Explicit Univariate Global Optimization with Piecewise Linear Support Functions

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Abstract. Piecewise linear convex and concave support functions combined with Pijavskii's method are proposed to be used for solving global optimization problems. Rules for constructing support functions are introduced.

Keywords: global minimum, support functions, piecewise concave and convex functions.

1 Introduction

We study the effectiveness of global optimization technique with nonlinear support functions. It is assumed that objective and constrained functions are explicit functions, and we take this term to mean the following. Given a set of elementary functions, an explicit function can be constructed from the elementary ones by means of the operations of adding, subtraction, multiplication, and division of two functions, multiplication by a constant, as well as the operations of composition and taking maximum or minimum of two functions. It is obvious that the problem of finding the global minimum of an explicit function may be quite complicated because, generally, an explicit function is not convex.

Possibilities of explicit functions optimization have already been studied. In [1] explicit functions are called factorable. Following step by step the process of construction of an explicit function, we can also track the global minimum or at least the range of a current function. The question is what is the best way to do this.

One of the most commonly used approach to global optimization is based on the properties of the Lipschitz constant of the function to be minimized. The Lipschitz constant is a uniform measure of function's rate of change over all the feasible set. This is both its advantage and drawback. The Lipschitz constant for any function over any segment can be found as the maximum of the absolute value of its derivative. After that, it is easy to define step by step an estimate of the true Lipschitz constant for any

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particular explicit function [2].

Methods of Lipschitz optimization ([3]–[9]) are reasonably easily and quickly coded. This is their indubitable advantage. The drawback is that every operation worsens the estimate of the true Lipschitz constant, immediately affecting the efficiency. In addition, the Lipschitz constant equally estimates the rate of change of the objective all over the segment, so, this constant is defined by segments of the fastest variation, which generally may contain no points of global minimum. To the contrary, the function may change more smoothly near points of global minimum. One of the ways out is to estimate Lipschitz constant according to the search region [10]. Also, some methods estimate Lipschitz constant on the basis of previously calculated values of the objective [11]. In this case, overestimation of Lipschitz constant and loss of the true global optimum may be a problem.

Global search is much more efficient if the objective is known to be differentiable and its derivative is known to satisfy the Lipschitz condition ([12]–[17]). Effective algorithms for global minimization of one-dimensional functions over a segment are presented in [18], [19], [14]. Obviously, for this class of functions the operations of taking maximum and minimum of two functions have to be excluded, because they do not retain the differentiability of the objective.

Another successfully developing field of global optimization is so called d.c. programming [20]–[22]. A d.c. function is a function representable as a difference of two convex continuous functions. A d.c. program is a problem of minimization of a d.c. function subject to equality and inequality constraints defined also by d.c. functions. Not considering in detail a wide variety of methods for solving d.c. problems, we only notice that finding a good representation (if it exists) of a given function in the form of a difference of two convex function may appear to be a quite complicated problem.

For global minimization of a univariate explicit function, we present a method of piecewise linear support functions [23], [24] which is a generalization of well-known Piyavskii's method [25]. It should be also noted the the proposed method belongs to the class of characteristical algorithms [26] and can be easily parallelized.

The optimization technique suggested in the paper differs from interval analysis approach used in global optimization [27] and is closely related to automatic differentiation [28], however not so much developed. In general, the approach of the paper can be considered as an initial extension of automatic differentiation to global optimization.

2 The problem and Piecewise linear support functions

We consider the global optimization problem

$$f(x) \to \min,\ x \in [\alpha, \beta], \tag{1.1}$$

where $\alpha, \beta \in R, \alpha \leq \beta$ and function $f(x)$ satisfies the following definition.

Definition 1.1. *A function $f(x)$ is said to have a convex support function-majorant and concave support function-minorant if there exist functions $\varphi^+(x, y)$ and $\varphi^-(x, y)$ such that*

1. $\varphi^+(x, y)$ is convex and continuous with respect to x at any fixed $y \in [\alpha, \beta]$;

2. $\varphi^-(x, y)$ is concave and continuous with respect to x at any fixed $y \in [\alpha, \beta]$;

3. $\varphi^-(x,y) \le f(x) \le \varphi^+(x,y)$ *for any* $x, y \in [\alpha,\beta]$;
4. $\varphi^-(y,y) = f(y) = \varphi^+(y,y)$ *for any* $y \in [\alpha,\beta]$.
The functions $\varphi^+(x,y)$ *and* $\varphi^-(x,y)$ *are called support function-majorant and support function-minorant, correspondingly .*

Any function satisfying Definition 1.1 is lower semicontinuous [24]. It is easy to see that any locally Lipschitz function satisfies this definition. In this paper we propose to construct $\varphi^+(x,y)$ and $\varphi^-(x,y)$ as piecewise linear functions.

Definition 2.1. *A convex piecewise linear support function-majorant of a function* $f(x)$ *and a concave piecewise linear support function-minorant of function* $f(x)$ *over a segment* $[\alpha,\beta]$ *at a support point* y *will denote such functions* $\varphi^+(x,y)$ *and* $\varphi^-(x,y)$ *that satisfy Definition 1.1 and have the form*

$$\varphi^+(x,y) = \max\left\{k_1^+(y)x + b_1^+(y), \quad k_2^+(y)x + b_2^+(y)\right\}, \tag{2.1}$$

$$\varphi^-(x,y) = \min\left\{k_1^-(y)x + b_1^-(y), \quad k_2^-(y)x + b_2^-(y)\right\} \tag{2.2}$$

where $k_1^+(y)$, $b_1^+(y)$, $k_2^+(y)$, $b_2^+(y)$, $k_1^-(y)$, $b_1^-(y)$, $k_2^-(y)$, *and* $b_2^-(y)$ *are certain numbers.*

Constructing the piecewise linear support functions for linear and convex functions is obvious, so we consider here one of the most trouble in global optimization function $f(x) = \sin(x)$ and analyze the following four special cases.

I. *Point* $y \in [-\frac{\pi}{2}, 0]$. The function $\sin(x)$ is convex over the segment $[-\frac{\pi}{2}, y]$, hence $\sin(x) \ge \sin(y)+\cos(y)(x-y), x \in [-\frac{\pi}{2}, y]$ and, consequently, $k_1^- = \cos(y), b_1^- = \sin(y) - k_1^- y$. Over the segment $[0, \frac{3}{2}\pi]$, the graph of the function $\sin(x)$ lies above the tangent line through the origin. The value $v_1 = 4.493409458$ is found from the transcendental equation $\sin(v) - \cos(v)v = 0$. Next, from the linear equation $\sin(v_1)+\cos(v_1)(z-v_1) = -1$ with respect to z the solution $z_1 = 4.603338848$ is obtained. Then, over the segment $[y, \frac{3}{2}\pi]$, the graph of the function $\sin(x)$ lies above the straight line through the points $(y, \sin(y))$ and $(z_1, -1)$: $\sin(x) \ge k_2^- x + b_2^-, x \in [y, \frac{3}{2}\pi], k_2^- = \frac{\sin(y)+1}{y-z_1}, b_2^- = \sin(y) - k_2^- y$.

II. *Point* $y \in [0, \frac{\pi}{2}]$. Over the segment $[-\frac{\pi}{2}, y]$, the graph of the function $\sin(x)$ lies above the straight line through the points $(-1, 1)$ and $(y, \sin(y))$: $\sin(x) \ge k_1^- x + b_1^-, x \in [-\frac{\pi}{2}, y]$, $k_1^- = \frac{\sin(y)+1}{y+1}, b_1^- = \sin(y) - k_1^- y$. Over the segment $[\frac{\pi}{2}, \frac{3}{2}\pi]$, the graph of the function $\sin(x)$ pases above the tangent line through the point $(\frac{\pi}{2}, 1)$. The value $v_2 = 3.901918697$ is a solution of the transcendent equation $\sin(v) + \cos(v)(\frac{\pi}{2} - v) = 1$ belonging to the segment $[\frac{\pi}{2}, \frac{3}{2}\pi]$. Find the value $z_2 = 4.330896607$ as a solution of the linear equation $\sin(v_2) + \cos(v_2)(z - v_2) = -1$ with respect to z. It is easy to see that, over the segment $[y, \frac{3}{2}\pi]$, the graph of the function $\sin(x)$ is located above the straight line through the points $(y, \sin(y))$ and $(-1, 1)$: $\sin(x) \ge k_2^- x + b_2^-, x \in [y, \frac{3}{2}\pi]$, $k_2^- = \frac{\sin(y)+1}{y-z_2}, b_2^- = \sin(y) - k_2^- y$.

III. *Point* $y \in [\frac{\pi}{2}, \pi]$. Over the segment $[-\frac{\pi}{2}, \frac{\pi}{2}]$, the graph of the function $\sin(x)$ lies above the tangent line through the point $(\frac{\pi}{2}, 1)$. The value $v_3 = -0.7603260437$ is a solution of the transcendent equation $\sin(v) + \cos(v)(\frac{\pi}{2} - v) = 1$ belonging to the segment $[-\frac{\pi}{2}, \frac{\pi}{2}]$. The value $z_3 = -1.189303953$ is obtained as a solution of the linear equation $\sin(v_3) + \cos(v_3)(z - v_3) = -1$ with respect to z. It is easy to see that, over the segment $[\frac{\pi}{2}, y]$, the graph of the function $\sin(x)$ is located above the straight line through $(z_3, -1)$ and $(y, \sin(y))$: $\sin(x) \ge k_1^- x + b_1^-, x \in [-\frac{\pi}{2}, y]$, $k_1^- = \frac{\sin(y)+1}{y-z_3}, b_1^- =$

$\sin(y) - k_1^- y$. Next, over the segment $[y, \frac{3}{2}\pi]$, the graph of the function $\sin(x)$ lies over the straight line through the points $(y, \sin(y))$ and $(\pi + 1, -1)$: $\sin(x) \geq k_2^- x + b_2^-, x \in [y, \frac{3}{2}\pi]$, $k_2^- = \frac{\sin(y)+1}{y-\pi-1}, b_2^- = \sin(y) - k_2^- y$.

IV. A support point $y \in [\pi, \frac{3}{2}\pi]$. Over the segment $[-\frac{\pi}{2}, \pi]$, the graph of the function lies above the tangent line through the point $(\pi, 0)$. The value $v_4 = -1.351816804$ is a solution of the transcendental equation $\sin(v) + \cos(v)(\pi - v) = 0$ belonging to the segment $[-\frac{\pi}{2}, 0]$. Again, the value $z_4 = -1.461746193$ is obtained as a solution of the linear equation $\sin(v_4) + \cos(v_4)(z - v_4) = -1$. with respect to z. Over the segment $[-\frac{\pi}{2}, y]$, the graph of the function $\sin(x)$ is located above the straight line through the points $(z_4, -1)$ and $(y, \sin(y))$: $\sin(x) \geq k_1^- x + b_1^-, x \in [-\frac{\pi}{2}, y]$, $k_1^- = \frac{\sin(y)+1}{y-z_4}, b_1^- = \sin(y) - k_1^- y$. Over the segment $[y, \frac{3}{2}\pi]$, the function $\sin(x)$ is convex therefore $k_2^- = \cos(y), b_2^- = \sin(y) - k_2^- y$.

To summarize, the rule for constructing a concave piecewise linear minorant of the function $\sin(x)$ over the segment $[-\frac{\pi}{2}, \frac{3}{2}\pi]$ is the following.

Input: the function $f(x) = \sin(x)$ and a point y.

Step 1.

If $-\frac{\pi}{2} \leq y \leq 0$, then $k_1^- = \cos(y), k_2^- = \frac{\sin(y)+1}{y-z_1}$.

If $0 \leq y \leq \frac{\pi}{2}$, then $k_1^- = \frac{\sin(y)+1}{y+1}, k_2^- = \frac{\sin(y)+1}{y-z_2}$.

If $\frac{\pi}{2} \leq y \leq \pi$, then $k_1^- = \frac{\sin(y)+1}{y-z3}, k_2^- = \frac{\sin(y)+1}{y-\pi-1}$.

If $\pi \leq y \leq \frac{3\pi}{2}$, then $k_1^- = \frac{\sin(y)+1}{y-z4}, k_2^- = \cos(y)$.

Step 2. $b_1^- = \sin(y) - k_1^- y, b_2^- = \sin(y) - k_2^- y$.

Step 3. Stop.

In general, rules for constructing support functions are based on similar geometrical ideas and can be found, for example, in [29].

3 General rules for constructing piecewise linear support functions

Let function $f(x)$ and its support function-majorant $\varphi_f^+(x, y)$ and support function-minorant $\varphi_f^-(x, y)$ be given:

$$\varphi_f^+(x, y) = \max\left\{k_{f1}^+(y)x + b_{f1}^+(y), \quad k_{f2}^+(y)x + b_{f2}^+(y)\right\}, \tag{3.1}$$

$$\varphi_f^-(x, y) = \min\left\{k_{f1}^-(y)x + b_{f1}^-(y), \quad k_{f2}^-(y)x + b_{f2}^-(y)\right\}. \tag{3.2}$$

Denote

$$l_{f1}^+(x, y) = k_{f1}^+(y)x + b_{f1}^+(y),\ l_{f2}^+(x, y) = k_{f2}^+(y)x + b_{f2}^+(y),$$
$$l_{f1}^-(x, y) = k_{f1}^-(y)x + b_{f1}^-(y),\ l_{f2}^-(x, y) = k_{f2}^-(y)x + b_{f2}^-(y).$$

Then

$$\varphi_f^+(x, y) = \max\{l_{f1}^+(x, y), \quad l_{f2}^+(x, y)\},\ \varphi_f^-(x, y) = \min\{l_{f1}^-(x, y), \quad l_{f2}^-(x, y)\}.$$

Assume that for another function $h(x)$ its support functions $\varphi_h^+(x,y)$ and $\varphi_h^-(x,y)$ are also known. The goal of this section is to present rules for constructing support functions-minorant and functions-majorant for the functions

$$F_1(x) = cf(x), F_2(x) = f(x) + h(x), F_3(x) = f(x) - h(x), F_4(x) = f^2(x),$$

$$F_5(x) = f(x) \cdot h(x), F_6(x) = \frac{1}{f(x)}, \quad F_7(x) = \max\{f(x), h(x)\},$$

$$F_8(x) = \min\{f(x), h(x)\}, F_9(x) = f(h(x))$$

where c is a constant.

3.1. The function $F_1(x) = cf(x)$. It is obvious that

$$\varphi_{F_1}^+(x,y) = \begin{cases} \max\{c \cdot l_1^+(x,y), \quad c \cdot l_2^+(x,y)\}, & c \geq 0 \\ \max\{c \cdot l_1^-(x,y), \quad c \cdot l_2^-(x,y)\}, & c < 0, \end{cases}$$

$$\varphi_{F_1}^-(x,y) = \begin{cases} \min\{c \cdot l_1^-(x,y), \quad c \cdot l_2^-(x,y)\}, & c \geq 0 \\ \min\{c \cdot l_1^+(x,y), \quad c \cdot l_2^+(x,y)\}, & c < 0. \end{cases}$$

3.2. The function $F_2(x) = f(x) + h(x)$. In this case

$$\varphi_{F_2}^+(x,y) = \max\{l_{f1}^+(x,y) + l_{h1}^+(x,y), \quad l_{f2}^+(x,y) + l_{h2}^+(x,y)\},$$

$$\varphi_{F_2}^-(x,y) = \min\{l_{f1}^-(x,y) + l_{h1}^-(x,y), \quad l_{f2}^-(x,y) + l_{h2}^-(x,y)\}.$$

3.3. The function $F_3(x) = f(x) - h(x)$. Obviously follows from 3.1 and 3.2.

3.4. The function $F_4(x) = f^2(x)$. From $(f(x) - f(y))^2 \geq 0$ we have $f^2(x) \geq 2f(x)f(y) - f^2(y)$. For obtaining $\varphi_{F_4}^-(x,y)$ the rule from 3.1 is used with $c = 2f(y)$. Find constants $\underline{f}$ and $\overline{f}$ such that $\underline{f} \leq f(x) \leq \overline{f} \ \ \forall x \in X$. Calculate $\hat{z} = f(y)$. Suppose that $\underline{f} < \hat{z} < \overline{f}$. Then

$$z^2 \leq (\underline{f} + \hat{z})z - \underline{f}\hat{z}, \quad z \in [\underline{f}, \hat{z}], \; z^2 \leq (\overline{f} + \hat{z})z - \overline{f}\hat{z}, \quad z \in [\hat{z}, \overline{f}].$$

Consequently, $z^2 \leq \max\left\{(\underline{f} + \hat{z})z - \underline{f}\hat{z}, \quad (\overline{f} + \hat{z})z - \underline{f}\hat{z}\right\}$, $z \in [\underline{f}, \overline{f}]$. After the substitution $z = f(x)$, $\hat{z} = f(y)$, we obtain

$$f^2(x) \leq \max\{(\underline{f} + f(y))f(x) - \underline{f}f(y), \quad (\overline{f} + f(y))f(x) - \overline{f}f(y)\}.$$

Then use support convex majorants for the functions $(\underline{f} + f(y))f(x) - \underline{f}f(y)$ and $(\overline{f} + f(y))f(x) - \overline{f}f(y)$.

3.5. The function $F_5(x) = f(x) \cdot h(x)$. Since $f(x) \cdot h(x) = \frac{(f(x)+h(x))^2}{2} - \frac{f^2(x)}{2} - \frac{h^2(x)}{2}$, support functions for $F_5(x)$ can be constructed by subsequent application of the rules from 3.1-3.4.

3.6. The function $F_6(x) = \frac{1}{f(x)}$. It is assumed that $f(x) > 0 \; \forall x \in [\alpha, \beta]$. The function $F_6(x)$ is a composition of the functions $\varphi(z) = \frac{1}{z}$ and $f(x)$. Let $0 < \underline{f} < f(x) < \overline{f} \quad \forall x \in [\alpha, \beta]$ and denote $\hat{z} = f(y)$. The function $\varphi(z)$ is convex over the segment $[\underline{f}, \overline{f}]$. Therefore $F(x) = \varphi(f(x)) \geq \frac{1}{f(y)} - \frac{f(x)}{f^2(y)}$. A concave minorant of the function

$-\frac{f(x)}{f^2(y)}$ is constructed by the rule presented in 3.1 with $c = -\frac{1}{f^2(y)}$. A convex majorant is constructed in the same way as in 3.4, with the auxiliary function z^2 substituted by the function $\frac{1}{z}$.

It is easy to see now that construction of support majorants and minorants of a concave function $f(x)$ consists in construction of support majorants and minorants of the convex function $-f(x)$, multiplying them by -1 and applying the rule from 3.1. In this way support majorants and minorants for the elementary functions $f_6(x) = \sqrt{x}$ and $f_7(x) = \ln(x)$ are constructed.

3.7. The function $F_7(x) = \max\{f(x), h(x)\}$**.** Without loss of generality, let us assume that $\max\{f(y), h(y)\} = f(y)$. Then $F_7(x) \geq f(x) \geq \varphi_f^-(x, y)$, what means that $\varphi_f^-(x, y)$ is a concave minorant of the function $F_7(x)$. On the other hand, $F_7(x) \leq \max\{\varphi_f^+(x, y), \varphi_h^+(x, y)\}$. Since the functions $\varphi_f^+(x, y)$ and $\varphi_h^+(x, y)$ are convex in x, the function $\psi(x) = \max\{\varphi_f^+(x, y), \varphi_h^+(x, y)\}$ is also convex in x. Constructing a support majorant of the function $\psi(x)$ we simultaneously obtain a support piecewise linear majorant for the function $F_7(x)$.

3.8. The function $F_8(x) = \min\{f(x), h(x)\}$**.**
Since $\min\{f(x), h(x)\} = -\max\{-f(x), -h(x)\}$, support functions can be found by application of the rules from 3.1 and 3.7.

3.9. The function $F_9(x) = f(h(x))$**.** From (3.1) and (3.2) we have

$$F_7(x) \leq \varphi_f^+(h(x), h(y)) = \max\left\{k_{f1}^+(h(y))h(x) + b_{f1}^+(h(y)), k_{f2}^+(h(y))h(x) + b_{f2}^+(h(y))\right\}, \tag{3.7}$$

$$F_7(x) \geq \varphi_f^-(h(x), h(y)) = \min\left\{k_{f1}^-(h(y))h(x) + b_{f1}^-(h(y)), k_{f2}^-(h(y))h(x) + b_{f2}^-(h(y))\right\}. \tag{3.8}$$

The right-hand expressions of inequalities (3.7) and (3.8) are piece-wise linear with respect to the function $h(x)$. Application of the rules presented in 3.7, 3.8, 3.1, and 3.2 suffices to construct support majorants and minorants.

The above-introduced rules for constructing support piecewise linear functions are applicable to any explicit function. The process of construction of a support functions is analogous in some way to the process of differentiation of a function: both can be automatized. Hence, the quite scrupulous procedure of finding support functions can be entrusted to a computer. The analytical representations of required functions are not always necessary, it is sufficient to have only algorithms of finding their values at any feasible point. That is the reason that a user needs nothing to do but explicitly set an objective, everything else can be automatized according to the above-described rules.

4 A Modification of Piyavskii's Method

The method of solving problem (1.1) proposed in this paper is a modification of the well-known Piyavskii's method [25]. Given points $x^1, ..., x^k$, determine next point x^{k+1} as a global solution of the problem

$$\psi_k(x) = \max_{1 \leq i \leq k} \{\varphi^-(x, x^i)\} \to \min, \; x \in [\alpha, \beta]. \tag{4.1}$$

After that, calculate the record $f_{k+1} = \min\{f(x^i) : i = 1, ..., k+1\}$, and if $f_{k+1} - \psi(x^{k+1}) \leq \varepsilon$ where $\varepsilon > 0$ is a required precision then the point $x^l, l = 1, ..., k+1$ such that $f(x^l) = f_k$ is the ε-optimal solution. A starting point $x^1 \in [\alpha, \beta]$ is arbitrary. If the functions $\psi_k(x)$ are uniformly bounded every limit point of sequence x_k is a global solution of problem (1.1) ([25]).

At the k-th iteration of the method the points $x^1, ..., x^k$ divide the segment $[\alpha, \beta]$ into p subsegments $[\alpha^j, \beta^j], j = 1, ..., p$ such that $\alpha^1 = \alpha, \beta^p = \beta, \beta^j = \alpha^{j+1}$. Next point x^{k+1} belongs to one of these subsegments, hence it is sufficient to build support functions only over the subsegment containing x^{k+1}. Such a strengthening of Piyavskii's method has been already used in Lipschitz optimization [10]. Another strengthening lies in the fact that a support function depends on the support point providing more accurate piecewise linear approximation of the objective over a segment, in contrast to [10] where approximation accuracy is determined by the Lipschitz constant (though over a smaller segment). The rules for constructing support functions presented in the previous section are aimed to improvement of accuracy of piecewise linear approximation through shortening the current subsegment.

Computational experiment performed in [29] shows acceleration in convergence from 3 to 10 times in comparison to results presented in [4].

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