About Local Optimum of the Weber Problem on Line with Forbidden Gaps

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Abstract. The location problem of interconnected facilities on the line with forbidden gaps is considered. The located facilities are connected among themselves and with gaps. Location in forbidden gaps is not allowed. It is need to minimize the total cost of connections between facilities and between facilities and gaps. It is known that the initial continuous problem is reduced to series discrete subproblems of smaller dimension. In this paper the definition of the local optimum of the problem is introduced. In order that to obtain the local optimum it is necessary to solve some subproblems. The variants of lower bounds of the goal function of the subproblems are proposed. The bounds can be used in the branch and bounds algorithm for solving the subproblems.

Keywords: forbidden gaps, local optimum, lower bounds, Weber problem

1 Introduction

The analysis and decision of location problems are intensively developing directions in operations research [2, 4, 5, 17, 18]. Problems of such class have important applications and arise in various fields of activity: at designing of master plans of the enterprisers, arrangement of processing equipment in shops, definition of the locations of the service stations, etc. The various statements of such problems are defined by size of facilities, area in which they are located, various restrictions and types of criteria and so on. An important subclass of the location problems of the interconnected facilities is the Weber problem with criterion of minimum of total cost of connections [8, 12, 17, 18]. The Weber problem is an adequate model of many practical applications. For example, in automation of design of the complex systems can be the problem of placement of structural elements in the area with implementation of certain requirements [7, 16]. At designing of the plan of the petrochemical enterprise the facilities are the equipment [7, 16]. The equipment can be connected among themselves by various communications, for example, systems of the pipelines. Regularity of location often is required to ensure the straightforward paths and convenient maintenance along so-called “red” lines [15].

* This work was supported by grant 16–01–00740 from the Russian Foundation for Basic Research.

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In: A. Kononov et al. (eds.): DOOR 2016, Vladivostok, Russia, published at http://ceur-ws.org

The Weber problem with restrictions on facility location and/or traveling has been considered widely in recent years [4, 6]. Depending on the type of restrictions, such problems are divided into three categories. The category considers planar facility location problems in which restrictions come from existence of forbidden regions (gaps). Forbidden gaps refer to prohibition of facility placement but allowance of free traveling. Moreover, some forbidden gaps may contain the equipment, for example, during modernization of the enterprise. In the second category, restrictions are imposed by barriers. Barriers are defined as regions where neither locating a facility on not passing through is allowed. The last category considers restricted planar location problems with congested regions. Congested regions are bounded areas in the plane that forbid facility location however passing through their interior is possible at some extra traveling cost. A broad overview on facility location problems with forbidden gaps is provided in [6].

If the facilities are commensurable to the area of location then they often are approximated by the rectangles; otherwise, we may consider them as material points. Various approaches to the problem of optimal location of the rectangles are developed [1, 8–10, 13, 15]. In case the rectangles are unconnected, the two–dimensional problem of packing of the rectangles into the strip of the minimal length is considered in [9]. The problem is formulated as a mixed integer nonlinear programming problem. For solving this problem the probabilistic tabu search algorithm is developed. In [15] the location problem of the rectangles on the parallel lines is considered. For constructing a set of Pareto–optimal solutions, integer optimization and dynamic programming are applied. In [8, 10], the algorithms of local optimization on the plane and dynamic programming on the line without forbidden gaps are described. For the problem in which the rectangles can be rotated and it is possible to create routes for establishing connections, the heuristic algorithm is proposed [1]. An important subclass of problems of optimal location of rectangles is the VLSI chip design problems. The wide review of practical applications and various approaches for solving such problems is provided in [13].

In this paper, we study the Weber problem on the line with forbidden gaps. The located facilities are segments, for example, projection of the rectangles onto the line. It is known that the initial continuous problem is reduced to series discrete subproblems of smaller dimension. All subproblems have identical structure. The local optimum of the problem is obtained by solving some subproblems. The variants of lower bounds of the goal function of the subproblems are proposed. The bounds can be used in algorithms for solving of the subproblems, for example, in the branch and bounds algorithm.

2 Problem Formulation and Basic Properties

The Weber problem on the line with forbidden gaps is formulated as follows. Consider a straight–line segment of length LS containing some fixed rectilinear areas (forbidden gaps). It is necessary to locate on this line facilities which are the segments, whose centers are connected with each other and with the forbidden gaps. Location in forbidden gaps is not allowed. The problem is to locate facilities on the segment of length LS outside the forbidden gaps so that they do not intersect each other and, moreover, the total cost of connections between the facilities and between facilities and gaps is minimized [17]. Without loss of generality, we can assume that the left border of the segment of length LS is the origin.

Let X_i and F_j denote the facilities and gaps with the centers at x_i and b_j and lengths of l_i and p_j, where $i \in I = \{1, \dots, n\}$ and $j \in J = \{1, \dots, m\}$; while $w_{ij} \geq 0$, $u_{ik} \geq 0$ are the specific costs of connections between X_i and F_j, X_i and X_k for i, $k \in I$, $j \in J$, and $i < k$. The target is to locate the facilities $X_1, \dots, X_n$ on the segment outside gaps $F_1, \dots, F_m$ and so that they do not intersect with each other and the total cost of the connections between the facilities and between facilities and gaps is minimized.

The mathematical model under consideration is as follows:

$$G(x) = \sum_{i=1}^{n} \sum_{j=1}^{m} w_{ij}|x_i - b_j| + \sum_{i=1}^{n-1} \sum_{k=i+1}^{n} u_{ik}|x_i - x_k| \to \min, \tag{1}$$

$$|x_i - b_j| \geq \frac{l_i + p_j}{2}, \quad i \in I, j \in J, \tag{2}$$

$$|x_i - x_k| \geq \frac{l_i + l_k}{2}, \quad i, k \in I, i < k, \tag{3}$$

$$\frac{l_i}{2} \leq x_i \leq LS - \frac{l_i}{2}, \quad i \in I. \tag{4}$$

The first component in (1) defines the total cost of connections between facilities and gaps; the second part defines the total cost of connections between the facilities themselves; and (2) and (3) are the conditions of disjointness.

The feasible area B is disconnected and consists of the set of r disjoint segments (blocks) B_k of length L_k that contain the facilities X_i, $i \in I$, $B = \bigcup_{k=\overline{1,r}} B_k$. The problem (1)–(4) is NP-hard; a feasible solution to the problem can be found by construction of a one–dimensional packing into containers [3]. In this case, the facilities with lengths of l_i, $i \in I$, are packed into containers with sizes L_k, $k = \overline{1,r}$. Moreover, if there are no gaps then (1)–(4) is an optimal linear ordering problem [11] which is NP-hard for an arbitrary set of connections between the facilities.

In [17] the algorithm for obtaining an approximate decision in two phases is offered. In the first phase, we find an feasible partition of the facilities into blocks, and in the second phase, the facilities in the blocks are interchanged for the purpose of minimization of total cost of connections. A computational experiment with this algorithm and solving of the problem using IBM ILOG CPLEX and a mixed-integer linear programming model is considered. This work is continuation of researches of the problem (1)–(4).

Given a feasible location, a *remainder* in the block B_k is a segment between two adjacent facilities that do not have a common border or between the border of B_k and an adjacent block. A block without facilities is called a *block with remainder*. Two elements (facilities, gaps, remainders) are called *glued* if they have a common border.

Let $J_L(B_k)$ and $J_R(B_k)$ denote the set of gaps to the left and to the right of the block B_k, let $I_L(A)$ and $I_R(A)$ denote the set of facilities to the left and to the right of the block (gap) A correspondingly. Given an facility X_i in B_k, let Lw_i and Rw_i denote the total cost of connections for X_i defined as

$$Lw_i = \sum_{j \in J_L(B_k)} w_{ij} + \sum_{k \in I_L(B_k)} u_{ik}, \quad Rw_i = \sum_{j \in J_R(B_k)} w_{ij} + \sum_{k \in I_R(B_k)} u_{ik}.$$

Let $x = (x_1, \dots, x_n)$ be a feasible solution of (1)–(4) that uniquely defines the partition of $X_1, \dots, X_n$ into blocks. Let $I_k(x)$ denote the set of the facility's numbers in B_k, $I = \bigcup_{k=\overline{1,r}} I_k(x)$, and let $H_k(x)$ denote the set of remainders in B_k for x. Let n_k denote the size of set $I_k(x)$, then $|H_k(x)| \le n_k + 1$. Note that x can be represented as $x = (x^1, \dots, x^r)$, where x^k is the coordinates of the centers of the facilities that are located in B_k and have numbers from $I_k(x)$.

Proposition 1. *(see [17]) Given a feasible solution x of (1)–(4), we can find a feasible solution x' such that*

$$|H_k(x')| \le 1, \quad k = 1, \dots, r, \quad G(x') \le G(x).$$

Thus in each block B_k it is possible to consider no more than one remainder, length of which we denote by Δ_k.

Let LB_k and RB_k denote the coordinates of the left and right borders of B_k. Then, for a fixed partition of facilities into blocks, the goal function $G(x)$ can be presented as

$$G(x) = \sum_{k=1}^{r} G_k(x^k) + \overline{C}, \tag{5}$$

where

$$G_k(x^k) = \sum_{s \in I_k(x)} \sum_{t \in I_k(x), t > s} u_{st} |x_s - x_t| + \sum_{s \in I_k(x)} |x_s - LB_k| \Bigg(\sum_{j \in J_L(B_k)} w_{sj} +$$

$$+ \sum_{i \in I_L(B_k)} u_{si} \Bigg) + \sum_{t \in I_k(x)} |x_t - RB_k| \Bigg(\sum_{j \in J_R(B_k)} w_{tj} + \sum_{i \in I_R(B_k)} u_{ti} \Bigg),$$

$\overline{C}$ is some constant.

The first component of $G_k(x^k)$ is the sum of costs of connections between the facilities in B_k, the second is the total cost of connections between the facilities from B_k and LB_k, and the third component is the total cost of connections between the facilities from B_k and RB_k. In the case when the area of location on the segment is bounded on the left and on the right by gaps, we have

$$\overline{C} = \frac{1}{2} p_1 \sum_{k \in I_R(F_1)} w_{k1} + \frac{1}{2} \sum_{j=2}^{m-1} p_j \Bigg(\sum_{k \in I_L(F_j)} w_{kj} + \sum_{k \in I_R(F_j)} w_{kj} +$$

$$+2 \sum_{k \in I_L(F_j)} \sum_{t \in J, t > j} w_{kt} + 2 \sum_{k \in I_R(F_j)} \sum_{s \in J, s < j} w_{ks} + 2 \sum_{k \in I_L(F_j)} \sum_{t \in I_R(F_j)} u_{kt} \Bigg) +$$

$$+\frac{1}{2}p_m \sum_{k\in I_L(F_m)} w_{km}.$$

Introduce the following definition.

Definition 1. *An admissible decision x of the problem (1)–(4) is called a local minimum if $G(x) \leq G(x')$ for all $x' : I_k(x) = I_k(x')$, $k = 1,\ldots,r$.*

Let the partition of the facilities into blocks is fixed. Then in each block B_k it is possible to consider the subproblem of location $n_k + 2$ facilities. In B_k the subproblem contains two fixed facilities and n_k located facilities. Denote by F_L and F_R the fixed facilities. The facilities are located in the points with coordinates LB_k and RB_k respectively. Denote by w_{iL} and w_{iR} the cost of connections between located facilities in B_k and fixed facilities LB_k and RB_k respectively for each $i \in I_k(x)$ and then

$$w_{iL} = \Bigg(\sum_{s\in J_L(B_k)} w_{is} + \sum_{t\in I_L(B_k)} u_{it}\Bigg),$$

$$w_{iR} = \Bigg(\sum_{s\in J_R(B_k)} w_{is} + \sum_{t\in I_R(B_k)} u_{it}\Bigg).$$

With the preceding designations the subproblem for B_k can be presented as

$$G_k(x^k) = \sum_{s\in I_k(x)} \sum_{t\in I_k(x),t>s} u_{st}|x_s - x_t| + \sum_{s\in I_k(x)} w_{sL}|x_s - LB_k|$$

$$+ \sum_{t\in I_k(x)} w_{tR}|x_t - RB_k| \to \min, \tag{6}$$

$$|x_i - x_k| \geq \frac{l_i + l_k}{2}, \quad i,k \in I_k(x), i < k, \tag{7}$$

$$LB_k + \frac{l_i}{2} \leq x_i \leq RB_k - \frac{l_i}{2}, \quad i \in I_k(x). \tag{8}$$

It is need to find coordinates x^k of the centers facilities in B_k, so that the total cost of connections between located facilities among themselves and with F_L and F_R is minimized.

As $I_k(x) \bigcap I_l(x) = \emptyset$ for all $k, l = 1,\ldots,r$, then to find a local optimum for some fixed partition of the facilities into blocks, it is sufficiently to find the minimum in r independent subproblems (6)–(8). These subproblems have identical structure.

Proposition 2. *For search of the local minimum of the problem (1)–(4) it is sufficiently to solve r subproblems (6)–(8).*

The proof of proposition 2 follows from representation of the goal function (5).

Note that the process of search of the local minimum can be parallelized as solving r independent subproblems (6)–(8).

The properties stated above allowed to reduce the initial continuous problem to series of discrete subproblems of smaller dimension. For finding of the local optimum of the problem it is need to solve r subproblems.

Solving of the problem (1)–(4) is sequential performance of two phases. In the first phase, we find feasible partition of the facilities into blocks by means of the algorithm [17]. In the second phase, for obtained partition the series of the subproblems of smaller dimension are solved. Thus we fined the local optimum of the problem (1)–(4). The local optimum with minimal value of the goal function is the exact decision of the problem (1)–(4). For the search of approximate decision of the problem (1)–(4) the stopping criteria of the algorithm may be running time, number of iterations, finding of the exact decision or given estimate of accuracy. Note that the number of possible partitions $X_i,\ i \in I$ into the blocks $B_1, \ldots, B_r$ is at most r^n. The remainder in B_k can be considered as the additional located facility X_{n_k+1}, for which $l_{n_k+1} = \Delta_k$ and $Lw_{n_k+1} = Rw_{n_k+1} = 0$.

Note that if $u_{st} = 0,\ \forall s, t \in I_k(x), s < t$, then in the second phase the exact decision for the subproblem (6)–(8) is found by the polynomial algorithm [17]. In this case the graph, whose tops are the left and the right borders of the block B_k and the facilities in B_k, has a series–parallel type with one top on each chain between the source and the sink. For such structure of the graph the effective algorithm is offered in [14].

Generally, if $\exists s, t \in I_k(x):\ u_{st} > 0$, then for solving the subproblem (6)–(8) for small value n_k, it is possible to use $n_k!$ permutations of the facilities into block. In general case it is possible to apply, for example, the branch and bounds algorithm. In the algorithm calculation of the lower bounds of the goal function is important.

3 Lower Bounds

Let some located facilities in B_k are glued to the left border of B_k, some located facilities are glued to the right border of B_k, and some facilities aren't located yet. Denote by NF_l, NF_r the sets of the located facility's numbers in B_k, then $I_k(x)\backslash\{NF_l \bigcup NF_r\}$ is the set of the unlocated facility's numbers. Without loss of generality, we shall consider that the facilities in the set NF_l have numbers from 1 to s, and in the set NF_r the facility's numbers are from $t+1$ to n_k. Denote by D the set of the admissible locations of the facilities in B_k when NF_l and NF_r are defined (partial location). Let $\xi(D)$ be the lower bound of function $G_k(x^k)$ for the set D. Then $\xi(D)$ can be presented as the sum of three components. The first component $\xi_1(D)$ is the total cost of connections between the located facilities themselves and with F_L and F_R. The second component $\xi_2(D)$ is lower bound of total cost of connections the unlocated facilities with F_L, F_R and with the located facilities in B_k. The third component $\xi_3(D)$ is lower bound of total cost of connections between the unlocated facilities themselves. Thus $\xi(D)$ can be presented as

$$\xi(D) = \xi_1(D) + \xi_2(D) + \xi_3(D).$$

The coordinates of the centers of the located facilities are known and $\xi_1(D)$ can be calculated as follows

$$\xi_1(D) = \sum_{s\in\{NF_l \bigcup NF_r\}} \ \sum_{t\in\{NF_l \bigcup NF_r\}, t>s} u_{st}|x_s - x_t| + \sum_{s\in\{NF_l \bigcup NF_r\}} |x_s - LB_k| \cdot$$

$$\cdot \left(\sum_{j\in J_L(B_k)} w_{sj} + \sum_{i\in I_L(B_k)} u_{si} \right) + \sum_{t\in\{NF_l \bigcup NF_r\}} |x_t - RB_k| \left(\sum_{j\in J_R(B_k)} w_{tj} + \sum_{i\in I_R(B_k)} u_{ti} \right).$$

Further two variants of calculation $\xi_2(D)$ are offered.

The first variant. For each $i \in I_k(x)\backslash\{NF_l \bigcup NF_r\}$ the total cost of connections between the located facilities themselves and with F_L and F_R is determined as follows

$$SL(i) = Lw_i + \sum_{k \in NF_l} u_{ik}, \quad SR(i) = Rw_i + \sum_{k \in NF_r} u_{ik}.$$

Further location of the unlocated facilities in B_k is defined in two ways. In the first way, the facilities are ordered by not increase of the relations $SL(i)/l_i$. The facilities consistently are pasted together in that order with the most left located facility in B_k. Let, for simplicity of designations, the glued unlocated facilities have numbers from $s+1$ to t.

In the second way, the unlocated facilities are ordered by not increase of the relations $SR(i)/l_i$. The facilities consistently are pasted together in that order with the most right located facility in B_k. Let the glued unlocated facilities have numbers from t to $s+1$. Then

$$\xi_2(D) = \xi_{2L}(D) + \xi_{2R}(D),$$

where $\xi_{2L}(D)$ and $\xi_{2R}(D)$ are the lower bounds of total cost of the connections unlocated facilities with F_L, F_R respectively and with the located facilities in B_k.

In both cases we receive some permutation π of facilities in the block B_k. Consider the permutation π and two distinct facilities X_i and X_j in B_k. The distance between X_i and X_j with respect to this permutation, assumed to be taken their centers, is equal to the half-length of X_i, plus the lengths l_k of all facilities which are between X_i and X_j in π, plus the half-length of X_j. The half-length of X_i plus the half-length of X_j is the constant. Thus, at the calculation of the distance between the facilities X_i and X_j it is sufficient to consider only the lengths l_k of all facilities which are between X_i and X_j in π. Then $\xi_{2L}(D)$ and $\xi_{2R}(D)$ can be calculated as follows:

$$\xi_{2L}(D) = \sum_{q=s+1}^{t} \left(Lw_q \sum_{g=1}^{q-1} l_g + \sum_{i=1}^{s} u_{qi} \sum_{k=i+1}^{q-1} l_k \right),$$

$$\xi_{2R}(D) = \sum_{q=s+1}^{t} \left(Rw_q \sum_{g=q+1}^{n_k} l_g + \sum_{i=t+1}^{n_k} u_{qi} \sum_{k=q+1}^{i-1} l_k \right).$$

The proof that $\xi_{2L}(D)$ and $\xi_{2R}(D)$ are the lower bounds of the total cost of connections unlocated facilities with F_L, F_R and with the located facilities in B_k is similar to the proof in [10].

The second variant. The set $I_k(x)\backslash\{NF_l \bigcup NF_r\}$ can be presented as union of non–crossing sets $N_L \bigcup N_C \bigcup N_R$, where by N_L, N_C, N_R are denoted the sets of facility's numbers for which $SL(i) > SR(i)$, $SL(i) = SR(i)$, $SL(i) < SR(i)$ respectively.

Further facilities with numbers from N_L are ordered by not increase of the relations $(SL(i) - SR(i))/l_i$. The facilities consistently are pasted together in that order with the most left located facility in B_k. The facilities with numbers from N_R are ordered by not increase of the relations $(SR(i) - SL(i))/l_i$, and they consistently are pasted together in that order with the most right located facility. The facilities with numbers from N_C are located between sets of the facilities with numbers from N_L and from N_R in any order.

Thus, for each $i \in I_k(x) \backslash \{NF_l \bigcup NF_r\}$ coordinate of the center is determined. Let $I_k(x) \backslash \{NF_l \bigcup NF_r\} = \{s+1, \dots, t\}$ as well as earlier. Determine the following value Z as

$$Z = \sum_{q=s+1}^{t} \left(Lw_q \sum_{g=1}^{q-1} l_g + \sum_{i=1}^{s} u_{qi} \sum_{k=i+1}^{q-1} l_k + Rw_q \sum_{h=q+1}^{n_k} l_h + \sum_{j=t+1}^{n_k} u_{qj} \sum_{v=q+1}^{j-1} l_v \right). \quad (9)$$

Proposition 3. *The value Z is the lower bound of the total cost of connections the unlocated facilities with F_L, F_R and with the located facilities in B_k.*

Proof. Let $N_L \neq \emptyset$ and $N_R = \emptyset$ and $N_C = \emptyset$. Without loss of generality, we consider the case when $NF_l = \emptyset$ and $NF_r = \emptyset$. Otherwise it is possible to redefine the cost of connections the unlocated facilities with F_L, F_R and with the located facilities. Then the set of the unlocated facility's numbers is $I_k(x)$ and $Lw_i > Rw_i$ for each $i \in I_k(x)$.

Let $(Lw_1 - Rw_1)/l_1 > \dots > (Lw_{n_k} - Rw_{n_k})/l_{n_k}$, then the facilities are located in order $X_1, \dots, X_{n_k}$ according to the second variant of the calculation $\xi_2(D)$. Denote by $\Pi^* = (1, \dots, n_k)$ the permutation of numbers facilities corresponding to order $X_1, \dots, X_{n_k}$ and denote by $Z(\Pi^*)$ the value Z for Π^*. On the formula (9) we receive

$$Z(\Pi^*) = Lw_2 l_1 + \dots + Lw_t(l_1 + \dots + l_{t-1}) + Lw_{t+1}(l_1 + \dots + l_t) +$$
$$+ \dots + Lw_{n_k}(l_1 + \dots + l_{n_k-1}) + Rw_1(l_2 + \dots + l_{n_k}) + \dots +$$
$$+ Rw_t(l_{t+1} + \dots + l_{n_k}) + Rw_{t+1}(l_{t+2} + \dots + l_{n_k}) + \dots + Rw_{n_k-1} l_{n_k}.$$

To prove that $Z(\Pi^*)$ is the lower bound for $Z(\Pi)$, where Π is any permutation of numbers facilities from $I_k(x)$. Assume, that $Z(\Pi^*)$ is not minimum. Then there is other permutation $\overline{\Pi}$, such that $Z(\Pi^*) > Z(\overline{\Pi})$. Any permutation can be received by series of transposition of adjacent numbers, then it is sufficiently to consider the proof for change the position of two adjacent facilities. Denote by $\overline{\Pi} = (1, \dots t-1, t+1, t, t+2, \dots, n_k)$, then on the formula (9) we receive

$$Z(\overline{\Pi}) = Lw_2 l_1 + \dots + Lw_{t+1}(l_1 + \dots + l_{t-1}) + Lw_t(l_1 + \dots + l_{t-1} + l_{t+1}) +$$
$$+ \dots + Lw_{n_k}(l_1 + \dots + l_{n_k-1}) + Rw_1(l_2 + \dots + l_{n_k}) + \dots +$$
$$+ Rw_{t+1}(l_t + l_{t+2} + \dots + l_{n_k}) + Rw_t(l_{t+2} + \dots + l_{n_k}) + \dots + Rw_{n_k-1} l_{n_k}.$$

By assumption $Z(\Pi^*) - Z(\overline{\Pi}) > 0$, then

$$Z(\Pi^*) - Z(\overline{\Pi}) = Lw_2 l_1 + \dots + Lw_t(l_1 + \dots + l_{t-1}) + Lw_{t+1}(l_1 + \dots + l_t) +$$
$$+ \dots + Lw_{n_k}(l_1 + \dots + l_{n_k-1}) + Rw_1(l_2 + \dots + l_{n_k}) + \dots +$$
$$+ Rw_t(l_{t+1} + \dots + l_{n_k}) + Rw_{t+1}(l_{t+2} + \dots + l_{n_k}) + \dots + Rw_{n_k-1} l_{n_k} -$$
$$- Lw_2 l_1 - \dots - Lw_{t+1}(l_1 + \dots + l_{t-1}) - Lw_t(l_1 + \dots + l_{t-1} + l_{t+1}) - \dots -$$
$$- Lw_{n_k}(l_1 + \dots + l_{n_k-1}) - Rw_1(l_2 + \dots + l_{n_k}) - \dots -$$

$$-Rw_{t+1}(l_t + l_{t+2} + \ldots + l_{n_k}) - Rw_t(l_{t+2} + \ldots + l_{n_k}) - \ldots - Rw_{n_k-1}l_{n_k} =$$
$$= (Lw_{t+1} - Rw_{t+1})l_t - (Lw_t - Rw_t)l_{t+1} > 0.$$

Hence
$$\frac{Lw_{t+1} - Rw_{t+1}}{l_{t+1}} > \frac{Lw_t - Rw_t}{l_t}.$$

It is the contradiction with inequalities $(Lw_1 - Rw_1)/l_1 > \ldots > (Lw_t - Rw_t)/l_t > (Lw_{t+1} - Rw_{t+1})/l_{t+1} > \ldots > (Lw_{n_k} - Rw_{n_k})/l_{n_k}$.

The proof for the subset $N_R \neq \emptyset$ is similarly. Note that facilities with numbers from N_C can be located in arbitrary order.

Proposition is proved.

Therefore, in the capacity of $\xi_2(D)$ it is possible to use the value Z.

Following way may be use for the calculation of $\xi_3(D)$. Let for the facilities X_s, X_t, X_q, $s, t, q \in I_k(x) \backslash \{NF_l \bigcup NF_r\}$ take place inequalities $u_{st} > 0$, $u_{sq} > 0$, $u_{tq} > 0$. Considering any three of the interconnected facilities, value of $\xi_3(D)$ can be calculated, for example, as follows

$$\xi_3(D) = \sum_{s,t,q \in I_k(x) \backslash \{NF_l \bigcup NF_r\}} \min\{u_{st}l_q;\ u_{sq}l_t;\ u_{tq}l_s\}.$$

4 Conclusion

The NP–hard location problem of interconnected facilities on the line with forbidden gaps is considered. It is need to minimize the total cost of connections between the facilities and between the facilities and gaps. It is known that the initial continuous problem is reduced to series discrete subproblems of smaller dimension. For finding of the local optimum of the problem it is need to solve some subproblems. Two variants of lower bounds of the goal function of the subproblems are proposed. The bounds can be used in the branch and bounds algorithm for solving the subproblems.

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