $\tilde{h}^{(0)} = h^{(0)} = \mathcal{O}$. As ${\mathfrak{G}_I}^{(0)}([\mathtt{t}_3) - {\mathfrak{G}_I}^{(0)}(\mathtt{t}_3\rangle) = 3$ it follows that

${h'}^{(0)}_{2,1} = 3$ and $\tilde{h}^{(0)} = h^{(0)} = \mathcal{O}$, ${h'}^{(0)} = \begin{pmatrix} 0&0&0&0 \\ 3&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$ and $h^{(1)} = \begin{pmatrix} 0&0&0&0 \\ 0&3&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}$ after a right shift. □

Analogously to states, we call *reachable time states* all after-tick and intermediate time states reached during the execution of arbitrary firing step sequences.

3.2 Algebraic calculus of the semantics

In the following we introduce all matrices which are necessary to obtain a state equation for some ITPN, starting with the so called *time incidence matrix.*

Let $\mathcal{Z}$ be an ITPN. The $(|P| \times (d+1) \cdot |T|)$-matrix C is called *the time incidence matrix* of $\mathcal{Z}$, if $C := (C_{(1)}, C_{(2)}, \ldots, C_{(|T|)})$ with $C_{(k)}$ being a $(|P| \times d)$-matrix for each $k \in \{1, \ldots, |T|\}$, such that $C_{(k)} = \left(c^{(k)}_{i,r}\right)_{\substack{i=1\cdots|P| \\ r=1\cdots d}}$ and

$$c^{(k)}_{i,r} := \begin{cases} -v(p_i, t_k) & \text{if } r = 1 \\ v(t_k, p_i) & \text{if } r - 2 = \mathit{lfd}(t_k) \\ 0 & \text{otherwise.} \end{cases} .$$

The matrix C consists of submatrices $C_{(k)}$ representing the transitions t_k of the net. Each $c^{(k)}_{i,1}$ is the number of tokens that will be changed (decremented) at place p_i immediately when the startfire event $[\mathtt{t_k}$ appears, and $c^{(k)}_{i,r}$ shows the number of tokens that will arrive at place p_i when the endfire event $\mathtt{t_k}\rangle$ appears after at most $(r-2)$ time units.

***E*xample 7** *The time incidence matrix of $\mathcal{Z}_o$ from Fig.2 is as follows:*

$$C = \left(\begin{smallmatrix} 0&0&0&0&0 & -1&0&0&0&0 & -2&0&0&0&0 & 0&0&0&1&0 \\ -3&0&0&1&0 & 0&0&0&0&2 & 0&0&1&0&0 & -1&0&0&2&0 \end{smallmatrix}\right).$$ □

Obviously the time incidence matrix takes into account the longest firing duration $lfd(t_i)$ for each transition ti.
The appearance of $\mathtt{t}\rangle_i^n$ in some $\mathfrak{G}_\rangle^{(l)}$ at a certain state $s^{(l)} = (m^{(l)}, h^{(l)})$ tells us that there are at least n active transitions. The matrix $G_\rangle^{(l)}$ associated to the end-step tells us which ones are going to endfire.

For subsequent computation we need to update the matrix C with respect to $\mathfrak{G}_\rangle^{(l)}$. This is achieved by matrix $C^{(l)}$ obtained from C where for each submatrix $C_{(i)}^{(l)}$ the first column represents the tokens consumed by the transitions to endfire and the j-th column represents the tokens arriving to the corresponding places after $j-2$ ticks at least.

Therefore, concerning $\mathfrak{G}_\rangle^{(l)}$ in the state $s^{(l)} = (m^{(l)}, h^{(l)})$, we define the matrix $C^{(l)} := \left(C_{(1)}^{(l)}, C_{(2)}^{(l)}, \ldots, C_{(|T|)}^{(l)}\right)$ as follows. Each $C_{(k)}^{(l)} = \left(c_{i,r}^{(l,k)}\right)_{\substack{i=1\cdots|P| \\ r=1\cdots d}}$ is a $(|P| \times (d+1))$-matrix with

$$c_{i,r}^{(l,k)} := \begin{cases} -v(p_i, t_k) \cdot \mathfrak{G}_\rangle(t_k) \text{ if } r = 1 \\ v(t_k, p_i) \cdot G\rangle_{k,r'} \quad \text{ f } r > 1 \text{ and } r' = lfd(t_k) - r + 3 \end{cases} . \tag{16}$$

***Example* 8** *In the ITPN $\mathcal{Z}_o$ let us consider the endfiring step* $s^{(l)} = (m^{(l)}, h^{(l)}) \xrightarrow{\mathfrak{G}_\rangle^{(l)}} \tilde{s}^{(l)} = (\tilde{m}^{(l)}, \tilde{h}^{(l)})$ *with* $m^{(l)} = \left(\begin{smallmatrix} 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 10 & 8 & 0 \end{smallmatrix}\right)$, $h^{(l)} = \begin{pmatrix} 0&0&0&0 \\ 0&4&1&0 \\ 0&0&0&0 \\ 0&0&4&0 \end{pmatrix}$ *and* $\mathfrak{G}_\rangle^{(l)} = \{t_2\rangle^4, t_4\rangle^3\}$. *Then its associated matrix is* $G_\rangle^{(l)} = \begin{pmatrix} 0&0&0&0 \\ 0&3&1&0 \\ 0&0&0&0 \\ 0&0&3&0 \end{pmatrix}$.
The time incidence matrix $C^{(l)}$ arises from the matrix C as follows:
$C^{(l)} = \left(\begin{smallmatrix} 0&0&0&0&0&-4&0&0&0&0&0&0&0&0&0&0&0&3&0&0 \\ 0&0&0&0&0&0&0&2&6&0&0&0&0&0&0&-3&0&6&0&0 \end{smallmatrix}\right)$. □

Our goal now is to introduce a sparse matrix U which allows us to calculate $C^{(l)}$ from C, such that $C^{(l)} = C \cdot U^{(l)}$ for its submatrix $U^{(l)}$. Let us consider $t_k\rangle \notin \mathfrak{G}_\rangle^{(l)}$ and $\boldsymbol{t_i}\rangle \in \boldsymbol{\mathfrak{G}_\rangle^{(l)}}$.
We define the square matrix $U^{(l)}$ with $(d+1) \cdot |T|$ rows and $(d+1) \cdot |T|$ columns

$$\begin{array}{c} \\ t_1 \\ \\ \\ \\ t_k \\ \\ t_i \\ \\ t_n \end{array} \begin{pmatrix} \overset{t_1}{A_1^{(l)}} & \mathcal{O} & \overset{t_k}{O_{d+1}} & \mathcal{O} & \overset{t_i}{O_{d+1}} & \mathcal{O} & \overset{t_n}{O_{d+1}} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathcal{O} & \ddots & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ O_{d+1} & \mathcal{O} & A_k^{(l)} = \mathcal{O}_{d+1} & \mathcal{O} & O_{d+1} & \mathcal{O} & O_{d+1} \\ \vdots\,\mathcal{O} & \vdots\,\mathcal{O} & \vdots\,\mathcal{O} & \ddots & \vdots\,\mathcal{O} & \mathcal{O} & \vdots\,\mathcal{O} \\ O_{d+1} & \mathcal{O} & O_{d+1} & \mathcal{O} & A_i^{(l)} & \mathcal{O} & O_{d+1} \\ \vdots\,\mathcal{O} & \vdots\,\mathcal{O} & \vdots\,\mathcal{O} & \vdots\,\mathcal{O} & \vdots\,\mathcal{O} & \ddots & \vdots\,\mathcal{O} \\ O_{d+1} & \mathcal{O} & O_{d+1} & \mathcal{O} & O_{d+1} & \mathcal{O} & A_n^{(l)} \end{pmatrix} .$$

$\mathcal{O}$ stands for a block of zeros, $A_i^{(l)}$ is a $(d+1 \times d+1)$ matrix obtained from E_{d+1} by:
(1) Multiplying the first column of E_{d+1} by $\mathfrak{G}_\rangle^{(l)}(\mathtt{t_i}\rangle)$ which is the number of occurrences of endfiring event $\mathtt{t_i}\rangle$ in the end-step $\mathfrak{G}_\rangle^{(l)}$.
(2) Superseding the $(lfd(t_i) - j + 3)$-th column of E_{d+1} by the $(lfd(t_i) + 2)$-th column multiplied by $G_{\rangle i,j}^{(l)}$ for each $j \in [0, d]$ as follows:

Example 9 *In $\mathcal{Z}_o$ from Fig. 2 we consider the same end-step $\mathfrak{G}_\rangle^{(l)} = \{t_2\rangle^4, t_4\rangle^3\}$ with $G_\rangle^{(l)} = \begin{pmatrix} 0&0&0&0\\ 0&3&1&0\\ 0&0&0&0\\ 0&0&3&0 \end{pmatrix}$. We obtain the corresponding matrices $A_2^{(l)}$, $A_4^{(l)}$ and $U^{(l)}$:*

$$A_2^{(l)} = \begin{pmatrix} 4&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&1&3&0 \end{pmatrix},\ A_4^{(l)} = \begin{pmatrix} 3&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&3&0&0\\ 0&0&0&0&0 \end{pmatrix},\ U^{(l)} = \begin{pmatrix} \mathcal{O}_5&\mathcal{O}_5&\mathcal{O}_5&\mathcal{O}_5\\ \mathcal{O}_5&A_2^{(l)}&\mathcal{O}_5&\mathcal{O}_5\\ \mathcal{O}_5&\mathcal{O}_5&\mathcal{O}_5&\mathcal{O}_5\\ \mathcal{O}_5&\mathcal{O}_5&\mathcal{O}_5&A_4^{(l)} \end{pmatrix}. \qquad \square$$

It is evident that matrix $U^{(l)}$ makes it possible to calculate $C^{(l)}$ because the values of each submatrix $C_{(k)}^{(l)}$ of $C^{(l)}$ verify with respect to the endfire events $t_k\rangle$

$$C_{(k)}^{(l)} = \begin{cases} C_{(k)} \cdot A_{(k)}^{(l)} & \text{if } t_k\rangle \in \mathfrak{G}_\rangle^{(l)} \\ C_{(k)} \cdot \mathcal{O}_{d+1} & \text{otherwise.} \end{cases} .$$

The $\Big(|P| \times (d+1) \cdot |T|\Big)$-matrix $C^{(l)} = C \cdot U^{(l)}$ is called *time incidence matrix with actual durations* for the end-step $\mathfrak{G}_\rangle^{(l)}$.

In the following calculi (just below and later) we need some matrices, all of them are sparse square $(d+1 \times d+1)$ matrices: Besides the already introduced *identity matrix* E_{d+1} and *zero-matrix* $\mathcal{O}_{d+1}$, we define here the matrices $L_{d+1} = (l_{ij})$, $W_{d+1} = (w_{ij})$ and the *progress matrix* $R_{d+1} = (r_{ij})$ by setting

$$l_{ij} := \begin{cases} 1 \text{if } i \geq 2 \\ \quad \text{and } i = j \\ 0 \text{otherwise} \end{cases},\ w_{ij} := \begin{cases} 1 \text{ if } i \geq 2 \\ \quad \text{and } j = 1 \\ 0 \text{ otherwise.} \end{cases},\ r_{i,j} := \begin{cases} 1 \text{if } \ (i = j = 1) \\ \quad \text{or } (i = j+1) \\ 0 \text{otherwise} \end{cases}.$$

For simplicity we write R instead of R_{d+1} if $d+1$ is clear from the context.

Example 10 *For the running example $\mathcal{Z}_o$ from Fig.1 with $d+1 = 5$ these square matrices are*

$$L_5 = \begin{pmatrix} 0&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, W_5 = \begin{pmatrix} 0&0&0&0&0\\ 1&0&0&0&0\\ 1&0&0&0&0\\ 1&0&0&0&0\\ 1&0&0&0&0 \end{pmatrix}, R_5 = \begin{pmatrix} 1&0&0&0&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0 \end{pmatrix}. \qquad \square$$

Now, let us observe the utility of these matrices. If we multiply an arbitrary $(l \times d+1)$- matrix A by L_{d+1} we obtain a $(l \times (d+1))$- matrix $B = A \cdot L_{d+1}$ whose first column is the l - dimensional zero-vector and the rest of its columns are the same as in the matrix A. If we multiply A by W_{d+1} we obtain a $(l \times (d+1))$-matrix $B' = A \cdot W$ whose first column is the sum of all but the first columns of A and all the other columns are zero-vectors. Finally, if we multiply A by R_{d+1} we obtain a $(l \times (d+1))$- matrix $B'' = A \cdot W$ whose i-th column is the $(i+1)$-th column of A, except the first one and the last one. Thus the multiplication by R insures a shift. The first column of B'' is the sum of the first and second columns of A and the last one is a zero-vector.

Now, for each *Endstep* $\mathfrak{G}_\rangle^{(l)} = \{t_{i_1}^{n_{i_1}}\rangle, \ldots, t_{i_\rho}^{n_{i_\rho}}\rangle\}$ and *Iteratedstep* $\mathfrak{G}_I^{(l)} = \{[t_{i_1}^{n_{i_1}}, t_{i_1}^{q_{i_1}}\rangle, \ldots, [t_{i_\kappa}^{n_{i_\rho}}, t_{i_\rho}^{q_{i_\rho}}\rangle\}$, with $q_s \leqslant n_s$ forall $s \in [1 \cdots \rho]$. we define a matrix $B_\rangle^{(l)}$, called the *bag matrix* of $\mathfrak{G}_\rangle^{(l)}$ as well as the matrices $B_{\mathrm{M}}^{(l)}$ and $B_{\mathrm{Z}}^{(l)}$

called the *bag matrices* of $\mathfrak{G}_I^{(l)}$, all being $(d+1\cdot|T|\times(d+1))$ matrices, by setting

$$B_{\rangle}^{(l)} = \begin{pmatrix} B_{\rangle(1)}^{(l)} \\ B_{\rangle(2)}^{(l)} \\ \vdots \\ B_{\rangle(|T|)}^{(l)} \end{pmatrix}, B_{\mathrm{M}}^{(l)} = \begin{pmatrix} B_{\mathrm{M}(1)}^{(l)} \\ B_{\mathrm{M}(2)}^{(l)} \\ \vdots \\ B_{\mathrm{M}(|T|)}^{(l)} \end{pmatrix} \text{ and } B_{\mathrm{Z}}^{(l)} = \begin{pmatrix} B_{\mathrm{Z}(1)}^{(l)} \\ B_{\mathrm{Z}(2)}^{(l)} \\ \vdots \\ B_{\mathrm{Z}(|T|)}^{(l)} \end{pmatrix} \text{ where}$$

$$B_{\rangle(s)}^{(l)} := \begin{cases} L_{d+1} & \text{if } s \in \{i_1,\ldots,i_\rho\} \\ 0\cdot E_{d+1} & \text{otherwise.} \end{cases}, \tag{17}$$

$$B_{\mathrm{M}(s)}^{(l)} := \begin{cases} \mathfrak{G}_I^{(l)}([\mathtt{t}_s)\cdot E_{d+1} & \text{if } s \in \{i_1,\ldots,i_\kappa\} \\ 0\cdot E_{d+1} & \text{otherwise.} \end{cases},$$

$$B_{\mathrm{Z}(s)}^{(l)} := \begin{cases} \mathfrak{G}_I^{(l)}(\mathtt{t_s}\rangle)\cdot L_{d+1} & \text{if } s \in \{i_1,\ldots,i_\kappa\} \\ 0\cdot E_{d+1} & \text{otherwise.} \end{cases} \tag{18}$$

***R*emark 1** *In the bag matrices for Endsteps $B_{\rangle}^{(l)}$ and $B_{\mathrm{Z}}^{(l)}$, the first column is obviously a zero vector.*

***E*xample 11** *The Iteratedstep $\mathfrak{G}_I^{(l)} = \{[t_2^6, [t_3, [t_1, t_3\rangle\}$ of the net $\mathcal{Z}_0$ from Fig.1 yields $B_{\mathrm{M}}^{(l)} = \begin{pmatrix} 1\cdot E_5 \\ 6\cdot E_5 \\ 1\cdot E_5 \\ 0\cdot E_5 \end{pmatrix}$ and $B_{\mathrm{Z}}^{(l)} = \begin{pmatrix} 0\cdot L_5 \\ 0\cdot L_5 \\ 1\cdot L_5 \\ 0\cdot L_5 \end{pmatrix}$.* □

Finally, we consider two $((d+1)\cdot|T|\times(d+1))$-matrices $K_{\rangle}^{(l)}$ and $B_I^{(l)}$ which help us to describe algebraically the effect of respectively an *Endstep* and an *Iteratedstep*.

We will prove that the following terms describe exactly this change.

$$-C^{(l)}\cdot B_{\rangle}^{(l)} + C^{(l)}\cdot B_{\rangle}^{(l)}\cdot R^d = -\underbrace{C\cdot U^{(l)}}_{C^{(l)}} B_{\rangle}^{(l)} + \underbrace{C\cdot U^{(l)}}_{C^{(l)}} \cdot B_{\rangle}^{(l)}\cdot R^d$$

$$= C\Big(\underbrace{-U^{(l)}B_{\rangle}^{(l)} + U^{(l)}\cdot B_{\rangle}^{(l)}\cdot R^d}_{:=K_{\rangle}^{(l)}}\Big) = C\cdot K_{\rangle}^{(l)} \quad . \tag{19}$$

$$\text{and} \quad C\cdot B_{\mathrm{M}}^{(l)} - C\cdot B_{\mathrm{Z}}^{(l)} + C\cdot B_{\mathrm{Z}}^{(l)}\cdot R^d = C\,(\underbrace{B_{\mathrm{M}}^{(l)} - B_{\mathrm{Z}}^{(l)} + B_{\mathrm{Z}}^{(l)}\cdot R^d}_{:=B_I^{(l)}}) = C\cdot B_I^{(l)}. \tag{20}$$

4 State equation

In this section we derive a state equation for an arbitrary ITPN that is analogous to the state equation (1) of time-less nets and which is consistent with the state equation for ITPNs without auto concurrency and zero durations [8].

We consider in the firing step sequence given in (10) the effects of the *Globalstep* appearing at the after-tick time state $s^{(l)}$, for some natural number $l \le n$, as well as of its subsequent tick event :

$$s^{(l)} \xrightarrow{\mathfrak{G}_{\rangle}^{(l)}} \tilde{s}^{(l)} \xrightarrow{\mathfrak{G}_I^{(l)}} s'^{(l)} \xrightarrow{\checkmark} s^{(l+1)}. \tag{21}$$

The following two remarks are easy to prove.

Remark 2 *For all $k \geq d$ it holds that $R^k =: (f_{i,j})_{\substack{i=1\cdots d\\ j=1\cdots d}}$ with*

$$f_{i,j} = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Remark 3 *Let be $W^{(l)} := B_{\rangle}^{(l)} \cdot R^d$. Then the matrix $W^{(l)}$ has the following structure:*

$$W^{(l)} = \begin{pmatrix} W_{(1)}^{(l)} \\ W_{(2)}^{(l)} \\ \vdots \\ W_{(|T|)}^{(l)} \end{pmatrix} \quad \text{and} \quad W_{(s)}^{(l)} := \begin{cases} W_d & \text{if } t_s \in \mathfrak{G}_{\rangle l} \\ O_d & \text{otherwise.} \end{cases}.$$

Lemma 12 *Let us consider the $(|P| \times d+1)$ - matrix $Q^{(l)} := C^{(l)} \cdot B_{\rangle}^{(l)}$. Then its elements $q_{i,j}$ have the following values:*

$$q_{i,j}^{(l)} = \begin{cases} 0 & \text{if } j = 1 \\ \sum_{k=1}^{|T|} G_{\rangle k,j'} \cdot v(t_k, p_i) & \text{if } 1 < j \leqslant d+1 \text{ and } j' = lfd(t_k) - j + 3 \end{cases}.$$

Proof. We compute the elements $q_{i,j}^{(l)}$.

Case 1: $j = 1$. Then $q_{i,1}^{(l)} = \left(C^{(l)} \cdot B_{\rangle}^{(l)}\right)_{i,1} = \left(\sum_{k=1}^{|T|} C_{(k)}^{(l)} \cdot B_{\rangle_{(k)}}^{(l)}\right)_{i,1} = \sum_{r=1}^{|T|} \sum_{k=1}^{(d+1)} c_{i,k}^{(l,r)} \cdot \underbrace{b_{k,1}^{(l,r)}}_{=0} = 0.$

Case 2: $1 < j \leqslant d+1$. Then

$$q_{i,j}^{(l)} = \left(C^{(l)} \cdot B_{\rangle}^{(l)}\right)_{i,j} = \left(\sum_{k=1}^{|T|} C_{(k)}^{(l)} \cdot B_{\rangle_{(k)}}^{(l)}\right)_{i,j} = \sum_{r=1}^{|T|} \sum_{k=1}^{d+1} \left(c_{i,k}^{(l,r)} \cdot b_{k,j}^{(l,r)}\right)$$

$$= \sum_{r=1}^{|T|} \left(c_{i,j}^{(l,r)} \cdot 1\right) \underset{(16)}{=} \sum_{k=1}^{|T|} G_{\rangle k, \left(lfd(t_k) - j + 3\right)} \cdot v(t_k, p_i).$$ □

We will first establish linear equations for the time markings around a firing step.

Theorem 13 *Let $\mathcal{Z}$ be an ITPN, and let the time states $s^{(l)} = (m^{(l)}, h^{(l)})$, $\tilde{s}^{(l)} = (\widetilde{m}^{(l)} \tilde{h}^{(l)})$, $s'^{(l)} = (m'^{(l)}, h'^{(l)})$ and $s^{(l+1)} = (m^{(l+1)}, h^{(l+1)})$ be defined as in (21). Then the time markings fulfil*

$$\widetilde{m}^{(l)} = m^{(l)} + C \cdot K_{\rangle}^{(l)} \tag{22}$$

$$m'^{(l)} = \widetilde{m}^{(l)} + C \cdot B_{I}^{(l)} \tag{23}$$

$$m^{(l+1)} = m'^{(l)} \cdot R \tag{24}$$

Proof of equation (22) :

In order to derive (22) we have to show that $\left(\widetilde{m}^{(l)}\right)_{i,j} = \left(m^{(l)}\right)_{i,j} + \left(C \cdot K_{\rangle}^{(l)}\right)_{i,j}$ for each $i \in \{1, \cdots, |P|\}$ and $j \in \{1, \cdots, d+1\}$.

Case 1: $j = 1$. According to the definition of time markings (13) it holds that

$$\left(\widetilde{m}^{(l)}\right)_{i,1} - \left(m^{(l)}\right)_{i,1} = \Big(\sum_{k=1}^{|T|}\Big(\sum_{r=1}^{d} G_{\rangle k,r}^{(l)}\Big)\cdot v(t_k,p_i)\Big).$$

Thus we have to prove that $\Big(\sum_{k=1}^{|T|}\Big(\sum_{r=1}^{d} G_{\rangle k,r}^{(l)}\Big)\cdot v(t_k,p_i)\Big) = \Big(C\cdot K_{\rangle}^{(l)}\Big)_{i,1}$. It holds

$$\Big(C\cdot K_{\rangle}^{(l)}\Big)_{i,1} \underset{(19)}{=} \Big(-C\cdot U^{(l)}\cdot B_{\rangle}^{(l)}\Big)_{i,1} + \Big(C\cdot U^{(l)}\cdot B_{\rangle}^{(l)}\cdot R^d\Big)_{i,1}. \qquad (25)$$

Now we first consider the term $\Big(-C\cdot U^{(l)}\cdot B_{\rangle}^{(l)}\Big)_{i,1}$. As the first column of the matrix $B_{\rangle}^{(l)}$ consists only of zeros, it holds that $\Big(-C\cdot U^{(l)}\cdot B_{\rangle}^{(l)}\Big)_{i,1} =$

$$\Big(-C^{(l)}\cdot B_{\rangle}^{(l)}\Big)_{i,1} = -\Big(Q^{(l)}\Big)_{i,1} = 0.\ \text{(cf. lemma 12)} \qquad (26)$$

Subsequently, we consider the second term $\big(C\cdot U^{(l)}\cdot B_{\rangle}^{(l)}\cdot R^d\big)_{i,1}$. By remark 3 we know that

$$\begin{aligned}\Big(C\cdot U^{(l)}\cdot B_{\rangle}^{(l)}\cdot R^{d-1}\Big)_{i,1} &= \Big(C^{(l)}\cdot W\Big)_{i,1} = \sum_{k=1}^{(d+1)\cdot|T|} c_{i,k}^{(l)}\cdot w_{k,1}^{(l)} = \\ \sum_{r=1}^{|T|}\sum_{k=1}^{d+1} c_{i,k}^{(l,r)}\cdot w_{k,1}^{(l,r)} &\underset{(16)}{=} \sum_{k=1}^{|T|}\Big(\sum_{r=1}^{d} G_{\rangle k,r}^{(l)}\Big)\cdot v(t_k,p_i).\end{aligned} \qquad (27)$$

Considering (25),(26) and (27) leads to the equation $\Big(\widetilde{m}^{(l)}\Big)_{i,1} = \Big(m^{(l)}\Big)_{i,1} + \Big(C\cdot K_{\rangle}^{(l)}\Big)_{i,1}$, as desired.

Case 2: $j > 1$.

According to the definition of time markings in (13) it holds that

$$\widetilde{m}_{i,j}^{(l)} - m_{i,j}^{(l)} = -\sum_{k=1}^{|T|} G_{\rangle\, k,\left(lfd(t_k)-j+3\right)}^{(l)}\cdot v(t_k,p_i). \qquad (28)$$

Thus, we have to prove that $\Big(C\cdot K_{\rangle}^{(l)}\Big)_{i,j} = -\sum_{k=1}^{|T|} G_{\rangle\, k,\left(lfd(t_k)-j+3\right)}^{(l)}\cdot v(t_k,p_i)$.

It holds that

$$\begin{aligned}\Big(C\cdot K_{\rangle}^{(l)}\Big)_{i,j} &\underset{(19)}{=} \Big(-C\cdot U^{(l)}\cdot B_{\rangle}^{(l)} + C\cdot U^{(l)}\cdot B_{\rangle}^{(l)}\cdot R^d\Big)_{i,j} \\ &= \Big(-C^{(l)}\cdot B_{\rangle}^{(l)} + C^{(l)}\cdot B_{\rangle}^{(l)}\cdot R^d\Big)_{i,j} \\ &= \Big(-Q^{(l)} + C^{(l)}\cdot W^{(l)}\Big)_{i,j} \quad \text{(cf. Rem. 3 and Lemma 12)} \\ &= -\Big(Q^{(l)}\Big)_{i,j} + \Big(C^{(l)}\cdot W^{(l)}\Big)_{i,j} \\ &= -\Big(Q^{(l)}\Big)_{i,j} + 0 = q_{i,j}^{(l)} \qquad \text{(cf. Rem. 3)} \\ &= -\sum_{k=1}^{|T|} G_{\rangle\, k,\left(lfd(t_k)-j+3\right)}^{(l)}\cdot v(t_k,p_i). \quad \text{(cf. Lemma 12)}\end{aligned}$$

The profs of equations (23) and (24) can be done similarly. □

Now we can deduce the main result, i.e., the equation for the sequence (10):

Theorem 14 *Let $\mathcal{Z}$ be an ITPN, $n \geq 1$ and σ a firing step sequence consisting of n Globalsteps, alternating with ticks, leading to the time state $s^{(n)} = (m^{(n)}, h^{(n)})$ as defined in (10). Then the time marking $m^{(n)}$ fulfils $m^{(n)} =$*

$$m^{(0)} \cdot R^n + C \cdot \Psi_\sigma \text{ where } \Psi_\sigma = \sum_{l=1}^{n} \left(K_{\rangle}^{(l-1)} + B_I^{(l-1)} \right) \cdot R^{n+1-l}. \tag{29}$$

The proof can be done by induction on n. □

We call Ψ_σ, which is a $((d+1) \cdot |T| \times |P|)$ - matrix, the *Parikh matrix* and equation (29) the *state equation* of the firing step sequence (10). Analogously to the Parikh vector, the Parikh matrix counts the number of appearances of startfire and endfire events in (10).

It is evident, that due to Theorems 13 and 14, we can analogously establish state equations for the other (intermediate) time markings, such as $m'^{(n)}$ and $\widetilde{m}^{(n)}$, that appear in the firing step sequences.

The last Theorem 14 provides a sufficient condition for the non-reachability of a given time marking. Let us explain what it means to show that there does not exist a sequence, nevertheless which length, such that after firing of the sequence from the initial time state, the net is in a time state whose time marking is the given one. For this reason, similar to the case for classic Petri nets, we have to solve an system of equalities defined by the equation (29). Of course, this system of equalities is much more difficult than that for the equation (1) for classic PNs.

The number of variables in the state equation is around $n \cdot |T| \cdot \big((d+1)^2/2+3\big)$, in total. Additionally, there are some more additional "local" equalities/inequalities.

Finally, we have to prove that for no n the obtained system of equalities of the state equation has an integer solution. In that case the given time marking is not reachable. In the other case - if there is an integer solution for some particular n - then no assertion can be done about the reachability of the time marking. It could be possible that the solution represents only non realizable sequences with, for instance, intermediate states which would have negative values.

5 Conclusion

In this article we have studied the class of Interval-Timed Petri nets with discrete delays in their most complex version. Firstly, zero duration is allowed (i.e. zero is possible as a lower bound of the duration interval of a transition), which has as consequence that in between two time ticks a certain number of transitions may start and end and provoke the start and perhaps ending of others, and so on. We consider only well formed nets where this number is always finite, i.e. where there is no undesired cycle of transitions of zero duration.

Then we allow auto-concurrency in the firing of transitions. This means that in maximal steps several instances of the same transition may start at the same moment and could have independent durations. Our notion of *Globalstep*, which consists of all startfire and endfire events in between two time ticks, is original.

When in a state a subbag of concurrently active instances of the same transition should end we could choose to end the oldest ones between them or arbitrary ones. We prove that both ways are equivalent, leading to sequences composed of the same *Globalsteps*. This result allows us to choose once for all in this article to end always the oldest active transitions.

To obtain adequate formalizations, original algebraic structures have been proposed for all defined concepts.

In this complex algebraic context, our goal was to construct state equations for the considered net class. We proposed a series of results which lead to the main theorem, which establishes that each reachable time state fulfils a certain nontrivial state equation. The paper contains all proofs.

By contraposition we may conclude, that a time state is unreachable in the considered Interval-Timed Petri net when the system of equalities associated to its state equation has no solution.

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