with the cost functional

$$J = \frac{1}{2}x^T(1)Fx(1) + \frac{1}{2}\int_0^1 (x^T(t)Q(t)x(t) + \varepsilon u^T(t)R(t)u(t))dt. \quad (18)$$

where $A,\ F_1,\ Q$ are $(n \times n)$-matrices, B is $(n \times m)$-matrix, and R is $(m \times m)$-matrix. Suppose that all these matrices have the following asymptotic presentations with respect to ε:

$$A(t,\varepsilon) = \sum_{j\geq 0} \varepsilon^j Aj(t),$$

$$B(t,\varepsilon) = \sum_{j\geq 0} \varepsilon^j B_j(t),$$

$$Q(t,\varepsilon) = \sum_{j\geq 0} \varepsilon^j Q_j(t),$$

$$R(t,\varepsilon) = \sum_{j\geq 0} \varepsilon^j R_j(t),$$

$$F(\varepsilon) = \sum_{j\geq 0} \varepsilon^j F_j$$

with smooth on t matrix coefficients, $t \in [0, 1]$.
The solution of this problem is the optimal linear feedback control law

$$u = -\varepsilon^{-1} R^{-1} B^T P(t,\varepsilon)x,$$

where P satisfies the differential matrix Riccati equation

$$\varepsilon\dot{P} = -PA - A^T P + PSP - \varepsilon Q, \quad P(1,\varepsilon) = F.$$

Setting $\varepsilon = 0$ we obtain the matrix algebraic equation

$$-MA_0 - A_0^T M + MS_0 M = 0,$$

where $S_0 = B_0 R_0^{-1} B_0^T$. It is clear that the main role plays the linear operator

$$\mathbf{L}X = XA_0 + A_0^T X.$$

For this class of systems the eigenvalues of A_0 are pure imaginary and the spectrum of the linear operator $\mathbf{L}$ has a nontrivial kernel, since sums $(\lambda_i(t) + \lambda_j(t))$, $i, j = 1, \ldots, n$, form its spectrum. This means that the Riccati equation is singular singularly perturbed. Thus, the problem under consideration is critical in this sense. Moreover, under taking into account that zero eigenvalues are multiple and all other, nonzero eigenvalues of $\mathbf{L}$, are pure imaginary, it is possible to say that this problem is thrice critical.

Example

Let

$$A = \begin{pmatrix} -\varepsilon & 1 \\ -1 & -\varepsilon \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad R = (1), \quad Q = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$S = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}.$$

Consider the corresponding differential system

$$\begin{aligned} \varepsilon\dot{p}_1 &= 2p_2 + 2\varepsilon p_1 + p_2^2 - \varepsilon, \\ \varepsilon\dot{p}_2 &= 2\varepsilon p_2 - p_1 + p_3 + p_2 p_3, \\ \varepsilon\dot{p}_3 &= -2p_2 + 2\varepsilon p_3 + p_3^2. \end{aligned} \tag{19}$$

First, we need to separate it into a slow and a fast subsystem. At first glance, all three equations are singularly perturbed. However, setting $\varepsilon = 0$ we obtain $p_1 = p_2 = p_3 = 0$, and we should consider the matrix of leading terms on the right hand side of the system, which has the form

$$\begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 1 \\ 0 & -2 & 0 \end{pmatrix}.$$

Obviously, this matrix has a zero eigenvalue and two pure imaginary eigenvalues, i.e. the problem under consideration is twice critical.Moreover, the trivial solution is multiple. This means that we have thrice critical case.
Let $\varepsilon = \mu^2$. Introducing the new variables

$$p_1 = \mu^2 q_1 + \mu, \quad p_2 = \mu^2 q_2 + \mu^2/2, \quad p_3 = \mu^2 q_3 + \mu,$$

and then $s = q_1 + q_3$, we obtain the differential system

$$\begin{aligned} \mu\dot{s} &= 2q_3 + \mu q_2 + 2\mu s + \mu q_2^2 + \mu q_3^2 + 4 + \mu/4, \\ \mu^2\dot{q}_2 &= -s + 2\mu^2 q_2 + 2q_3 + \mu q_2 + \mu^2 q_2 q_3 + \mu/2 + \mu^2, \\ \mu^2\dot{q}_3 &= -2q_2 + 2\mu q_3 + 2\mu^2 q_3 + \mu^2 q_3^2 + 2\mu \end{aligned} \tag{20}$$

with the slow variable s and two fast variables $q_2,\ q_3$.
The last system possesses one-dimensional slow invariant manifold which is weakly attractive with respect to argument $1 - t$ because the main matrix of the fast subsystem is

$$\begin{pmatrix} \mu & 2 \\ -2 & 2\mu \end{pmatrix}.$$

Thus, the dimension of the system of Riccati differential equations can be reduced from three to one. Let us construct the slow integral manifold using the fact that it can be asymptotically expanded in powers of the small parameter. Setting

$$q_2 = \varphi(s, \mu) = \mu\varphi_1(s) + \mu^2 \dots\ ,$$

$$q_3 = \psi(s, \mu) = \psi_0(s) + \mu\psi_1(s) + \mu^2 \dots ,$$

we obtain

$$\psi_0(s) = s/2, \quad \varphi_1(s) = s/2, \quad \psi_1(s) = -1/4.$$

Thus we obtain the slow invariant manifold

$$q_2 = \mu s/2 + O(\mu^2), \quad q_3 = s/2 - \mu/4 + O(\mu^2),$$

with the equation on the integral manifold

$$\mu\dot{s} = s + 2\mu s + \mu s^2/4 + O(\mu^2).$$

Figure 1 demonstrates the closeness of solutions of the original system and the system on the slow invariant manifolds for $q_1(t)$. The similar situation takes place for q_2 and q_3.

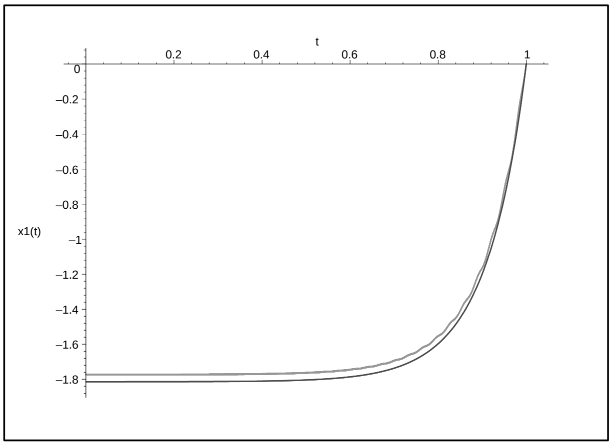

Fig. 1. The graphs of q_1 for original differential system (grey line) and for the equation on the slow invariant manifold (black line).

Conclusion

Critical cases for singularly perturbed differential systems are studied in the paper. We have considered singularly perturbed control problems as applications. It has been shown that the reduction of dimensions of these problems can be done by means of the integral manifold method. The slow integral manifolds for the matrix Riccati equation of linear-quadratic control problem are constructed and it is shown that the method of integral manifolds allows us to reduce the dimension of control problems. This approach was used for the investigation of optimal filtering problems in [28, 29].

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