$$\begin{cases} x^* = \dfrac{b}{vn}\left(1-\dfrac{2\alpha-1}{\alpha n}\right)\left(1+\dfrac{1-\alpha}{\alpha}\dfrac{n-k}{k}\right), \\ y^* = \dfrac{b}{vn}\cdot\dfrac{2\alpha-1}{\alpha}\cdot\left(1-\dfrac{2\alpha-1}{\alpha n}\right). \end{cases} \tag{9}$$

However, is this point stable?

Proposition 2. For any set b, $v>0$ and α ($0\le\alpha\le 1$) Nash equilibrium (9) is unstable for sufficiently large number of firms n if $\left|\dfrac{k}{n}\right|>\varepsilon$ and $\left|\dfrac{k}{n}-\dfrac{3}{4}\right|>\varepsilon$ for any $\varepsilon>0$.

The destabilizing role of number of players n is well known for the evolution of firms' strategies in oligopoly games [8]. However, in this case, according to calculations, point (9) is unstable even at $n\ge 5$.

Proof. We show that in dynamic system (6) at Nash equilibrium point (9) modulus of Jacobian J is greater than 1: |det J|>1. This implies that at least one eigenvalue of the Jacobian is greater than 1 in absolute value, which means instability of the fixed point (9). Here, the Jacobian of the system (6):

$$J=\begin{pmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy}\end{pmatrix}=\begin{pmatrix} \dfrac{\partial x_{t+1}}{\partial x_t} & \dfrac{\partial x_{t+1}}{\partial y_t} \\ \dfrac{\partial y_{t+1}}{\partial x_t} & \dfrac{\partial y_{t+1}}{\partial y_t}\end{pmatrix}.$$

$$J_{xx}=\frac{\frac{b}{v}(k-1)}{2\sqrt{\frac{b}{v}((k-1)x_t+(n-k)y_t+d^2}}-(k-1), \quad J_{xy}=\frac{\frac{b}{v}(n-k)}{2\sqrt{\frac{b}{v}((k-1)x_t+(n-k)y_t+d^2}}-(n-k),$$

$$J_{yx}=\frac{\frac{b}{v}k}{2\sqrt{\frac{b}{v}(kx_t+(n-k)y_t}}-k, \quad J_{yy}=\frac{\frac{b}{v}(n-k-1)}{2\sqrt{\frac{b}{v}((k-1)x_t+(n-k-1)y_t}}-(n-k-1),$$

where $d=\dfrac{1}{2}\dfrac{1-\alpha}{\alpha}\dfrac{b}{vk}$, then $\det J = J_{xx}\cdot J_{yy}-J_{xy}\cdot J_{yx}=$

$$=(1-n)\cdot\left(\frac{\frac{b}{v}}{2\sqrt{\frac{b}{v}((k-1)x_t+(n-k)y_t+d^2}}-1\right)\cdot\left(\frac{\frac{b}{v}}{2\sqrt{\frac{b}{v}((k-1)x_t+(n-k-1)y_t}}-1\right).$$

But for point (9) in the denominator $\dfrac{b}{v}((k-1)x^*+(n-k-1)y^*)=$

$$=\left(\frac{b}{v}\right)^2\left[\frac{1-\alpha}{\alpha}(1-\frac{k}{n})+\frac{2\alpha-1}{\alpha}(1-\frac{k}{n})\right]+o(\frac{1}{n})=\left(\frac{b}{v}\right)^2\cdot(1-\frac{k}{n})+o(\frac{1}{n}),$$

where $o(\frac{1}{n})\to 0$ for $n\to\infty$. Similarly, we obtain for the second denominator:

$$\frac{b}{v}((k-1)x^*+(n-k)y^*)+d^2=\left(\frac{b}{v}\right)^2\cdot(1-\frac{k}{n})+o(\frac{1}{n}).$$

But by the data $\left|\frac{k}{n}-\frac{3}{4}\right|>\varepsilon$ at a certain $\varepsilon>0$, which guarantees that the factors $\dfrac{\frac{b}{v}}{2\sqrt{\frac{b}{v}((k-1)x_t+(n-k)y_t+d^2}}-1$ and $\dfrac{\frac{b}{v}}{2\sqrt{\frac{b}{v}((k-1)x_t+(n-k-1)y_t}}-1$ do not equal zero for all possible n, k, b, v> 0 and α (0≤α≤1), Q.E.D.

2.3 Dynamic Model Equations with Adaptive Expectations

Since all selfish firms are assumed as identical, it is natural to suggest that they have the same planning at moment t, so their production quantities y_{t+1} at moment $t+1$ will be equal too. Given these expectations, each selfish firm is looking for such value y_{t+1} at which it obtains the highest profit, suggesting that production quantity of SR firms will remain unchanged:

$$\pi_Y=\left(\frac{b}{kx_t+(n-k)y_{t+1}}-v\right)\cdot y_{t+1}. \tag{10}$$

Obviously, the maximum point for y_{t+1} is found from the condition $\frac{\partial\pi_Y}{\partial y_{t+1}}=0$, which gives us:

$$\left(kx_t+(n-k)y_{t+1}\right)^2=\frac{b}{v}kx_t. \tag{11}$$

Then $kx_t+(n-k)y_{t+1}=\sqrt{\frac{b}{v}kx_t}$, from here response function of PI firms is:

$$(n-k)y_{t+1}=\sqrt{\frac{b}{v}kx_t}-kx_t. \tag{12}$$

Similarly, firm-reciprocator naturally expects that the quantity of production of all these firms at moment $t+1$ would be the same. Based on this expectation, each firm-reciprocator finds the value of x_{t+1} at which the objective function is maximal, assuming that the output of PI firms does not change:

$$\Pi_X=\alpha\left(\frac{b}{kx_{t+1}+(n-k)y_t}x_{t+1}-vx_{t+1}\right)+(1-\alpha)\frac{b\gamma}{k}\ln\left(\frac{kx_{t+1}+(n-k)y_t}{\varepsilon}\right). \tag{13}$$

Here we can find the maximum point for x_{t+1} from the condition $\frac{\partial \Pi_X}{\partial x_{t+1}} = 0$, hereof:

$$(kx_{t+1} + (n-k)y_t)^2 = \frac{b}{v}(n-k)y_t + \frac{b\gamma}{v}\frac{1-\alpha}{\alpha} \cdot (kx_{t+1} + (n-k) \cdot y_t) . \tag{14}$$

Let $z = kx_{t+1} + (n-k) \cdot y_t$, represent (14) as:

$$\left(z - \frac{1}{2}\frac{b\gamma}{v}\frac{1-\alpha}{\alpha}\right)^2 = \frac{b}{v}(n-k) \cdot y_t + \left(\frac{1}{2}\frac{b\gamma}{v}\frac{1-\alpha}{\alpha}\right)^2 .$$

Hence $z - \frac{1}{2}\frac{b\gamma}{v}\frac{1-\alpha}{\alpha} = \sqrt{\frac{b}{v}(n-k)y_t + (\frac{1}{2}\frac{b\gamma}{v}\frac{1-\alpha}{\alpha})^2}$. Thus, in view of (12), we obtain a system of dynamics equations of the model, taking into account the forecast:

$$\begin{cases} kx_{t+1} = \sqrt{\frac{b}{v}(n-k)y_t + (\frac{1}{2}\frac{1-\alpha}{\alpha}\frac{b\gamma}{v})^2} - (n-k)y_t + \frac{1}{2}\frac{1-\alpha}{\alpha}\frac{b\gamma}{v}, \\ (n-k)y_{t+1} = \sqrt{\frac{b}{v}kx_t} - kx_t. \end{cases} . \tag{15}$$

2.4 Equilibrium Conditions for the Model with Adaptive Expectations

In the Nash equilibrium point $x_{t+1}=x_t=x$, $y_{t+1}=y_t=y$ for all $t = 0,1,\ldots$. Therefore, at this point in view of (11) and (14) we get:

$$(kx + (n-k)y)^2 = \frac{b}{v}kx = \frac{b}{v}(n-k)y + \frac{b\gamma}{v}\frac{1-\alpha}{\alpha} \cdot (kx + (n-k) \cdot y) . \tag{16}$$

From the second equation we get $x - \frac{1-\alpha}{\alpha}\gamma x = \frac{n-k}{k} \cdot (1 + \frac{1-\alpha}{\alpha}\gamma)y$, whence response functions for selfish and reciprocator firms are, respectively:

$$y = \frac{k}{n-k}\frac{\alpha - (1-\alpha)\gamma}{\alpha + (1-\alpha)\gamma} \cdot x \qquad x = \frac{n-k}{k}\frac{\alpha + (1-\alpha)\gamma}{\alpha - (1-\alpha)\gamma} \cdot y \tag{17}$$

To calculate the coordinates of the fixed point, we substitute this expression y in terms of x at first equation (16):

$$\left(kx + (n-k)\frac{k}{n-k}\frac{\alpha - (1-\alpha)\gamma}{\alpha + (1-\alpha)\gamma} \cdot x\right)^2 = \frac{b}{v}kx \qquad (kx)^2 \cdot \left(\frac{\alpha - (1-\alpha)\gamma}{\alpha + (1-\alpha)\gamma} + 1\right) = \frac{b}{v}kx$$

Hence, we obtain:

Proposition 3. There is unique Nash equilibrium point in the dynamic system (15) with adaptive expectations:

$$\begin{cases} x^* = \frac{b}{vk}(\frac{\alpha + (1-\alpha)\gamma}{2\alpha})^2, \\ y^* = \frac{b}{v(n-k)}\frac{\alpha^2 - ((1-\alpha)\gamma)^2}{(2\alpha)^2}. \end{cases} . \tag{18a}$$

As before, without loss of generality, let $\gamma = 1$, otherwise we can override the share of profit as $\hat{\alpha} = \dfrac{\alpha}{\alpha + (1-\alpha)\gamma}$. At $\gamma = 1$ system (18) takes the form:

$$\begin{cases} x^* = \dfrac{b}{vk}\left(\dfrac{1}{2\alpha}\right)^2, \\ y^* = \dfrac{b}{v(n-k)}\dfrac{2\alpha - 1}{(2\alpha)^2} = \dfrac{b}{v(n-k)}\dfrac{1}{2\alpha}\left(1 - \dfrac{1}{2\alpha}\right). \end{cases} \qquad (18b)$$

Proposition 4. The equilibrium point (18) is stable for all possible values of the parameters.

Proof. To prove the stability of dynamic system (15) in Nash equilibrium point (18) it is necessary and sufficient to demonstrate that for Jacobian J of this system in (18) the following conditions named after Shur were satisfied:

$$\begin{cases} 1 + tr\,J + \det J > 0, \\ 1 - tr\,J + \det J > 0, \\ 1 - \det J > 0. \end{cases}$$

Here, the Jacobian of system (15)

$$J = \begin{pmatrix} J_{xx} & J_{xy} \\ J_{yx} & J_{yy} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x_{t+1}}{\partial x_t} & \dfrac{\partial x_{t+1}}{\partial y_t} \\ \dfrac{\partial y_{t+1}}{\partial x_t} & \dfrac{\partial y_{t+1}}{\partial y_t} \end{pmatrix},$$

obviously, $J_{xx} = J_{yy} = 0$ where $trJ = J_{xx} + J_{yy} = 0$. Thus, to test Shur conditions it is sufficient to establish that $\det J < 1$. But at point (18) $y^* = \dfrac{b}{v(n-k)}\dfrac{\alpha^2 - ((1-\alpha)\gamma)^2}{(2\alpha)^2}$ and therefore

$$kJ_{xy} = \frac{\frac{b}{v}(n-k)}{2\sqrt{\frac{b}{v}(n-k)y^* + d^2}} - (n-k) = \frac{n-k}{2\sqrt{\frac{\alpha^2}{4\alpha^2}}} - (n-k) = 0$$

Consequently, $\det J = J_{xx} \cdot J_{yy} - J_{xy} \cdot J_{yx} = 0$, Q.E.D.

The price of product P in the market is given by the inverse market demand function $P = P(Q) = \dfrac{b}{Q}$ ($b > 0$), and the price is not less than a cent, i.e. $P \geq 0.01$.

Therefore, the product quantity of each firm-reciprocator is $x \leq \dfrac{100b}{k}$. Similarly, the product quantity of each selfish firm is $y \leq \dfrac{100b}{n-k}$.

Corollary. The trajectories of the dynamical system (15) converge to a Nash equilibrium (18) for any initial values $x_0 \leq \dfrac{100b}{k}$, $y_0 \leq \dfrac{100b}{n-k}$.

3 Dynamic Model Equations in a General Case

Suppose that in planning under the given market model adaptive expectations are used with probability p, naïve ones - with probability $q = 1 - p$. Then the profit function for a typical (representative) firm-egoist has the form:

$$\pi_Y = \left(\frac{b}{y_{t+1} + kx_t + p(n-k-1)y_{t+1} + q(n-k-1)y_t} - v \right) \cdot y_{t+1}, \tag{19}$$

and the objective function for the representative firm-reciprocator

$$\Pi_X = \alpha \cdot \left(\frac{b}{x_{t+1} + p(k-1)x_{t+1} + q(k-1)x_t + (n-k)y_t} x_{t+1} - vx_{t+1} \right) + ,$$
$$+ (1-\alpha)\frac{b\gamma}{k}\ln\left(\frac{x_{t+1} + p(k-1)x_{t+1} + q(k-1)x_t + (n-k)y_t}{\varepsilon}\right). \tag{20}$$

Obviously, for $p = 0$ ($q = 1$) objective functions π_Y and Π_X are consistent with the results of naive model (1) and (4), for $p = 1$ ($q = 0$), they are consistent with the results of the adaptive model (10) and (13) respectively. Let us assume

$z_{\pi_X} = y_{t+1} + kx_t + (n-k-1)(py_{t+1} + qy_t)$ $\quad z_{\Pi_X} = x_{t+1} + (n-k)y_t + (k-1)(px_{t+1} + qx_t)$

In this notation $\pi_Y = \left(\frac{b}{z_{\pi_X}} - v \right) \cdot y_{t+1}$; $\Pi_X = \alpha \cdot \left(\frac{b}{z_{\Pi_X}} x_{t+1} - vx_{t+1} \right) + (1-\alpha)\frac{b\gamma}{k}\ln z_{\Pi_X}$.

Then the point y_{t+1} of maximum profit function π_X is found from the condition $\frac{\partial \pi_X}{\partial y_{t+1}} = 0$, here

$$z^2_{\pi_X} = \frac{b}{v}(kx_t + q(n-k-1)y_t \tag{21}$$

whence

$$y_{t+1} \cdot (1 + p(n-k-1)) = \sqrt{\frac{b}{v}(kx_t + q(n-k-1)y_t} - k \cdot x_{t+1} - (n-k-1) \cdot q \cdot y_t \tag{22}$$

The maximum point x_{t+1} for the objective function Π_X is found from the first order condition $\frac{\partial \Pi_X}{\partial x_{t+1}} = 0$. Forth without loss of generality we assume here $\gamma = 1$, otherwise as above we redefine the share of profit as $\tilde{\alpha} = \frac{\alpha}{\alpha + (1-\alpha)\gamma}$. Then

$$z^2_{\Pi_X} = \frac{b}{v} \cdot ((n-k) \cdot y_t + (k-1) \cdot qx_t) + \frac{1-\alpha}{\alpha}\frac{b(1 + p(k-1))}{vk} z_{\Pi_X} \tag{23}$$

Thus, in view of (22), we obtain the dynamics model of equations system of in the general case:

$$\begin{cases} (1+p(k-1))x_{t+1} = \sqrt{\dfrac{b}{v}w_x + d^2} - w_x + d, \\ (1+p(n-k-1))y_{t+1} = \sqrt{\dfrac{b}{v}w_y} - w_y, \end{cases} \tag{24}$$

where $\begin{cases} w_x = q(k-1)x_t + (n-k)y_t, \\ w_y = kx_t + q(n-k-1)y_t. \end{cases}$

3.1 Equilibrium Conditions in a General Case

Since Nash equilibrium point is $x_{t+1}=x_t=x$, $y_{t+1}=y_t=y$ for all $t = 0,1,...$, then at this point in view of (21) and (23) we obtain:

$$z_{\pi_Y} = z_{\Pi_X} = (kx+(n-k)y)^2 = \frac{b}{v}(kx+q(n-k-1)y) = \frac{b}{v}((k-1)qx+(n-k)y) + \\ + \frac{1-\alpha}{\alpha}\frac{b(1+p(k-1))}{vk}\cdot(kx+(n-k)\cdot y) \tag{25}$$

From the second equation we get:

$$y\left[(n-k)+\frac{1-\alpha}{\alpha}\frac{1+p(k-1)}{k}(n-k)-q(n-k-1)\right] = x\left[k-(k-1)q-\frac{1-\alpha}{\alpha}\frac{1+p(k-1)}{k}k\right]$$

Thus,

$$y\left[p\frac{n-k}{\alpha}+q(1+\frac{1-\alpha}{\alpha}\cdot\frac{n-k}{k})\right] = x\left[(1+p(k-1))\cdot\frac{2\alpha-1}{\alpha}\right],$$

where response function in this case

$$\frac{x}{y} = G = \frac{p(n-k)+q(\alpha+(1-\alpha)\dfrac{n-k}{k})}{(2\alpha-1)(1+p(k-1))}. \tag{26}$$

To calculate the coordinates of the fixed point we substitute from (26) expression for $\frac{x}{y}$ in the first equation of (25) $y^2(kG+(n-k))^2 = \frac{b}{v}y(kG+q(n-k-1))$. Hence

Proposition 5. There is unique Nash equilibrium point in a general dynamical system (24):

$$y^* = \frac{\dfrac{b}{v}(kG+q(n-k-1))}{(kG+(n-k))^2} \qquad x^* = Gy^* = \frac{\dfrac{b}{v}(k+q(1/G)(n-k-1))}{(k+(1/G)(n-k))^2} \tag{27}$$

where the function $G = G(p,q,n,k,\alpha)$ is given in (26).

Proposition 6. For $p=0$ ($q=1$) the equilibrium point $(x^*;y^*)$ coincides with point (9) of a dynamic system with naive expectations. When $p=1$ ($q=0$) the equilibrium point coincides with point (18b) of the dynamic system with adaptive expectations.

4 C# - Application Model for Numerical Investigation

C# window application *Model* has been created specifically for the numerical investigation of the model of this paper, using a graphical interface of C# system libraries System.Drawing and System.Windows.Forms. Note that all the calculations associated with the model, are localized in the method *calc* of the application *Model* that makes it easy to modify the equations of the model and use the *Model* to study the other two-dimensional dynamical systems. Fig. 1 shows the application window.

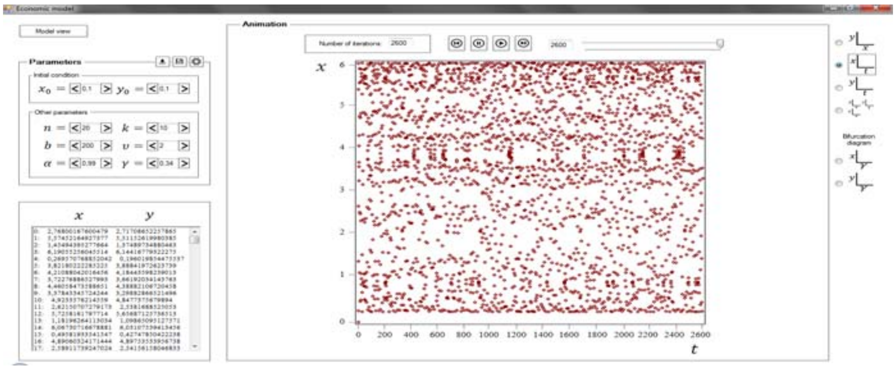

Fig. 1. Application *Model* for two-dimensional model

The right side presents 6 kinds of graphs displayed by the application; their examples are set forth in the paper. Selected switch indicates that here the graph of trajectory $x(t)$ is selected. On the left side counters allow us to specify the parameters of the model and the initial values of the trajectory. After their setting the calculation results of the iterations' coordinates below and their image in the center of the window. This displays an animation of a selected path, the number of iterations been set on the scroll bar above. Pressing the button *Model view* left displays information about the model, its equations and parameter information.

4.1 Numerical Experiment: from Stability to Chaos with Increasing of Naive Expectations

With the increasing probability of naive expectations q, that is with decreasing p, the market becomes unstable, evolving from simple dynamics (15) with a single stable equilibrium point to the unpredictable behavior of system (6). From the proof of Proposition 2 it follows that the market volatility is proportional to the number n of firms in the market. Therefore, for fixed q market instability increases with increasing n. Thus, model (24) has two parameters: the number of firms n and the probability of a naive approach q, whose growth leads to instability. The transition from stability to chaos is the same in both cases. Consider this transition for parameter q.

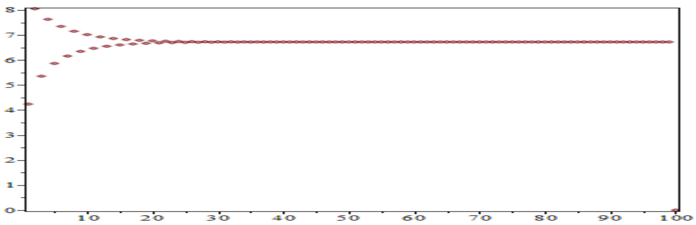

Fig. 2. Quantity trajectory of selfish firm under probability of naive expectations $q = 0.5$

Let n n=20, k=6, b=200, v=2, α=0.9, q=0.5. The trajectory of the dynamical system (24) with the following parameters and the initial point x_0=0.1, y_0=0.1 is shown in the following figures 2 and 3. In Fig. 2 on the x-axis of the system are given iterations of system (24) from m = 1 to m = 100, on the y-axis – corresponding quantity product of selfish firm y_m.

As we can see from the graph, the path quickly converges to the equilibrium value y*≈2.488. The graph for the trajectory of firm-reciprocator x_m on y-axis is similar. The equilibrium value of x* is about 6.72. Let us consider the graph of the trajectory for the same parameters except q. Now q = 0.55 (Fig. 3).

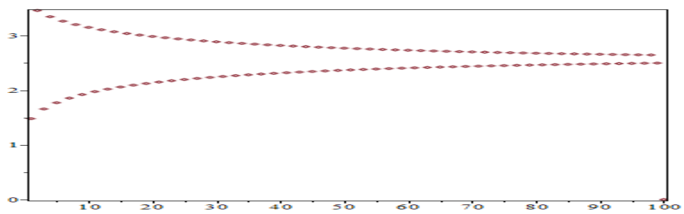

Fig. 3. Quantity trajectory of selfish firm under probability of naive expectations $q = 0.55$

It still has stable Nash equilibrium, but 100 iterations does not suffice for convergence. Further, let q = 0.6 (Fig. 4).

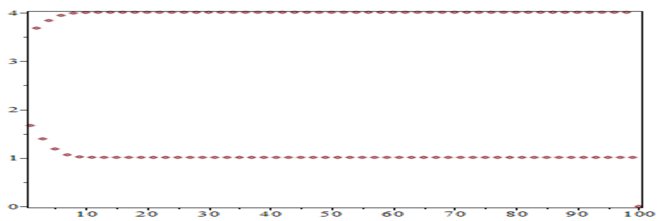

Fig. 4. Bifurcation quantity trajectory of selfish firm under probability of naive expectations $q = 0.6$

As we can see, bifurcation occurred, and instead of equilibrium point there was a steady cycle, where values of y_m are approaching the point of $y^*\approx 4$ for even m and the point of y*≈1 for odd m. By doubling the lag between iterations only even or only odd iterations will be considered, and thus either point y*≈4, or y*≈1 respectively would be the equilibrium steady state.

Stable cycle has four cycles for q=0.64 (fig. 5). There was a new cycle doubling bifurcation. Calculations show that with increasing parameter q doubling bifurcation cycle continues, following Sharkovskii's scale. According to this scale, when q≈0.675 there is the state of dynamic chaos (fig. 6). Similarly, the graph of product x_m on y-axis by firm-reciprocator looks like trajectory of a selfish firm.

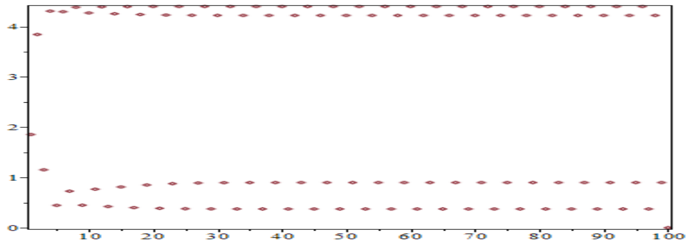

Fig. 5. Doubling bifurcation cycle of quantity by selfish firm under probability of naive expectations $q = 0.64$

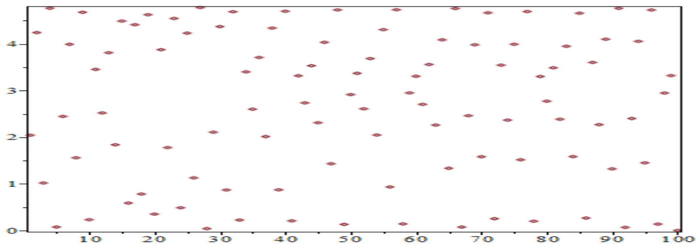

Fig. 6. The state of dynamic chaos of quantity by selfish firms under probability of naive expectations $q \approx 0.675$

Note that the ratio between the quantity of output by selfish firms and reciprocators remains almost unchanged. It is demonstrated in the graph of fig. 9, where each iteration on x-axis shows the value of output by firms-reciprocators x_m, and the vertical axis - the appropriate output of quantity y_m of selfish firms (fig. 7.).

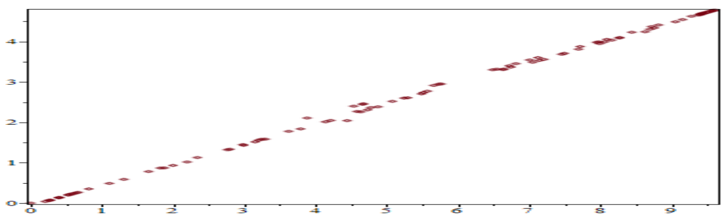

Fig. 7. The ratio between the quantity of product of selfish firms (horizontal axis) and reciprocator ones (vertical axis)

4.2 Bifurcation diagram

In detail the process of loss of stability and transition to chaos of dynamic system (24) can be presented in the following bifurcation diagram (fig. 8).

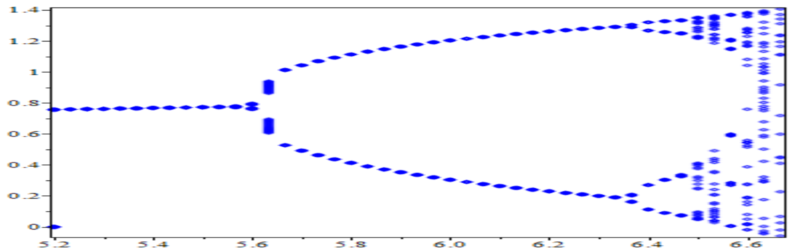

Fig. 8. The bifurcation diagram of dependence quantity product of selfish firm (y) on the probability of naive expectations (q) in a general dynamical system

Here the horizontal axis represents the parameter value of q multiplied by 10. The ordinate values quantity volumes of selfish firm on stable cycle, multiplied by 0.3. This rescaling is done for the sake of clarity. The values of the other parameters are the same as above. The bifurcation diagram, where on vertical axis are placed the values of output of firms-reciprocators x_m looks similar.

As noted in numerical simulations, the bifurcation may be interpreted as separation of equilibrium into several ways, one of which is selected by the market due to evolution of firms' strategies, such as repeated interactions and adaptations. Numerical experiments with n firms as the variable parameter are analogous to those described above.

5 Conclusion

Thus, we have designed the strategic model of cooperation between the two types of firms in the market of homogeneous product, where reciprocator and selfish firms plan their output using the adaptive approach with probability p and naïve (bounded rationality) one with a probability of $q = 1-p$, which distinguishes this model from existing analogues, where each type of firm adheres to one strategy rather than their combination and maximizes only its own profit rather than social welfare.

Desktop $C^{\#}$ application *Model* using a graphical interface to animate the model trajectories has been created specifically for the numerical investigation of the model.

It has been proved that in the model with adaptive expectations the unique Nash equilibrium in a dynamic system is stable for all possible values of the parameters. The trajectories of the dynamical system converge to the fixed point for any possible initial values. In the model with naive expectations the unique Nash equilibrium is unstable for sufficiently large values of n for all possible values of other parameters. According to the calculations, this point is unstable even at $n \geq 5$.

As a result of numerical experiment we have found that bifurcations of cycle doubling occur with an increase in naive expectations. This bifurcation can be interpreted as separation of equilibrium state into several ways, one of which is selected by the market in the evolution of firms' strategies. If two-thirds of firms use naive expectation ($q \approx 0.675$), then in accordance with the Sharkovskii scale there appears the state of dynamic chaos in the market, leading to degeneration of the existing competition model between two types of firms.

Thus, the crucial factor, which ensures sustainable equilibrium in the market and the ability to predict the product quantity of firms, is the adaptive approach, i.e. the one taking into account adaptive expectations of the firms when they plan their production.

Similar results are obtained if instead of q we use parameter n - number of firms in the market, where system also moves from stability to chaos if n increases.

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