Fuzzy logic models in some categories

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***Abstract*—Models of a fuzzy logic in two categories of sets with similarity relations are introduced. Interpretations of formulas in these models are defined and some relations between different interpretations are investigated.**

Keywords*—*Many-Valued and Fuzzy Logics.

I. Introduction

Fuzzy logic can be simply characterized as a special many-valued logic with special properties aiming at modeling of the vagueness phenomenon and some parts of the meaning of natural language by using graded approach, where grades are (in general) from some lattice structure Q. From formal point of view fuzzy logic provides a theoretical background for the graded approach to vagueness. As a mathematical object fuzzy logic has classical structure of a logic, i.e. it consists of a first order language J which consists (as classically) of a set of predicate symbols $P \in \mathcal{P}$, a set of functional symbols $f \in \mathcal{F}$ and a set of logical connectives $\{\wedge, \vee, \Rightarrow, \neg, \otimes\}$. Moreover J contains also a set Q of logical constants. In that language, terms and formulas can be defined (by using of inductive principle) in the same way as for a classical first order predicate logic. With any classical logic a syntactic structure is connected. It means that for any formula ψ of a logic we can derive if that formula is provable (i.e. truth) in that logic (in symbol $\vdash \psi$) or not. Principal tools for calculations are *deduction rules* which are used in the logic. In a fuzzy logic graded versions of deduction rules are used and it means that we receive also a graded notion of a provability of a formula, i.e. $\vdash_\alpha \psi$ means that ψ is *true* in the logic in a degree α, where $\alpha \in Q$. There is another tool for verification of a provability $\vdash \psi$. Instead of *syntactic methods* (i.e. formal rules for handling with formulas) we can use *semantic methods*, i.e. methods based on interpretations of formulas in models. A model $\mathcal{E}$ of a logic J (not important if a logic is a fuzzy logic or classical one) is based on some concrete structure $\mathbf{A}$ (in general, an object of some category) and interpretations of predicate and functional symbols in that structure. As a result of an interpretation we can define a truth valuation of a formula in a model $\mathcal{E}$. For a fuzzy logic a truth valuation can be principally defined in two different ways:

(i) As a *fuzzy set object* $\|\psi\|$, i.e. a special morphism $\|\psi\| : \mathbf{A} \to Q$ in a corresponding category, or
(ii) As a *cut object* $(|\psi|_\alpha)_{\alpha \in Q}$, where $|\psi|_\alpha$ are special (nested) subobjects in $\mathbf{A}$ in a corresponding category.

Intuitively,

1) If $\|\psi\|$ is a fuzzy set object in $\mathbf{A}$, it means that for any "element" $\mathbf{a} \in \mathbf{A}$, $\|\psi\|(\mathbf{a}) \in Q$ is a **degree** in which a formula ψ is true in model $\mathcal{E}$, if the value of free variables x in ψ is substituted by elements $a \in \mathbf{A}$.
2) If $(|\psi|_\alpha)_\alpha$ is a cut object in $\mathbf{A}$, then $|\psi|_\alpha$ is a **"subset"** of all interpretations in $\mathbf{A}$ of free variables, for which a formula ψ is true in a model $\mathcal{E}$ in a degree at least α.

It can be then proved that for some types of models and some fuzzy logic (based on some special deduction rules) a *completeness theorem* is true, i.e. $\vdash_{1_Q} \psi$ if and only if $\bigvee_{\mathbf{a} \in \mathbf{A}} \|\psi\|(\mathbf{a}) = 1_Q$ for any model $\mathcal{E}$. For more details concerning fuzzy logic and its models see e.g. [12].

In the paper we will be interested in constructions of fuzzy logic models in general settings - in some categories. That approach enables us to extend significantly a variety of possible models of fuzzy logic and to create tools for calculating values $\|\psi\|_\mathcal{E}$ depending on models $\mathcal{E}$. An idea to construct a model of a logic in categories is not new. A comprehensive study has been done in [6], nevertheless all important results were received for a very special category only, namely for a topos, which seems not to be very useful for fuzzy set theory. In the paper we will be interested in more general categories which are based on sets with similarity relations (i.e. a graded identity relation) with values in residuated lattice. In general, such categories are not topoi, but as generalizations of fuzzy sets seem to be very useful for fuzzy logic models constructions.

As a result, we construct 2 types of models of a fuzzy logic based on two different categories $\mathrm{Set}(Q)$ and $\mathrm{SetS}(Q)$ and for each category we define two different types on formula interpretations and we show some relationships between these interpretations.

II. Preliminary notions and results

In this section we present some preliminary notions and definitions which could be helpful for better understanding of results concerning sets with similarity relations. Most of these notions can be found e.g. in [9], [8], [7]. A principal structure used in the paper is a *complete residuated lattice* (see e.g. [12]), i.e., a structure $Q = (Q, \wedge_Q, \vee_Q, \otimes_Q, \rightarrow_Q, 0_Q, 1_Q)$ such that $(Q, \wedge_Q, \vee_Q)$ is a complete lattice, $(Q, \otimes_Q, 1_Q)$ is a commutative monoid with operation $\otimes_Q$ isotone in both arguments and $\rightarrow_Q$ is a binary operation which is residuated with respect to $\otimes_Q$, i.e.,

$$\alpha \otimes_Q \beta \leq \gamma \quad \text{iff} \quad \alpha \leq \beta \rightarrow_Q \gamma.$$

For simplicity the index Q will be sometimes omitted.

Any classical set A can be considered as a pair $(A, =)$, where $=$ is the equality relation. It is then natural to consider a generalization of that pair, i.e., a pair (A, δ), where δ is a similarity relation. Recall that a similarity relation in A is a map $\delta : A \times A \to \Omega$ such that

(a) $(\forall x \in A) \quad \delta(x, x) = 1$,
(b) $(\forall x, y \in A) \quad \delta(x, y) = \delta(y, x)$,
(c) $(\forall x, y, z \in A) \quad \delta(x, y) \otimes \delta(y, z) \le \delta(x, z)$ (generalized transitivity).

A pair (A, δ) will be called Q-set. In the paper, we will be working not only with Q-sets, but also with "mappings" between Q-sets, i.e., it seems to be useful to use a category theory tools for further investigation of such structures. We basically use two categories with Q-sets as objects and with differently defined morphisms. A morphism $f : (A, \delta) \to (B, \gamma)$ in the first category $\mathrm{Set}(Q)$ is a map $f : A \to B$ such that $\gamma(f(x), f(y)) \ge \delta(x, y)$ for all $x, y \in A$. The other category $\mathrm{SetS}(Q)$ is an analogy of the category of sets with relations between sets as morphisms. Objects of the category $\mathrm{SetS}(Q)$ are the same as in the category $\mathrm{Set}(Q)$ and morphisms $f : (A, \delta) \to (B, \gamma)$ are maps $f : A \times B \to \Omega$ (i.e., Q-valued relations) such that

(a) $(\forall x, z \in A)(\forall y \in B) \quad \delta(z, x) \otimes f(x, y) \le f(z, y)$,
(b) $(\forall x \in A)(\forall y, z \in B) \quad f(x, y) \otimes \gamma(y, z) \le f(x, z)$.
(c) $(\forall a \in A) \quad 1 = \bigvee\{f(x, y) : y \in B\}$.

If $f : (A, \delta) \to (B, \gamma)$ and $g : (B, \gamma) \to (C, \omega)$ are two morphisms in $\mathrm{SetS}(Q)$, then their composition is a relation $g \circ f : A \times C \to \Omega$ such that

$$g \circ f(x, z) = \bigvee_{y \in B} (f(x, y) \otimes g(y, z)).$$

It should be mentioned that there is another category of sets with similarity relations as objects, which was intensively investigated. Namely the category $\mathrm{SetS}(Q)$ with morphisms satisfying previous conditions (a) and (b) only. Unfortunately, that category is not appropriate for logic interpretation, because of a lack of categorical products, which are important for models constructions.

Lemma II.1 *There exists a functor $F : Set(Q) \to SetS(Q)$.*

As we mentioned in Introduction, a fuzzy set f in an Q-set (A, δ) in a category $\mathbf{K}$ of Q-sets (in symbol: $f \subseteq_{\sim \mathbf{K}} (A, \delta)$) is a morphism $f : (A, \delta) \to (Q, \leftrightarrow)$ in a category $\mathbf{K}$, where $\leftrightarrow$ is the biresiduation operation in Q defined by $\alpha \leftrightarrow \beta = (\alpha \to \beta) \wedge (\beta \to \alpha)$. Hence, $f \subseteq_{\sim \mathrm{Set}(Q)} (A, \delta)$, if $f : A \to \Omega$ is a map such that $\delta(x, y) \le f(x) \leftrightarrow f(y)$, or equivalently, $f(x) \otimes \delta(x, y) \le f(y)$ for all $x, y \in A$. Analogously, a fuzzy set in the category $\mathrm{SetS}(Q)$ is a map $f : A \times Q \to Q$ which has to satisfy the following conditions:

1) $f(x, \alpha) \otimes \delta(x, y) \le f(y, \alpha)$, for all $x, y \in A, \alpha \in Q$,
2) $f(x, \alpha) \otimes (\alpha \leftrightarrow \beta) \le f(x, \beta)$, for all $x \in A, \alpha, \beta \in Q$,
3) $1 = \bigvee_{\alpha \in Q} f(x, \alpha)$, for any $x \in A$.

A set of all fuzzy sets $f \subseteq_{\sim \mathbf{K}} (A, \delta)$ in a Q-set (A, δ) is an object function of a functor, for $\mathbf{K} = \mathrm{Set}(Q)$ or $\mathrm{SetS}(Q)$. In fact, there exists a functor $\mathcal{F}_{\mathbf{K}} : \mathbf{K} \to \mathrm{Set}$ that is defined by $\mathcal{F}_{\mathbf{K}}(A, \delta) = \{s : s \subseteq_{\sim \mathbf{K}} (A, \delta)\}$. If $f : (A, \delta) \to (B, \gamma)$ is a morphism in $\mathbf{K}$, then the map $\mathcal{F}_{\mathbf{K}}(f) : \mathcal{F}_{\mathbf{K}}(A, \delta) \to \mathcal{F}_{\mathbf{K}}(B, \gamma)$ is defined differently for categories $\mathbf{K} = \mathrm{Set}(Q)$ and $\mathbf{K} = \mathrm{SetS}(Q)$. We have,

$$\mathcal{F}_{\mathrm{Set}(Q)}(f)(s)(b) = \bigvee_{x \in A} s(x) \otimes \gamma(b, f(x)),$$

for all $b \in B$ and any $s \subseteq_{\sim \mathrm{Set}(Q)} (A, \delta)$ (see [8]) and for the category $\mathrm{SetS}(Q)$ we have,

$$(\forall b \in B)(\forall \alpha \in Q)\mathcal{F}_{\mathrm{SetS}(Q)}(f)(s)(b, \alpha) = \bigvee_{a \in A} f(a, b) \otimes s(a, \alpha),$$

for all $b \in B, \alpha \in Q$ (see [9]).

Proposition II.1 *There exists a natural transformation*

$$\sigma : \mathcal{F}_{Set(Q)} \to \mathcal{F}_{SetS(Q)} \circ F.$$

Not every maps $A \to Q$ or $A \times Q \to Q$, respectively, are morphisms in category $\mathrm{Set}(Q)$ or $\mathrm{SetS}(Q)$, respectively. On the other hand, any such maps can be extended to morphisms, according the following methods.

Lemma II.2 *Let $(A, \delta), (B, \gamma)$ be Q-sets and let $g : A \times B \to Q$ be a map, such that $1 = \bigvee_{b \in B} g(a, b)$, for any $a \in A$. Let $\tilde{g} : A \times B \to Q$ be defined by the formula*

$$\tilde{g}(a, b) = \bigvee_{x \in A} \bigvee_{y \in B} g(x, y) \otimes \delta(a, x) \otimes \gamma(b, y).$$

Then

1) *$\tilde{g}$ is a morphism in $SetS(Q)$,*
2) *$\tilde{g} = \bigwedge\{h : h$ is a morphism $(A, \delta) \to (B, \gamma)$ in $SetS(Q), h \ge g\}$,*
3) *If g is a morphism in $SetS(Q)$, then $\tilde{g} = g$.*

Lemma II.3 *Let (A, δ) be an Q-set and let $s : A \to Q$ be a map. Then we define a map $\widehat{s} : A \to Q$ such that $\widehat{s}(a) = \bigvee_{x \in A} \delta(a, x) \otimes s(x)$ for all $a \in A$. Then*

1) *$\widehat{s} : (A, \delta) \to (Q, \leftrightarrow)$ is a morphism in $Set(Q)$,*
2) *If $s : (A, \delta) \to (Q, \leftrightarrow)$ is a morphism in $Set(Q)$ then $\widehat{s} = s$,*
3) *$\widehat{s} = \bigwedge\{t : t$ is a morphism $(A, \delta) \to (Q, \leftrightarrow), t \ge s$ in $Set(Q)\}$.*

It is well known that any classical fuzzy set (with values in a residuated lattice Q) in a set A can be alternatively expressed as a system of α-cuts $\mathbf{C} = (C_\alpha)_\alpha$, where $\mathbf{C}$ is a nested system of subsets of A. In our previous papers ([8], [7]) we proved that analogical representations of fuzzy sets by special cuts, named f-cuts, exists in categories $\mathrm{Set}(Q), \mathrm{SetS}(Q)$. The definitions of these f-cuts is as follows.

Definition II.1 *Let (A, δ) be a Q-set. Then a system $\mathbf{C} = (C_\alpha)_{\alpha \in Q}$ of subsets in A is called an* **f-cut** *in (A, δ) in the category $Set(Q)$ if*

1) $\forall a, b \in A, \quad a \in C_\alpha \Rightarrow b \in C_{\alpha \otimes \delta(a,b)}$,
2) $\forall a \in A, \forall \alpha \in Q, \quad \bigvee_{\{\beta : a \in C_\beta\}} \beta \geq \alpha \Rightarrow a \in C_\alpha$.

Definition II.2 *Let (A, δ) be a Q-set. Then a system $(C_\alpha)_{\alpha \in Q}$ is called an* **f-cut** *in (A, δ) in the category SetS(Q), if*

1) *$C_\alpha \subseteq A \times Q$, for any $\alpha \in Q$,*
2) $\forall a, b \in A, \quad (a, \beta) \in C_\alpha \Rightarrow (b, \beta) \in C_{\alpha \otimes \delta(a,b)}$,
3) $\forall a \in A, \forall \gamma \in Q, \quad \bigvee_{\{\beta : (a,\gamma) \in C_\beta\}} \beta \geq \alpha \Rightarrow (a, \gamma) \in C_\alpha$,
4) $\forall a \in A, \forall \alpha, \gamma \in Q, \quad (a, \alpha) \in C_\beta \Rightarrow (a, \gamma) \in C_{\beta \otimes (\alpha \leftrightarrow \gamma)}$,
5) $\forall a \in A, \quad 1 = \bigvee_{\{(\alpha,\beta) : (a,\alpha) \in C_\beta\}} \beta$

It is cleat that not every system of subsets $(C_\alpha)_\alpha$ from A or $A \times Q$ is an f-cut. On the other hand, analogously as for fuzzy set, such systems can be extended to f-cuts, as the following lemmas show.

Proposition II.2 *Let (A, δ) be a Q-set and let $(C_\alpha)_\alpha$ be a system of subsets in a set A. For any $\alpha \in Q$ we set*

$$\overline{C_\alpha} = \{a \in A : \bigvee_{\{(x,\beta) : x \in C_\beta\}} \beta \otimes \delta(a, x) \geq \alpha\}.$$

Then

1) *$C_\alpha \subseteq \overline{C_\alpha}$ for all $\alpha \in Q$,*
2) *$(\overline{C_\alpha})_\alpha$ is an f-cut system in (A, δ) in the category Set(Q),*
3) *If $(C_\alpha)_\alpha$ is an f-cut system in (A, δ) in the category Set(Q), then $\overline{C_\alpha} = C_\alpha$ for all $\alpha \in Q$.*
4) *If $(D_\alpha)_\alpha$ is a system of subsets in A such that $C_\alpha \subseteq D_\alpha$ for all $\alpha \in Q$, then $\overline{C_\alpha} \subseteq \overline{D_\alpha}$ for all $\alpha \in Q$.*

An analogical result holds for the category SetS(Q).

Proposition II.3 *Let (A, δ) be a Q-set and let $(C_\alpha)_\alpha$ be a system of subsets in a set $A \times Q$, such that $1 = \bigvee_{\{(\alpha,\beta) : (a,\alpha) \in C_\beta\}} \beta$, for all $a \in A$. For any $\alpha \in Q$ we set*

$$\overline{C_\alpha} =$$
$$\{(a, \beta) \in A \times Q : \bigvee_{\{(x,\tau,\rho) : (x,\tau) \in C_\rho\}} \rho \otimes \delta(a, x) \otimes (\tau \leftrightarrow \beta) \geq \alpha\}.$$

Then

1) *$C_\alpha \subseteq \overline{C_\alpha}$ for all $\alpha \in Q$,*
2) *$(\overline{C_\alpha})_\alpha$ is an f-cut in (A, δ) in the category SetS(Q),*
3) *If $(C_\alpha)_\alpha$ is an f-cut in (A, δ) in the category SetS(Q), then $\overline{C_\alpha} = C_\alpha$ for all $\alpha \in Q$.*
4) *If $(D_\alpha)_\alpha$ is a system of subsets in A such that $C_\alpha \subseteq D_\alpha$ for all $\alpha \in Q$, then $\overline{C_\alpha} \subseteq \overline{D_\alpha}$ for all $\alpha \in Q$.*

Analogously as for fuzzy sets, there exists a functor $\mathcal{C}_{\mathbf{K}} : \mathbf{K} \to \mathbf{Set}$, such that $\mathcal{C}_{\mathbf{K}}(A, \delta)$ is the set of all f-cuts in (A, δ) in a category $\mathbf{K} = \mathrm{Set}(\mathbf{Q}), \mathrm{SetS}(\mathbf{Q})$ (for details see [10], [7]). In the same papers the following theorem is proved.

Theorem II.1 *For a category $\mathbf{K} = Set(\mathbf{Q}), SetS(\mathbf{Q})$, there exists a natural isomorphism $\Psi_{\mathbf{K}} : \mathcal{F}_{\mathbf{K}} \to \mathcal{C}_{\mathbf{K}}$.*

It means, especially, that for any Q-set (A, δ) and any category $\mathbf{K} = \mathrm{Set}(Q), \mathrm{SetS}(Q)$, there exists a bijection $\Psi_{\mathbf{K},(A,\delta)} : \mathcal{C}_{\mathbf{K}}(A, \delta) \to \mathcal{F}_{\mathbf{K}}(A, \delta)$.

Proposition II.4 *There exists a natural transformation*

$$\tau : \mathcal{C}_{Set(Q)} \to \mathcal{C}_{SetS(Q)} \circ F.$$

We show only how that transformation is defined. Let (A, δ) be a Q-set and let $\mathbf{C} = (\mathbf{C}_\alpha)_\alpha \in \mathcal{C}_{\mathrm{Set}(\mathbf{Q})}(\mathbf{A}, \delta)$ be an f-cut in (A, δ) in the category Set(Q). Then $\tau_{(A,\delta)}(\mathbf{C}) = (D_\alpha)_\alpha$, where $D_\alpha = \{(a, \gamma) : \bigvee_{a \in C_\beta} \beta \leftrightarrow \gamma \geq \alpha\}$.

III. Construction of models of a fuzzy logic in categories Set(Q) and SetS(Q)

Let us recall some definitions and results concerning interpretation of a fuzzy logic in models based on Q-sets (see [12], [11]). Recall that a first order predicate fuzzy logic is based on a first order language J which consists (as classically) of a set of predicate symbols $P \in \mathcal{P}$, a set of functional symbols $f \in \mathcal{R}$ and a set of logical connectives $\{\wedge, \vee, \Rightarrow, \neg, \otimes\}$. Terms and formulas are defined analogously as for the classical predicate logic by using of the inductive principle.

Let $\mathbf{K}$ be a category with Q-sets as objects and such that for any set of objects $\{(A_i, \delta_i) : i \in I\}$ there exists a product $(A, \delta) = \prod_{i \in I}(A_i, \delta_i)$. Recall that a product of $\{(A_i, \delta_i) : i \in I\}$ in a category $\mathbf{K}$ is an object (A, δ) with morphisms $pr_i : (A, \delta) \to (A_i, \delta_i)$ such that for any other object (B, γ) with morphisms $q_i : (B, \gamma) \to (A_i, \delta_i)$ there exists the unique morphism $q = \prod_i q_i$ such that the diagram commutes:

$$\begin{array}{ccc} (A, \delta) & = & (A, \delta) \\ {\scriptstyle pr_i}\downarrow & & \uparrow{\scriptstyle q = \prod_i q_i} \\ (A_i, \delta_i) & \xleftarrow{q_i} & (B, \gamma). \end{array}$$

Recall, how a product is constructed in our categories Set(Q) and SetS(Q). Let (A_i, δ_i) be Q-sets, $i \in I$. Let us consider the category Set(Q), firstly. Then $\prod_{i \in I}(A_i, \delta_i) = (A_I, \delta_I)$, where A_I is the cartesian product of sets A_i and $\delta_I(\mathbf{a}, \mathbf{b}) = \bigwedge_{i \in I} \delta_i(a_i, b_i)$, for any $\mathbf{a}, \mathbf{b} \in A$. The projection morphisms $pr_i : (A_I, \delta_I) \to (A_i, \delta_i)$ are classical projection maps and if $q_i : (B, \gamma) \to (A_i, \delta_i)$ are morphisms, for $i \in I$, the unique morphism $\prod_i q_i$ is such that $\prod_i q_i(b) = (q_i(b))_i \in A_I$, for any $b \in B$.

Now, let us consider the category SetS(Q). A product (A_I, δ_I) is the same object as in the category Set(Q), but with different projection morphisms pr_i defined such that $pr_i : (A_I, \delta_I) \to (A_i, \delta_i)$ in the category SetS(Q), i.e. $pr_i : A_I \times A_i \to Q$, such that $pr_i(\mathbf{a}, b) = \delta_i(a_i, b)$, for $\mathbf{a} \in A_I, b \in A_i$. If $q_i : (B, \gamma) \to (A_i, \delta_i)$ are morphisms, for $i \in I$, the unique morphism $\prod_i q_i$ is such that $\prod_i q_i : B \times A_I \to Q$, $\prod_i q_i(b, \mathbf{a}) = \bigwedge_i q_i(b, a_i)$. If $(A_i, \delta_i) = (A, \delta)$ for every $i = 1, \ldots, n$, then $\prod_i(A_i, \delta_i)$ will be denoted by (A^n, δ_n).

We now introduce two types of models of a fuzzy logic in a category $\mathbf{K} = \mathrm{Set}(Q)$ or SetS(Q).

Definition III.1 *A* **fuzzy set model** *of a language J in a category $\mathbf{K}$ is*

$$\mathcal{E}_{\mathbf{K}} = ((A,\delta), \{P_{\mathcal{E},\mathbf{K}} : P \in \mathcal{P}\}, \{f_{\mathcal{E},\mathbf{K}} : f \in \mathcal{R}\}),$$

where

(1) *(A,δ) is an Q-set from a category $\mathbf{K}$,*
(2) *$P_{\mathcal{E},\mathbf{K}}$ is a fuzzy set in $(A,\delta)^n$ in a category $\mathbf{K}$, i.e., a morphism $(A,\delta)^n \to (Q,\leftrightarrow)$,*
(3) *$f_{\mathcal{E},\mathbf{K}} : (A,\delta)^n \to (A,\delta)$ is a morphism in a category $\mathbf{K}$.*

Definition III.2 *A* **cut model** *of a language J in a category $\mathbf{K}$ is*

$$\mathcal{D}_{\mathbf{K}} = ((A,\delta), \{P_{\mathcal{D},\mathbf{K}} : P \in \mathcal{P}\}, \{f_{\mathcal{D},\mathbf{K}} : f \in \mathcal{R}\}),$$

where

(1) *(A,δ) is an Q-set from a category $\mathbf{K}$,*
(2) *$P_{\mathcal{D},\mathbf{K}}$ is an f-cut in $(A,\delta)^n$ in a category $\mathbf{K}$, $P_{\mathcal{D},\mathbf{K}} = (P_\alpha)_\alpha$,*
(3) *$f_{\mathcal{D},\mathbf{K}} : (A,\delta)^n \to (A,\delta)$ is a morphism in a category $\mathbf{K}$.*

If ψ is a formula in a fuzzy logic with a set X of free variables, then an interpretation $\|\psi\|$ of ψ will be different for different types of a model.

(a) Interpretation $\|\psi\|_{\mathcal{E},\mathbf{K}}$ in a model $\mathcal{E}_{\mathbf{K}}$ is a fuzzy set in a category $\mathbf{K}$,
(b) Interpretation $\|\psi\|_{\mathcal{D},\mathbf{K}}$ in a model $\mathcal{D}_{\mathbf{K}}$ is an f-cut in a category $\mathbf{K}$.

Intuitively,

1) If $\|\psi\|$ is a fuzzy set in $(A,\delta)^X$, it means that for any $\mathbf{a} = (a_x)_{x\in X} \in A^X$, $\|\psi\|(\mathbf{a}) \in Q$ is a **degree** in which a formula ψ is true in model $\mathcal{E}$, if the value of a variable x is substituted by element $a_x \in A$.
2) If $\|\psi\|$ is an f-cut $(|\psi|_\alpha)_\alpha$ in $(A,\delta)^X$, then $|\psi|_\alpha$ is a **set** of all interpretations (in A) of free variables from X, for which a formula ψ is true in a model $\mathcal{E}$ in a degree at least α.

Now we present these definitions of $\|\psi\|$ in our two types of models. First of all, we need to define an interpretation of terms in our models. The definition will be the same for all two types of models. Let $\mathcal{G} = \mathcal{E}_{\mathbf{K}}$ or $\mathcal{D}_{\mathbf{K}}$ be a model of a language J in a category $\mathbf{K}$. An interpretation of a term with a set of variables contained in a set X is a morphism $\|t\|_{\mathcal{G},\mathbf{K}} : (A,\delta)^X \to (A,\delta)$ in a category $\mathbf{K}$, defined as follows:

1) Let $t = x$, where $x \in X$. Then $\|t\|_{\mathcal{G},\mathbf{K}} := pr_x : (A,\delta)^X \to (A,\delta)$ is the x-projection morphism in the category $\mathbf{K}$.
2) Let $t = f(t_1,\dots,t_n)$. Then $\|t\|_{\mathcal{G},\mathbf{K}}$ is a composition (in $\mathbf{K}$) of morphisms

$$(A,\delta)^X \xrightarrow{\prod_i \|t_i\|_{\mathcal{G},\mathbf{K}}} (A,\delta)^n \xrightarrow{f_{\mathcal{G},\mathbf{K}}} (A,\delta)\cdot$$

For example, for $\mathbf{K} = \mathrm{SetS}(Q)$ we have $\|t\|_{\mathcal{G},\mathrm{SetS}(Q)}(\mathbf{a},b) = \bigvee_{\mathbf{x}\in A^n} \bigwedge_{i=1}^n \|t_i\|(\mathbf{a},x_i) \otimes f_{\mathcal{G}}(\mathbf{x},b)$, where $\mathbf{a} \in A^X, b \in A$.

A definition of $\|\psi\|$ will differs for different types of models and different categories $\mathbf{K} = \mathrm{Set}(\mathbf{Q}), \mathrm{SetS}(\mathbf{Q})$. Definitions will be done by the induction principle depending on a structure of ψ.

Definition III.3 (Interpretation in models $\mathcal{E}_{\mathbf{Set}(Q)}, \mathcal{E}_{\mathbf{SetS}(Q)}$) *Let $\mathbf{K}$ be the category Set(Q) or SetS(Q), respectively. An interpretation $\|\psi\| = \|\psi\|_{\mathcal{E},\mathbf{K},X}$ of ψ with free variables in a set X in a category $\mathbf{K}$ is defined as follows.*

1) *Let $\psi \equiv P(t_1,\dots,t_n)$. Then $\|\psi\|_{\mathcal{E},\mathbf{K}}$ is defined as the composition of the following morphisms in $\mathbf{K}$:*

$$(A,\delta)^X \xrightarrow{\prod_i \|t_i\|_{\mathcal{E},\mathbf{K}}} (A,\delta)^n \xrightarrow{P_{\mathcal{E},\mathbf{K}}} (Q,\leftrightarrow).$$

2) *Let $\psi \equiv t_1 = t_2$. Then $\|\psi\|_{\mathcal{E},\mathbf{K}}$ is the composition of the following morphisms in $\mathbf{K}$:*

$$(A,\delta)^X \xrightarrow{\|t_1\|_{\mathcal{E},\mathbf{K}}\times\|t_2\|_{\mathcal{E},\mathbf{K}}} (A,\delta)^2$$
$$(A,\delta)^2 \xrightarrow{\Delta_{\mathbf{K},\mathcal{E}}} (Q,\leftrightarrow),$$

where $\Delta_{\mathbf{K},\mathcal{E}}$ is a morphism in $\mathbf{K}$, which interprets equality in a model $\mathcal{E}_{\mathbf{K}}$.

3) *Let $\psi \equiv \sigma\nabla\tau$, where ∇ represents logical connectives $\wedge, \vee, \implies, \otimes$, respectively. Then $\|\psi\|_{\mathcal{E},\mathbf{K}}$ is the composition of the following morphisms:*

$$(A,\delta)^X \xrightarrow{\|\sigma\|_{\mathcal{E},\mathbf{K}}\times\|\tau\|_{\mathcal{E},\mathbf{K}}} (Q,\leftrightarrow)^2$$
$$(Q,\leftrightarrow)^2 \xrightarrow{\odot_{\mathbf{K},\mathcal{E}}} (Q,\leftrightarrow),$$

where $\odot_{\mathbf{K},\mathcal{E}}$ is a morphism in $\mathbf{K}$, which interprets logical operations $\wedge, \vee, \to, \otimes$, respectively, in a model $\mathcal{E}_{\mathbf{K}}$.

4) *Let $\psi \equiv \neg\sigma$. Then $\|\psi\|_{\mathcal{E},\mathbf{K}}$ is the composition of the following morphisms:*

$$(A,\delta)^X \xrightarrow{\|\sigma\|_{\mathcal{E},\mathbf{K}}} (Q,\leftrightarrow) \xrightarrow{\neg_{\mathbf{K},\mathcal{E}}} (Q,\leftrightarrow),$$

where $\neg_{\mathbf{K},\mathcal{E}}$ is a morphism in $\mathbf{K}$, which interprets logical negation in a model $\mathcal{E}_{\mathbf{K}}$.

5) *Let $\psi \equiv (\forall x)\sigma$. Then $\|\sigma\|_{\mathcal{E},\mathbf{K},X\cup\{x\}}$ is already defined as a morphism $(A,\delta)^X \times (A,\delta) \to (Q,\leftrightarrow)$ in $\mathbf{K}$. Then we set*

$$\|\psi\|_{\mathcal{E},\mathbf{K},X}(\mathbf{a}) = \bigwedge_{x\in A} \|\sigma\|_{\mathcal{E},\mathbf{K},X\cup\{x\}}(\mathbf{a},x).$$

Let us now consider some examples of interpretations of logical connectives in categories Set(Q) and SetS(Q), presented in the definition.

Example III.1

1) A reasonable example of a morphism $\Delta_{\mathrm{Set}(Q),\mathcal{E}}$ could be the morphism $\widehat{\delta} : (A,\delta)^2 \to (Q,\leftrightarrow)$ (see Lemma I.3), i.e.

$$\widehat{\delta}(a,b) = \bigvee_{x,y\in A} \delta(x,y) \otimes (\delta(a,x) \wedge \delta(b,y)).$$

2) A morphism $\Delta_{\mathrm{SetS}(Q),\mathcal{E}}$ can be also defined as a map $A^2 \times Q \to Q$ by using results from Lemmas I.1-I.3 in several natural ways, i.e.:

(i) $\Delta_{\mathrm{SetS}(Q),\mathcal{E}}((a,b),\alpha) = F(\widehat{\delta})((a,b),\alpha),$

(ii) $\Delta_{\mathrm{SetS}(Q),\mathcal{E}}((a,b),\alpha)=\widetilde{\widehat{F(\delta)}}((a,b),\alpha)$.

3) $\wedge_{\mathrm{Set}(Q),\mathcal{E}}$ which interprets logical connective $\wedge$ can be $\wedge_Q$. In fact, the following holds for any $a,a',b,b'\in A$:

$$(a\wedge_Q a')\leftrightarrow(b\wedge_Q b')\geq(a\leftrightarrow b)\wedge_Q(a'\leftrightarrow b'),$$

which represents a fact that $\wedge_Q:(Q,\leftrightarrow)^2\rightarrow(Q,\leftrightarrow)$ is a morphism in $\mathrm{Set}(Q)$.

4) $\wedge_{\mathrm{SetS}(Q),\mathcal{E}}$ can be defined by $\wedge_{\mathrm{SetS}(Q),\mathcal{E}}((\beta,\gamma),\alpha)=\alpha\leftrightarrow(\beta\wedge\gamma)$. In fact, $\wedge_{\mathrm{SetS}(Q),\mathcal{E}}:(Q,\leftrightarrow)^2\rightarrow(Q,\leftrightarrow)$ is a morphism in $\mathrm{SetS}(Q)$, as it can be proved by a simple calculation.

5) On the other hand, the operation $\rightarrow_Q$ is not a morphism in $\mathrm{Set}(Q)$ and a reasonable candidate for interpretation $\rightarrow_{\mathrm{Set}(Q),\mathcal{E}}$ of a logical implication in $\mathrm{Set}(Q)$ can be then $\widehat{\rightarrow}$.

6) Analogously, $\rightarrow_{\mathrm{SetS}(Q),\mathcal{E}}$ can be defined naturally in two different ways:

(i) $\rightarrow_{\mathrm{SetS}(Q),\mathcal{E}}=F(\widetilde{\rightarrow_{\mathrm{Set}(Q),\mathcal{E}}})$,
(ii) $\rightarrow_{\mathrm{SetS}(Q),\mathcal{E}}=F(\widehat{\rightarrow_{\mathrm{Set}(Q),\mathcal{E}}})$.

7) $\neg_{\mathrm{Set}(Q),\mathcal{E}}$ can be defined as a morphism $(Q,\leftrightarrow)\rightarrow(Q,\leftrightarrow)$, such that $\neg_{\mathrm{Set}(Q),\mathcal{E}}(\alpha)=\widehat{\neg}(\alpha)=\bigvee_{\beta\in Q}(\beta\rightarrow 0_Q)\otimes(\alpha\leftrightarrow\beta)$.

Proposition III.1 *$\|\psi\|_{\mathcal{E},\mathbf{K},X}$ is a fuzzy set in $(A,\delta)^X$ in any category $\mathbf{K}=Set(Q),SetS(Q)$.*

Definition III.4 (Interpretation in model $\mathcal{D}_{\mathrm{Set}(Q)}$) *An interpretation $\|\psi\|=\|\psi\|_{\mathcal{D},Set(Q),X}$ of ψ with free variables in a set X in the category $Set(Q)$ is an f-cut $(|\psi|_\alpha)_\alpha$ in $(A,\delta)^X$ defined as follows (instead of $\|t\|_{\mathcal{D},Set(Q)}$ we write $\|t\|$, for any term t).*

1) *Let $\psi\equiv P(t_1,\dots,t_n)$. Then*

$$|\psi|_\alpha=\{\mathbf{a}\in A^X:(\|t_1\|(\mathbf{a}),\dots,\|t_n\|(\mathbf{a}))\in P_\alpha\}.$$

2) *Let $\psi\equiv t_1=t_2$. Then*

$$|\psi|_\alpha=\{\mathbf{a}\in A^X:\Delta_{Set(Q),\mathcal{D}}(\|t_1\|(\mathbf{a}),\|t_2\|(\mathbf{a}))\geq\alpha\},$$

where $\Delta_{Set(Q),\mathcal{D}}$ is a morphism in $Set(Q)$, which interprets equality in a model $\mathcal{D}_{Set(Q)}$.

3) *Let $\psi\equiv\sigma\nabla\tau$, where ∇ represents logical connectives $\wedge,\vee,\implies,\otimes$, respectively. Let $\|\sigma\|=(|\sigma|_\beta)_\beta,\|\tau\|=(|\tau|_\gamma)_\gamma$ be f-cuts in $(A,\delta)^X$. Then*

$$|\psi|_\alpha=\{\mathbf{a}\in A^X:(\bigvee_{\mathbf{a}\in|\sigma|_\beta}\beta)\odot_{Set(Q),\mathcal{D}}(\bigvee_{\mathbf{a}\in|\tau|_\gamma}\gamma)\geq\alpha\},$$

where $\odot_{Set(Q),\mathcal{D}}$ is a morphism $(Q,\leftrightarrow)^2\rightarrow(Q,\leftrightarrow)$ in $Set(Q)$, which interprets logical operations $\wedge,\vee,\rightarrow,\otimes$, respectively, in a model $\mathcal{D}_{Set(Q)}$.

4) *Let $\psi\equiv\neg\sigma$. Let $\|\sigma\|=(|\sigma|_\beta)_\beta$ be an f-cut. Then*

$$|\psi|_\alpha=\{\mathbf{a}\in A^X:\neg_{Set(Q),\mathcal{D}}(\bigvee_{\mathbf{a}\in|\sigma|_\beta}\beta)\geq\alpha\},$$

where $\neg_{Set(Q),\mathcal{D}}$ is a morphism in $Set(Q)$, which interprets logical negation in a model $\mathcal{D}_{Set(Q)}$.

5) *Let $\psi\equiv(\forall x)\sigma$. Then $\|\sigma\|_{\mathcal{D},Set(Q),X\cup\{x\}}$ is already defined as an f-cut $(|\sigma|_\beta)_\beta$ in $(A,\delta)^{X\cup\{x\}}$ and we set*

$$|\psi|_\alpha=\{\mathbf{a}\in A^X:\bigwedge_{\{(x,\beta):x\in A,\beta\in Q,(\mathbf{a},x)\in|\sigma|_\beta\}}\beta\geq\alpha\}.$$

Proposition III.2 *$\|\psi\|_{\mathcal{D},Set(Q),X}=(|\psi|_\alpha)_\alpha$ is an f-cut in $(A,\delta)^X$ in the category $Set(Q)$.*

Finally, we will describe an interpretation in the model $\mathcal{D}_{\mathrm{SetS}(Q)}$.

Definition III.5 (Interpretation in model $\mathcal{D}_{\mathrm{SetS}(Q)}$) *An interpretation $\|\psi\|=\|\psi\|_{\mathcal{D},SetS(Q),X}$ of ψ with free variables in a set X in the category $SetS(Q)$ is an f-cut $(|\psi|_\alpha)_\alpha$ in $(A,\delta)^X$ in the category $SetS(Q)$, defined as follows (instead of $\|t\|_{\mathcal{D},SetS(Q)}$ we write $\|t\|$ for any term t).*

1) *Let $\psi\equiv P(t_1,\dots,t_n)$. Then*

$$|\psi|_\alpha=\{(\mathbf{a},\beta)\in A^X\times Q:\bigvee_{\mathbf{x}\in A^n}\bigwedge_{i=1}^n\|t_i\|(\mathbf{a},x_i)\otimes\bigvee_{\{\gamma:(\mathbf{x},\beta)\in P_\gamma\}}\gamma\geq\alpha\}.$$

2) *Let $\psi\equiv t_1=t_2$. Then*

$$|\psi|_\alpha=\{(\mathbf{a},\beta)\in A^X\times Q:\bigvee_{(x,y)\in A^2}(\|t_1\|(\mathbf{a},x)\wedge\|t_2\|(\mathbf{a},y))\otimes\Delta_{SetS(Q),\mathcal{D}}((x,y),\beta)\geq\alpha\},$$

where $\Delta_{SetS(Q),\mathcal{D}}$ is a morphism in $SetS(Q)$, which interprets equality in a model $\mathcal{D}_{SetS(Q)}$.

3) *Let $\psi\equiv\sigma\nabla\tau$, where ∇ represents logical connectives $\wedge,\vee,\implies,\otimes$, respectively. Let $\|\sigma\|=(|\sigma|_\beta)_\beta,\|\tau\|=(|\tau|_\gamma)_\gamma$ be f-cuts in $(A,\delta)^X$ in $SetS(Q)$. Then*

$$|\psi|_\alpha=\{(\mathbf{a},\varphi)\in A^X\times Q:\bigvee_{\rho,\varepsilon}(\bigvee_{\{(\gamma,\omega):(\mathbf{a},\rho)\in|\sigma|_\gamma,(\mathbf{a},\varepsilon)\in|\tau|_\omega\}}\gamma\wedge\omega)\otimes\odot_{SetS(Q),\mathcal{D}}((\rho,\varepsilon),\varphi)\geq\alpha\},$$

where $\odot_{SetS(Q),\mathcal{D}}$ is a morphism $(Q,\leftrightarrow)^2\rightarrow(Q,\leftrightarrow)$ in $SetS(Q)$, which interprets logical operations $\wedge,\vee,\rightarrow,\otimes$, respectively, in a model $\mathcal{D}_{SetS(Q)}$.

4) *Let $\psi\equiv\neg\sigma$. Let $\|\sigma\|=(|\sigma|_\beta)_\beta$ be an f-cut. Then*

$$|\psi|_\alpha=\{(\mathbf{a},\rho)\in A^X\times Q:\bigvee_{\{(\beta,\gamma):(\mathbf{a},\beta)\in|\sigma|_\gamma\}}\gamma\otimes\neg_{SetS(Q),\mathcal{D}}(\beta,\rho)\geq\alpha\},$$

where $\neg_{SetS(Q),\mathcal{D}}$ is a morphism $(Q,\leftrightarrow)\rightarrow(Q,\leftrightarrow)$ in $SetS(Q)$, which interprets logical negation in a model $\mathcal{D}_{SetS(Q)}$.

5) *Let $\psi\equiv(\forall x)\sigma$. Then $\|\sigma\|_{\mathcal{D},SetS(Q),X\cup\{x\}}$ is already defined as an f-cut $(|\sigma|_\beta)_\beta$ in $(A,\delta)^{X\cup\{x\}}\times(Q,\leftrightarrow)$ and we set*

$$|\psi|_\alpha=\{(\mathbf{a},\rho)\in A^X\times Q:\bigwedge_{\{(x,\beta):x\in A,\beta\in Q,((\mathbf{a},x),\rho)\in|\sigma|_\beta\}}\beta\geq\alpha\}.$$

Proposition III.3 *$\|\psi\|_{\mathcal{D},SetS(Q),X} = (|\psi|_\alpha)_\alpha$ is an f-cut in $(A,\delta)^X$ in the category SetS(Q).*

IV. Relations between interpretations

As we know from Section II, there are some relations between categories Set(Q) and SetS(Q) and between fuzzy sets and f-cuts in these categories. Roughly speaking, fuzzy sets and f-cuts in one of these categories represent the same objects. It is then natural to ask a question, if some relations exist also between interpretations of formulas in models $\mathcal{E}_{\text{Set}(Q)}, \mathcal{E}_{\text{SetS}(Q)}, \mathcal{D}_{\text{Set}(Q)}$ and $\mathcal{D}_{\text{SetS}(Q)}$. In that section we show some principal relations between these interpretations in the case, that corresponding models are derived from one generic model. In the following definition we use a notation from Lemma II.1 and Theorem II.1.

Definition IV.1 *We say that models $\mathcal{E}_{SetS(Q)}, \mathcal{D}_{Set(Q)}$ or $\mathcal{D}_{SetS(Q)}$ are associated with a model $\mathcal{E}_{Set(Q)}$, if the following hold:*

1) *Model $\mathcal{E}_{SetS(Q)}$ is associated with $\mathcal{E}_{Set(Q)}$, if*
   a) $P_{\mathcal{E},SetS(Q)} = F(P_{\mathcal{E},Set(Q)})$, *for any* $P \in \mathcal{P}$,
   b) $f_{\mathcal{E},SetS(Q)} = F(f_{\mathcal{E},Set(Q)})$, *for any* $f \in \mathcal{R}$,
   c) $\Delta_{SetS(Q),\mathcal{E}} = F(\Delta_{Set(Q),\mathcal{E}})$,
   d) $\odot_{SetS(Q),\mathcal{E}} = F(\odot_{Set(Q),\mathcal{E}})$,
   e) $\neg_{SetS(Q),\mathcal{E}} = F(\neg_{Set(Q),\mathcal{E}})$.
2) *Model $\mathcal{D}_{Set(Q)}$ is associated with $\mathcal{E}_{Set(Q)}$, if*
   a) $P_{\mathcal{D},Set(Q)} = \Psi_{(A,\delta)^n}(P_{\mathcal{E},Set(Q)})$, *for any* $P \in \mathcal{P}$,
   b) $f_{\mathcal{D},Set(Q)} = f_{\mathcal{E},Set(Q)}$, *for all* $f \in \mathcal{R}$,
   c) $\Delta_{Set(Q),\mathcal{D}} = \Delta_{Set(Q),\mathcal{E}}$,
   d) $\odot_{Set(Q),\mathcal{D}} = \odot_{Set(Q),\mathcal{E}}$,
   e) $\neg_{Set(Q),\mathcal{D}} = \neg_{Set(Q),\mathcal{E}}$.
3) *Model $\mathcal{D}_{SetS(Q)}$ is associated with $\mathcal{E}_{Set(Q)}$, if*
   a) $P_{\mathcal{D},SetS(Q)} = \Psi_{(A,\delta)^n}(F(P_{\mathcal{E},Set(Q)}))$,
   b) $f_{\mathcal{D},SetS(Q)} = F(f_{\mathcal{E},Set(Q)})$,
   c) $\Delta_{SetS(Q),\mathcal{D}} = F(\Delta_{Set(Q),\mathcal{E}})$,
   d) $\odot_{SetS(Q),\mathcal{D}} = F(\odot_{Set(Q),\mathcal{E}})$,
   e) $\neg_{SetS(Q),\mathcal{D}} = F(\neg_{Set(Q),\mathcal{E}})$.

Proposition IV.1 *Let t be a term and let $\mathcal{E}_{SetS(Q)}$, $\mathcal{D}_{Set(Q)}$ and $\mathcal{D}_{SetS(Q)}$ be associated with model $\mathcal{E}_{Set(Q)}$. Then we have*

$$\|t\|_{\mathcal{E},Set(Q)} = \|t\|_{\mathcal{D},Set(Q)},$$
$$\|t\|_{\mathcal{E},SetS(Q)} = \|t\|_{\mathcal{D},SetS(Q)} = F(\|t\|_{\mathcal{E},Set(Q)}).$$

Theorem IV.1 *Let ψ be a formula and let model $\mathcal{E}_{SetS(Q)}$ be associated with model $\mathcal{E}_{Set(Q)}$.*

(i) *Let ψ does not contain quantifier $\forall$. Then*
$$\|\psi\|_{\mathcal{E},SetS(Q)} = F(\|\psi\|_{\mathcal{E},Set(Q)}).$$
(ii) *Let $\psi = (\forall x)\sigma$. Then*
$$\|\psi\|_{\mathcal{E},SetS(Q)} \le F(\|\psi\|_{\mathcal{E},Set(Q)}).$$

Theorem IV.2 *Let a model $\mathcal{D}_{Set(Q)}$ be associated with a model $\mathcal{E}_{Set(Q)}$. Let for a formula ψ with free variables contained in a set X, $\|\psi\|_{\mathcal{D},Set(Q)}$ be an f-cut $(|\psi|_\alpha)_{\alpha\in Q}$ in the category Set(Q). Then*

$$(\forall \alpha \in Q) \quad |\psi|_\alpha = \{\mathbf{a} \in A^X : \|\psi\|_{\mathcal{E},Set(Q)}(\mathbf{a}) \ge \alpha\}.$$

An analogical theorem holds for interpretations in the category SetS(Q).

Theorem IV.3 *Let models $\mathcal{D}_{SetS(Q)}$ and $\mathcal{E}_{SetS(Q)}$ be associated with a model $\mathcal{E}_{Set(Q)}$. Let for a formula ψ with free variables contained in a set X, $\|\psi\|_{\mathcal{D},SetS(Q)}$ be an f-cut $(|\psi|_\alpha)_{\alpha\in Q}$ in the category SetS(Q). Then*

$$|\psi|_\alpha = \{(\mathbf{a},\beta) \in A^X \times Q : \|\psi\|_{\mathcal{E},SetS(Q)}(\mathbf{a},\beta) \ge \alpha\},$$

for all $\alpha \in Q, \mathbf{a} \in A^X$.

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