Reducts in Multi-Adjoint Concept Lattices

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Abstract. Removing redundant information in databases is a key issue in Formal Concept Analysis. This paper introduces several results on the attributes that generate the meet-irreducible elements of a multi-adjoint concept lattice, in order to provide different properties of the reducts in this framework. Moreover, the reducts of particular multi-adjoint concept lattices have been computed in different examples.
Keywords: attribute reduction, reduct, multi-adjoint concept lattice

1 Introduction

Attribute reduction is an important research topic in Formal Concept Analysis (FCA) [1, 4, 10, 15]. Reducts are the minimal subsets of attributes needed in order to compute a lattice isomorphic to the original one, that is, that preserve the whole information of the original database. Hence, the computation of these sets is very interesting. For example, they are useful in order to obtain attribute implications and, since the complexity to build concept lattices directly depend on the number of attributes and objects, if a reduct can be detected before computing the whole concept lattice, the complexity will significantly be decreased.

Different fuzzy extensions of FCA have been introduced [2, 3, 9, 14]. One of the most general is the multi-adjoint concept lattice framework [11, 12]. Based on a characterization of the meet-irreducible elements of a multi-adjoint concept lattice, a suitable attribute reduction method has recently been presented in [6]. In this paper the notions of absolutely necessary, relatively necessary and absolutely unnecessary attribute, as in Rough Set Theory (RST) [13], have been considered in order to classify the set of attributes. This classification provides a procedure to know whether an attribute should be considered or not. Consequently, it can be used to extract reducts. In addition, when the attribute classification verifies that the set of relatively necessary attributes is not empty several reducts can be obtained.

Due to the relation between the given attribute classification and the meet-irreducible elements of a concept lattice, this paper studies the attributes that generate the meet-irreducible elements of a multi-adjoint concept lattice. From the introduced results, different properties of the corresponding reducts have been presented. In addition, two examples in which the reducts of particular multi-adjoint concept lattices have been included.

2 Preliminaries

A brief summary with the basic notions and results related to attribute classification in the fuzzy framework of multi-adjoint concept lattices is presented.

2.1 Multi-adjoint concept lattices

First of all, we will recall the definitions of multi-adjoint frame and context where the operators to carry out the calculus are adjoint triples [7, 8].

Definition 1. *A* multi-adjoint frame *is a tuple* $(L_1, L_2, P, \&_1, \dots, \&_n)$ *where* $(L_1, \preceq_1)$ *and* $(L_2, \preceq_2)$ *are complete lattices,* $(P, \leq)$ *is a poset and* $(\&_i, \swarrow^i, \nwarrow_i)$ *is an adjoint triple with respect to* L_1, L_2, P*, for all* $i \in \{1, \dots, n\}$*.*

Definition 2. *Let* $(L_1, L_2, P, \&_1, \dots, \&_n)$ *be a multi-adjoint frame, a* context *is a tuple* (A, B, R, σ) *such that* A *and* B *are nonempty sets (usually interpreted as attributes and objects, respectively),* R *is a* P*-fuzzy relation* $R\colon A \times B \to P$ *and* $\sigma\colon A \times B \to \{1, \dots, n\}$ *is a mapping which associates any element in* $A \times B$ *with some particular adjoint triple in the frame.*

In order to introduce the *multi-adjoint concept lattice* associated with this frame and this context, two concept-forming operators ${}^\uparrow\colon L_2^B \to L_1^A$ and ${}^\downarrow\colon L_1^A \to L_2^B$ are considered. These operators are defined as

$$g^\uparrow(a) = \inf\{R(a,b) \swarrow^{\sigma(a,b)} g(b) \mid b \in B\} \tag{1}$$

$$f^\downarrow(b) = \inf\{R(a,b) \nwarrow_{\sigma(a,b)} f(a) \mid a \in A\} \tag{2}$$

for all $g \in L_2^B$, $f \in L_1^A$ and $a \in A$, $b \in B$, where L_2^B and L_1^A denote the set of mappings $g\colon B \to L_2$ and $f\colon A \to L_1$, respectively, which form a Galois connection [12].

By using the concept-forming operators, a *multi-adjoint concept* is defined as a pair $\langle g, f\rangle$ with $g \in L_2^B$, $f \in L_1^A$ satisfying $g^\uparrow = f$ and $f^\downarrow = g$. The fuzzy subsets of objects g (resp. fuzzy subsets of attributes f) are called *extensions* (resp. *intensions*) of the concepts.

Definition 3. *The* multi-adjoint concept lattice *associated with a multi-adjoint frame* $(L_1, L_2, P, \&_1, \dots, \&_n)$ *and a context* (A, B, R, σ) *given, is the set*

$$\mathcal{M} = \{\langle g, f\rangle \mid g \in L_2^B, f \in L_1^A \text{ and } g^\uparrow = f, f^\downarrow = g\}$$

where the ordering is defined by $\langle g_1, f_1\rangle \preceq \langle g_2, f_2\rangle$ *if and only if* $g_1 \preceq_2 g_2$ *(equivalently* $f_2 \preceq_1 f_1$*).*

A classification of the attributes of a multi-adjoint context from a characterization of the $\wedge$-irreducible elements of the corresponding concept lattice $(\mathcal{M}, \preceq)$ was given in [5, 6]. Before introducing this classification, the characterization theorem must be recalled. First and foremost, it is necessary to define the following specific family of fuzzy subsets of attributes.

Definition 4. *For each $a \in A$, the fuzzy subsets of attributes $\phi_{a,x} \in L_1^A$ defined, for all $x \in L_1$, as*

$$\phi_{a,x}(a') = \begin{cases} x & \text{if } a' = a \\ \bot_1 & \text{if } a' \neq a \end{cases}$$

will be called fuzzy-attributes*, where $\bot_1$ is the minimum element in L_1. The set of all fuzzy-attributes will be denoted as $\Phi = \{\phi_{a,x} \mid a \in A, x \in L_1\}$.*

Theorem 1 ([5]). *The set of $\wedge$-irreducible elements of $\mathcal{M}$, $M_F(A)$, is formed by the pairs $\langle \phi_{a,x}^{\downarrow}, \phi_{a,x}^{\downarrow\uparrow} \rangle$ in $\mathcal{M}$, with $a \in A$ and $x \in L_1$, such that*

$$\phi_{a,x}^{\downarrow} \neq \bigwedge \{\phi_{a_i,x_i}^{\downarrow} \mid \phi_{a_i,x_i} \in \Phi, \phi_{a,x}^{\downarrow} \prec_2 \phi_{a_i,x_i}^{\downarrow}\}$$

and $\phi_{a,x}^{\downarrow} \neq g_{\top_2}$, where $\top_2$ is the maximum element in L_2 and $g_{\top_2} \colon B \to L_2$ is the fuzzy subset defined as $g_{\top_2}(b) = \top_2$, for all $b \in B$.

2.2 Attribute classification

The main results, related to the attribute classification in a multi-adjoint concept lattice framework, were established by meet-irreducible elements of the concept lattice and the notions of consistent set and reduct [6]. For that reason, we will recall the following definitions.

Definition 5. *A set of attributes $Y \subseteq A$ is a* consistent *set of (A, B, R, σ) if the following isomorphism holds:*

$$\mathcal{M}(Y, B, R_Y, \sigma_{Y \times B}) \cong_E \mathcal{M}(A, B, R, \sigma)$$

This is equivalent to say that, for all $\langle g, f \rangle \in \mathcal{M}(A, B, R, \sigma)$, there exists a concept $\langle g', f' \rangle \in \mathcal{M}(Y, B, R_Y, \sigma_{Y \times B})$ such that $g = g'$.

Moreover, if $\mathcal{M}(Y \setminus \{a\}, B, R_{Y \setminus \{a\}}, \sigma_{Y \setminus \{a\} \times B}) \not\cong_E \mathcal{M}(A, B, R, \sigma)$, for all $a \in Y$, then Y is called a reduct *of (A, B, R, σ).*

The core *of (A, B, R, σ) is the intersection of all the reducts of (A, B, R, σ).*

A classification of the attributes can be given from the reducts of a context.

Definition 6. *Given a formal context (A, B, R, σ) and the set $\mathcal{Y} = \{Y \subseteq A \mid Y \text{ is a reduct}\}$ of all reducts of (A, B, R, σ). The set of attributes A can be divided into the following three parts:*

1. *Absolutely necessary attributes (core attribute) $C_f = \bigcap_{Y \in \mathcal{Y}} Y$.*
2. *Relatively necessary attributes $K_f = (\bigcup_{Y \in \mathcal{Y}} Y) \setminus (\bigcap_{Y \in \mathcal{Y}} Y)$.*

3. Absolutely unnecessary attributes $I_f = A \setminus (\bigcup_{Y \in \mathcal{Y}} Y)$.

The attribute classification theorems introduced in [6] are based on the previous notions and are recalled below.

Theorem 2 ([6]). *Given* $a_i \in A$*, we have that* $a_i \in C_f$ *if and only if there exists* $x_i \in L_1$*, such that* $\langle \phi^{\downarrow}_{a_i,x_i}, \phi^{\downarrow\uparrow}_{a_i,x_i} \rangle \in M_F(A)$*, satisfying that* $\langle \phi^{\downarrow}_{a_i,x_i}, \phi^{\downarrow\uparrow}_{a_i,x_i} \rangle \neq \langle \phi^{\downarrow}_{a_j,x_j}, \phi^{\downarrow\uparrow}_{a_j,x_j} \rangle$*, for all* $x_j \in L_1$ *and* $a_j \in A$*, with* $a_j \neq a_i$*.*

Theorem 3 ([6]). *Given* $a_i \in A$*, we have that* $a_i \in K_f$ *if and only if* $a_i \notin C_f$ *and there exists* $\langle \phi^{\downarrow}_{a_i,x_i}, \phi^{\downarrow\uparrow}_{a_i,x_i} \rangle \in M_F(A)$ *satisfying that* E_{a_i,x_i} *is not empty and* $A \setminus E_{a_i,x_i}$ *is a consistent set, where the sets* $E_{a_i,x}$ *with* $a_i \in A$ *and* $x \in L_1$ *are defined as:*

$$E_{a_i,x} = \{a_j \in A \setminus \{a_i\} \mid \textit{there exist } x' \in L_1, \ \textit{satisfying } \phi^{\downarrow}_{a_i,x} = \phi^{\downarrow}_{a_j,x'}\}$$

Theorem 4 ([6]). *Given* $a_i \in A$*, it is absolutely unnecessary,* $a_i \in I_f$*, if and only if, for each* $x_i \in L_1$*, we have that* $\langle \phi^{\downarrow}_{a_i,x_i}, \phi^{\downarrow\uparrow}_{a_i,x_i} \rangle \notin M_F(A)$*, or in the case that* $\langle \phi^{\downarrow}_{a_i,x_i}, \phi^{\downarrow\uparrow}_{a_i,x_i} \rangle \in M_F(A)$*, then* $A \setminus E_{a_i,x_i}$ *is not a consistent set.*

The classification of the set of attributes in absolutely necessary, relatively necessary and absolutely unnecessary attributes, provided by the previous theorems, will allow us to obtain reducts (minimal sets of attributes) in the following section. Determining the reducts can entail an important reduction of the computational complexity of the concept lattice.

3 Computing the reducts of a multi-adjoint concept lattice

This section is focused on analyzing the construction process of reducts from the attribute classification shown in the previous section. To begin with, the attributes in the core, that is, the absolutely necessary attributes, are included in all reducts and the unnecessary attributes must be removed.

The choice of the relatively necessary attributes is the main task in the process, because several reducts are obtained when the set of relatively necessary attributes is nonempty.

Hence, several issues raise, such as, how should we select the set of relatively necessary attributes? What is the most efficient way to perform this process? Do all the reducts have the same cardinality? How can we get a reduct with a minimal number of attributes? This work establishes the first steps in order to answer these questions.

Regarding a simplification in the selection of the relatively necessary attributes, a subset of attributes associated with each concept will be considered.

Definition 7. *Given a multi-adjoint frame* $(L_1, L_2, P, \&_1, \ldots, \&_n)$ *and a context* (A, B, R, σ) *with the associated concept lattice* $(\mathcal{M}, \preceq)$*. Let* C *be a concept of* $(\mathcal{M}, \preceq)$*, we define* the set of attributes generating C *as the set:*

$$\mathit{Atg}(C) = \{a_i \in A \mid \textit{there exists} \quad \phi_{a_i,x} \in \Phi \quad \textit{such that} \quad \langle \phi^{\downarrow}_{a_i,x}, \phi^{\downarrow\uparrow}_{a_i,x} \rangle = C\}$$

Now, we will present several properties about the attributes of the context which will be useful to build reducts in our context, together with some example which illustrate them.

Proposition 1. *If C is a meet-irreducible concept of $(\mathcal{M}, \leq)$, then $Atg(C)$ is a nonempty set.*

The following example was introduced in [6], in which an attribute classification was given. Now, we will use it in order to clarify the previous result.

Example 1. Let $(L, \preceq, \&_G)$ be a multi-adjoint frame, where $\&_G$ is the Gödel conjunctor with respect to $L = \{0, 0.5, 1\}$. In this framework, the context is (A, B, R, σ), where $A = \{a_1, a_2, a_3, a_4, a_5\}$, $B = \{b_1, b_2, b_3\}$, $R\colon A \times B \to L$ is given by the table in Figure 1, and σ is constant.

R	b_1	b_2	b_3
a_1	1	1	0
a_2	0.5	1	0
a_3	0.5	1	0
a_4	1	1	0.5
a_5	1	1	1

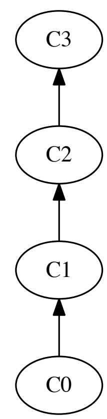

Fig. 1. Relation R and Hasse diagram of Example 1.

The concept lattice of the considered framework and context are displayed in Figure 1, from which it is easy to see that the meet-irreducible elements are C_0, C_1 and C_2. Now, we will show that the sets $\mathrm{Atg}(C_0)$, $\mathrm{Atg}(C_1)$ and $\mathrm{Atg}(C_2)$ are not empty. For that, the fuzzy-attributes associated with the meet-irreducible concepts need to be obtained. Applying the concept-forming operators to the fuzzy-attributes we have

$$\begin{aligned}
&\langle \phi^{\downarrow}_{a_4,1.0}, \phi^{\downarrow\uparrow}_{a_4,1.0} \rangle = C_2\\
\langle \phi^{\downarrow}_{a_1,0.5}, \phi^{\downarrow\uparrow}_{a_1,0.5} \rangle = \langle \phi^{\downarrow}_{a_1,1.0}, \phi^{\downarrow\uparrow}_{a_1,1.0} \rangle = \langle \phi^{\downarrow}_{a_2,0.5}, \phi^{\downarrow\uparrow}_{a_2,0.5} \rangle = &\langle \phi^{\downarrow}_{a_3,0.5}, \phi^{\downarrow\uparrow}_{a_3,0.5} \rangle = C_1\\
\langle \phi^{\downarrow}_{a_2,1.0}, \phi^{\downarrow\uparrow}_{a_2,1.0} \rangle = &\langle \phi^{\downarrow}_{a_3,1.0}, \phi^{\downarrow\uparrow}_{a_3,1.0} \rangle = C_0
\end{aligned}$$

obtaining the association which is written in Table 1.

$M_F(A)$	Fuzzy-attributes generating the meet-irreducible concept
C_0	$\phi_{a_2,1}, \phi_{a_3,1}$
C_1	$\phi_{a_1,0.5}, \phi_{a_1,1}, \phi_{a_2,0.5}, \phi_{a_3,0.5}$
C_2	$\phi_{a_4,1}$

Table 1. Fuzzy-attributes generating the meet-irreducible concepts of Example 1.

From this table, the sets of attributes generating these concepts are straightforwardly determined:

$$\begin{aligned}\text{Atg}(C_0) &= \{a_2, a_3\}\\ \text{Atg}(C_1) &= \{a_1, a_2, a_3\}\\ \text{Atg}(C_2) &= \{a_4\}\end{aligned}$$

Hence, these subsets of attributes are nonempty as Proposition 1 shows. □

The following proposition characterizes the singleton sets of attributes generating a concept.

Proposition 2. *If C is a meet-irreducible concept of $(\mathcal{M}, \leq)$ satisfying that card$(Atg(C)) = 1$, then $Atg(C) \subseteq C_f$.*

Example 2. In the framework of Example 1, if we consider the concept C_2 then we see that the hypothesis given in Proposition 2 are satisfied, that is $\text{card}(\text{Atg}(C_2)) = 1$, and consequently $\text{Atg}(C_2) = \{a_4\} \subseteq C_f$.

This can be checked from the attribute classification given from Table 1 and the classification theorems:

$$\begin{aligned}I_f &= \{a_1, a_5\}\\ K_f &= \{a_2, a_3\}\\ C_f &= \{a_4\}\end{aligned}$$

□

Note that the counterpart of the previous proposition is not true, in general. That is, we can find $a \in C_f$ such that $a \in \text{Atg}(C)$ and satisfying that $\text{card}(\text{Atg}(C)) \geq 1$. What we can assert is that we can always find a meet-irreducible element C satisfying that $\text{card}(\text{Atg}(C)) = 1$, if the core is nonempty, as the following proposition explains.

Proposition 3. *If the attribute $a \in C_f$ then there exists $C \in M_F(A)$ such that $a \in Atg(C)$ and card$(Atg(C)) = 1$.*

Example 3. Coming back to Example 1, we can ensure that the attribute a_4 belongs to C_f and, as Proposition 3 shows, there exists a concept in $M_F(A)$, which is C_2, verifying that $a_4 \in \text{Atg}(C_2)$ and $\text{card}(\text{Atg}(C_2)) = 1$. □

As a consequence of the above properties, the following corollary holds.

Corollary 1. *If C is a meet-irreducible concept of $(\mathcal{M}, \leq)$ and $Atg(C) \cap K_f \neq \varnothing$ then $card(Atg(C)) \geq 2$.*

Example 4. In Example 1, the concept C_1 is a meet-irreducible element such that $\text{Atg}(C_1) \cap K_f = \{a_1, a_2, a_3\} \cap \{a_2, a_3\} = \{a_2, a_3\} \neq \varnothing$. As a consequence, we have that $\text{card}(\text{Atg}(C_1)) = 3 \geq 2$ as Corollary 1 shows. □

The next proposition guarantees that, if a meet-irreducible concept C is obtained from a relatively necessary attribute, then there does not exist an attribute in the core belonging to $\text{Atg}(C)$.

Proposition 4. *Let C be a meet-irreducible concept. $Atg(C) \cap K_f \neq \varnothing$ if and only if $Atg(C) \cap C_f = \varnothing$.*

Example 5. Considering the meet-irreducible concept C_0 of Example 1, we have that $\text{Atg}(C_0) \cap K_f = \{a_2, a_3\}$. Since this intersection is nonempty, applying Proposition 4, we obtain that $\text{Atg}(C_0) \cap C_f = \{a_2, a_3\} \cap \{a_4\} = \varnothing$. A similar situation is given if we take into account C_1. □

A lower bound and a upper bound of the cardinality of the reducts in a multi-adjoint concept lattice framework are provided.

Proposition 5. *Given $\mathcal{G}_K = \{Atg(C) \mid C \in M_F(A) \quad and \quad Atg(C) \cap K_f \neq \varnothing\}$ and any reduct Y of the context (A, B, R, σ). Then, the following chain is always satisfied:*

$$card(C_f) \leq card(Y) \leq card(C_f) + card(\mathcal{G}_K)$$

Example 6. From Example 1, we can ensure that either attribute a_2 or a_3 is needed (the attribute a_1 is absolutely unnecessary) in order to obtain the meet-irreducible concepts C_0 and C_1. Hence, since $a_4 \in C_f$, two reducts $Y_1 = \{a_2, a_4\}$ and $Y_2 = \{a_3, a_4\}$ exist. Thus, only two attributes are needed in order to consider a concept lattice isomorphic to the original one. Now, we will see that these reducts satisfy the previous proposition.

Since the set $\mathcal{G}_K$ is composed by the attributes generating C_0 and C_1, we have that $\mathcal{G}_K = \{\{a_2, a_3\}, \{a_1, a_2, a_3\}\}$. Therefore, both reducts Y_1 and Y_2 satisfy the inequalities in Proposition 5:

$$1 = \text{card}(C_f) \leq \text{card}(Y_1) = \text{card}(Y_2) \leq \text{card}(C_f) + \text{card}(\mathcal{G}_K) = 3$$

□

The proposition below is fundamental in order to provide a sufficient condition to ensure that all the reducts have the same cardinality.

Proposition 6. *If $\mathcal{G}_K = \{Atg(C) \mid C \in M_F(A) \quad and \quad Atg(C) \cap K_f \neq \varnothing\}$ is a partition of K_f, each attribute in K_f generates only one meet-irreducible element of the concept lattice.*

The following result states several conditions to guarantee that all the reducts have the same cardinality.

Theorem 5. *When the set*

$$\mathcal{G}_K = \{Atg(C) \mid C \in M_F(A) \quad and \quad Atg(C) \cap K_f \neq \varnothing\}$$

is a partition of K_f, then:

(a) All the reducts $Y \subseteq A$ have the same cardinality and, specifically, the cardinality is:

$$card(Y) = card(C_f) + card(\mathcal{G}_K)$$

(b) The number of different reducts obtained from the multi-adjoint context is

$$\prod_{Atg(C) \in \mathcal{G}_K} card(Atg(C))$$

Note that the previous theorem provides a sufficient condition in order to ensure that the cardinality of the reducts is the same, however it is not a necessary condition as Example 6 reveals.

4 Worked out examples

This section begins with an illustrative example of Proposition 6 and Theorem 5 that computes the reducts of a particular multi-adjoint concept lattice framework, and shows that these reducts have the same cardinality.

Example 7. Let $(L_1, L_2, L_3, \preceq, \&_G^*)$ be a multi-adjoint frame, where $L_1 = [0,1]_{10}$, $L_2 = [0,1]_4$ and $L_3 = [0,1]_5$ are regular partitions of $[0,1]$ in 10, 4 and 5 pieces, respectively, and $\&_G^*$ is the discretization of the Gödel conjunctor defined on $L_1 \times L_2$. We consider a context (A, B, R, σ), where $A = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, $B = \{b_1, b_2, b_3\}$, $R\colon A \times B \to L_3$ is given by the table shown in the left side of Figure 2 and σ is constantly $\&_G^*$.

In order to obtain reducts, we will study the meet-irreducible elements of the concept lattice displayed in the right side of Figure 2 and the fuzzy-attributes associated with them. From the corresponding Hasse diagram, we can assert that $M_F(A) = \{C_1, C_8, C_9, C_{10}, C_{13}, C_{14}\}$. The fuzzy-attributes related to these concepts are shown in Table 2.

Applying the attribute classification theorems, we obtain:

$$C_f = \{a_1, a_2\}$$
$$K_f = \{a_3, a_4, a_5, a_6\}$$

Once we have classified the attributes, we are going to construct all possible reducts. Clearly, the attributes a_1 and a_2 must be included in all reducts. Hence, it only remains to choose the relatively necessary attributes that should

R	a_1	a_2	a_3	a_4	a_5	a_6
b_1	0.6	0.2	0.2	0	1	0.6
b_2	0.8	0.4	0.6	0.6	1	0.8
b_3	0.6	0.6	0.2	0	0	0

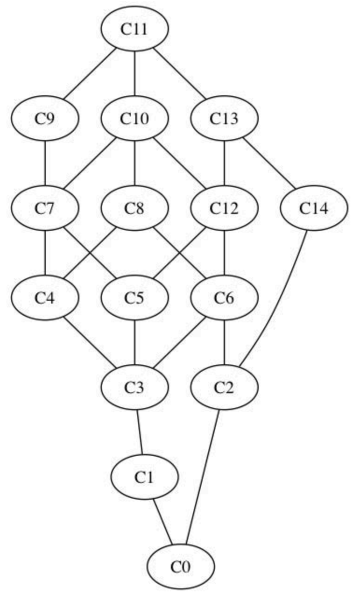

Fig. 2. Relation R (left side) and Hasse diagram of $(\mathcal{M}, \preceq)$ (right side) of Example 7.

be contained in each reduct. For that purpose, we will analyze the attributes generating each meet-irreducible concept:

$$\begin{aligned}
\text{Atg}(C_1) &= \{a_3, a_4\} \\
\text{Atg}(C_8) &= \{a_1\} \\
\text{Atg}(C_9) &= \{a_5, a_6\} \\
\text{Atg}(C_{10}) &= \{a_1\} \\
\text{Atg}(C_{13}) &= \{a_2\} \\
\text{Atg}(C_{14}) &= \{a_2\}
\end{aligned}$$

Since $\text{Atg}(C_1)$ and $\text{Atg}(C_9)$ are disjoint subsets of K_f, we can guarantee that $\mathcal{G}_K$ is a partition of K_f and therefore:

(1) By Proposition 6, each attribute in K_f generates only one meet-irreducible element of the concept lattice. From Table 2, it is easy to prove that the attributes a_3 and a_4 only generate the meet-irreducible concept C_1. The concept C_9 is uniquely generated by a_5 and a_6.

$M_F(A)$	Fuzzy-attributes generating the meet-irreducible concept
C_1	$\phi_{a_3,0.7}, \phi_{a_3,0.8}, \phi_{a_3,0.9}, \phi_{a_3,1}$ $\phi_{a_4,0.7}, \phi_{a_4,0.8}, \phi_{a_4,0.9}, \phi_{a_4,1}$
C_8	$\phi_{a_1,0.9}, \phi_{a_1,1}$
C_9	$\phi_{a_5,0.1}, \phi_{a_5,0.2}, \phi_{a_5,0.3}, \phi_{a_5,0.4}, \phi_{a_5,0.5}, \phi_{a_5,0.6}, \phi_{a_5,0.7}, \phi_{a_5,0.8}, \phi_{a_5,0.9}, \phi_{a_5,1}$ $\phi_{a_6,0.1}, \phi_{a_6,0.2}, \phi_{a_6,0.3}, \phi_{a_6,0.4}, \phi_{a_6,0.5}, \phi_{a_6,0.6}$
C_{10}	$\phi_{a_1,0.7}, \phi_{a_1,0.8}$
C_{13}	$\phi_{a_2,0.3}, \phi_{a_2,0.4}$
C_{14}	$\phi_{a_2,0.5}, \phi_{a_2,0.6}$

Table 2. Fuzzy-attributes generating the meet-irreducible concepts of Example 7.

(2) By Theorem 5, all the reducts have the same cardinality. Thus, since

$$\begin{aligned}\mathcal{G}_K &= \{\text{Atg}(C) \mid C \in M_F(A) \quad \text{and} \quad \text{Atg}(C) \cap K_f \neq \varnothing\} \\ &= \{\text{Atg}(C_1), \text{Atg}(C_9)\} \\ &= \{\{a_3, a_4\}, \{a_5, a_6\}\}\end{aligned}$$

we have that $\text{card}(Y) = \text{card}(C_f) + \text{card}(\mathcal{G}_K) = 2 + 2 = 4$, for any reduct Y of the context. Moreover, the number of reducts that we obtain from this context is

$$\prod_{\text{Atg}(C) \in \mathcal{G}_K} \text{card}(\text{Atg}(C)) = 2 \cdot 2 = 4$$

Specifically, the whole set of reducts are listed below:

$$\begin{aligned}Y_1 &= \{a_1, a_2, a_3, a_5\} \\ Y_2 &= \{a_1, a_2, a_3, a_6\} \\ Y_3 &= \{a_1, a_2, a_4, a_5\} \\ Y_4 &= \{a_1, a_2, a_4, a_6\}\end{aligned}$$

From the previous reducts, we obtain the following isomorphic concept lattices:

$$(\mathcal{M}, \preceq) \cong (\mathcal{M}^{Y_1}, \preceq) \cong (\mathcal{M}^{Y_2}, \preceq) \cong (\mathcal{M}^{Y_3}, \preceq) \cong (\mathcal{M}^{Y_4}, \preceq)$$

□

Now, we will present a situation where the elements belonging to the set $\mathcal{G}_K$ are not a partition of K_f, and we will see that in this particular example several reducts with different cardinality are obtained.

Example 8. Considering the same framework that in the previous example, we fix a context (A, B, R, σ) where the set A consists of seven attributes, the set B contains three objects and R is obtained from the relation of the previous example with a few of changes shown in Table 3. Hence, we obtain an isomorphic concept lattice to the one shown in Figure 2, but a different attribute classification arises.

Table 3. Definition of R

R	a_1	a_2	a_3	a_4	a_5	a_6	a_7
b_1	0.6	0.2	0.2	1	0.6	0.2	0
b_2	0.8	0.4	0.4	1	0.8	0.6	0.6
b_3	0.6	0.6	0.2	0	0	0.2	0

The attributes are classified as follows:

$$C_f = \{a_1, a_2\}$$
$$K_f = \{a_4, a_5, a_6, a_7\}$$
$$I_f = \{a_3\}$$

As a consequence, a_1 and a_2 must belong to all the reducts and a_3 should be removed. Analyzing the meet-irreducible elements and the fuzzy-attributes generating them, we obtain:

$$\mathrm{Atg}(C_1) = \{a_6, a_7\}$$
$$\mathrm{Atg}(C_8) = \{a_1\}$$
$$\mathrm{Atg}(C_9) = \{a_4, a_5, a_6\}$$
$$\mathrm{Atg}(C_{10}) = \{a_1\}$$
$$\mathrm{Atg}(C_{13}) = \{a_2\}$$
$$\mathrm{Atg}(C_{14}) = \{a_2\}$$

Now, we have to select one attribute of $\mathrm{Atg}(C_1)$ and another one of $\mathrm{Atg}(C_9)$ in order to obtain the whole set of meet-irreducible concepts and compute the reducts. However, in this case, $\mathrm{Atg}(C_1) \subseteq K_f$ and $\mathrm{Atg}(C_9) \subseteq K_f$ and the intersection $\mathrm{Atg}(C_1) \cap \mathrm{Atg}(C_9) = a_6$ is nonempty. Therefore, the set $\mathcal{G}_K = \{\mathrm{Atg}(C_1), \mathrm{Atg}(C_9)\}$ is not a partition of K_f.

Consequently, we can obtain the following different reducts whose sizes depend on the chosen attributes as we can see below:

$$Y_1 = \{a_1, a_2, a_6\}$$
$$Y_2 = \{a_1, a_2, a_4, a_7\}$$
$$Y_3 = \{a_1, a_2, a_5, a_7\}$$

□

This example provides the idea that, in order to compute a minimal reduct, with respect to the number of attributes, the relatively necessary attributes to be taken into account must be the ones given in the intersection of the sets $\mathrm{Atg}(C)$, with $\mathrm{Atg}(C) \in \mathcal{G}_K$.

5 Conclusion and future work

Based on the attribute classification introduced in [6], a construction process of the reducts of a multi-adjoint concept lattice has been shown. Several properties have been stated together with examples that illustrate the shown results. The importance of the choice of the relatively necessary attributes for computing the reducts has also been highlighted.

More properties related to reducts will be investigated in the future in order to find the most profitable way to generate them. We are also interested in obtaining an algorithm that provides a reduct with a minimal number of attributes for any multi-adjoint concept lattice framework given.

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