We can choose the block V_1 from the condition of asymptotic stability of the slow subsystem, which for an autonomous system takes the form

$$\operatorname{Re}\lambda_j(A_1 - V_1\tilde{C}_1) < 0. \tag{12}$$

For the estimation of state vector of the initial system we have

$$m_1 = m_v + \varepsilon P m_z, \quad m_2 = m_z + L m_1.$$

Let us construct the full order observer for slow subsystem of the block diagonal system (10)

$$\begin{aligned} \dot{v} &= A_1 v, \\ y &= \tilde{C}_1 v \end{aligned} \tag{13}$$

as

$$\dot{n_v} = (A_1 - V_1\tilde{C}_1)n_v + V_1 y.$$

We can choose the block V_1 from the condition of the asymptotic stability which for an autonomous system has a form of the inequalities (12).
It can be proved that

$$\lim_{t\to\infty} \|n_v - m_v\| = 0.$$

As an estimation of fast variable we can use any solution of the system

$$\varepsilon \dot{m}_z = A_2 m_z.$$

For estimation of state vector of the initial system we have

$$m_1 = n_v + \varepsilon P m_z, \quad m_2 = m_z + L m_1.$$

The similar reasoning can be used for construction of the Luenberger observer. For the slow subsystem (13) we choose matrix W such that matrix $Q = \begin{pmatrix} \tilde{C}_1 \\ W \end{pmatrix}$ is nonsingular. Let $Q^{-1} = (R \;\; D)$. The estimations of state vector take the form

$$m_1 \;\; = \;\; D\alpha + (R + DV_1)y, \quad m_2 = m_z,$$

where

$$\dot{\alpha} = (W - V_1\tilde{C}_1)(A_1 D\alpha + A_1(DV_1 + R)y), \quad \alpha(0) = \alpha_0,$$

$$\varepsilon \dot{m}_z = A_2 m_z.$$

Aircraft model

Consider the model of a longitudinal motion of an aircraft, see Figure 1, [10]

$$\begin{aligned} \ddot{v} &= d_1\alpha - d_2\delta \\ \dot{\theta} &= d_3\alpha, \\ T\dot{\delta} + \delta &= K_{rm} u. \end{aligned} \tag{14}$$

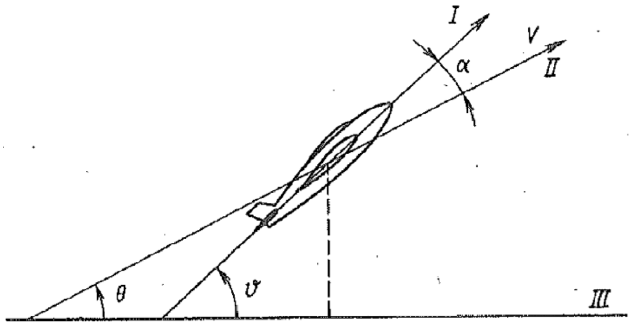

Fig. 1. Aircraft model

where ν is a pitch angle, θ is a flight path angle, $\alpha = v - \theta$ is an angle of attack, δ is a deviation of the elevator, d_i is the aerodynamic coefficients, T and K_{rm} are the characteristics of control–surface actuator.
The typical values of the parameters are $d_1 = 36$, $d_2 = 18$, $d_3 = 1.2$, $T = 0.1$.
Let $\varepsilon = T$ and

$$x_1 = \begin{pmatrix} \dot{\nu} \\ \nu \\ \theta \end{pmatrix}, \quad x_2 = \delta.$$

The system (14) takes the form

$$\begin{aligned} \dot{x_1} &= A_{11}x_1 + A_{12}x_2 \\ \varepsilon \dot{x_2} &= -x_2, \end{aligned} \tag{15}$$

where

$$A_{11} = \begin{pmatrix} 0 & -d_1 & d_1 \\ 1 & 0 & 0 \\ 0 & d_3 & -d_3 \end{pmatrix} \quad A_{12} = \begin{pmatrix} -d_2 \\ 0 \\ 0 \end{pmatrix},$$

$$A_{21} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}, \quad A_{22} = -1.$$

Let the outputs be ν and θ, then

$$y = C \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Using the coordinate transformation

$$x_2 = z, \quad x_1 = v + \varepsilon P z,$$

where $P = P(\varepsilon)$ is the matrix function, which satisfies the equation

$$-P = \varepsilon A_{11} P + A_{12},$$

we can transform the system (15) to the block diagonal form

$$\dot{v} = A_{11} v, \quad \varepsilon \dot{z} = -z. \tag{16}$$

The matrix function $P(\varepsilon)$ may be constructed with any degree of accuracy as asymptotic series in small parameter ε

$$P = P(\varepsilon) = P^{(0)} + \varepsilon P^{(1)} + \dots,$$

where

$$P^{(0)} = -A_{12} = \begin{pmatrix} d_1 \\ 0 \\ 0 \end{pmatrix}, \quad P^{(1)} = -A_{11}P^{(0)} = \begin{pmatrix} 0 \\ -d_2 \\ 0 \end{pmatrix}.$$

Output vector y takes the form

$$y = \tilde{C}\begin{pmatrix} v \\ z \end{pmatrix},$$

were

$$\tilde{C} = (\tilde{C}_1 \quad \tilde{C}_2), \quad \tilde{C}_1 = C_1, \quad \tilde{C}_2 = \varepsilon C_1 P + C_2,$$

$$\tilde{C}_1 = C_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \tilde{C}_2 = \begin{pmatrix} -\varepsilon^2 d_2 \\ 1 \end{pmatrix}$$

Let us construct the full order observer for the slow subsystem of the block diagonal system (16)

$$\dot{v} = A_{11}v,$$
$$y = \tilde{C}_1 v,$$

in the form

$$\dot{n}_v = (A_1 - V_1\tilde{C}_1)n_v + V_1 y.$$

We can choose the block V_1 from the condition of the asymptotic stability in the form

$$V_1 = \begin{pmatrix} a & 0 \\ b & 0 \\ c & 0 \end{pmatrix}.$$

Then the estimation of the state vector v must satisfy the equation

$$\dot{n}_v = \begin{pmatrix} 0 & -d_1 & d_1 - a \\ 1 & 0 & -b \\ 0 & d_3 & -d_3 - c \end{pmatrix} n_v + \begin{pmatrix} a \\ b \\ c \end{pmatrix} y_1(t).$$

For example, let us put $a = 0$, $b = 1$, $c = 1$.
As an estimation of the fast variable z we can use any solution of the system

$$\varepsilon\dot{m}_z = -m_z.$$

For the estimation of the state vector of the initial system we have

$$m_1 = n_v + \varepsilon P m_z, \quad m_2 = m_z.$$

The Figures 2 – 5 demonstrate the dynamic of the state vector and its estimation. Similar reasoning can be used for construction the Luenberger observer.

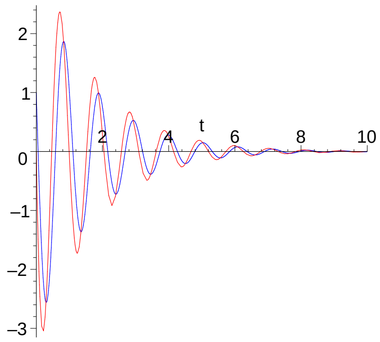

Fig. 2. $\dot{v}$, $m_{\dot{v}}$

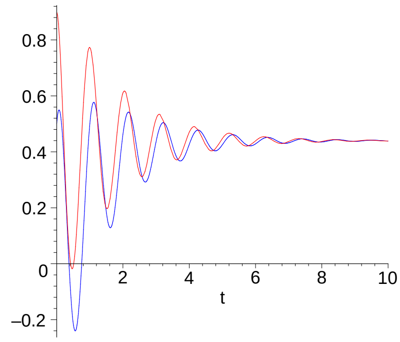

Fig. 3. v, m_v

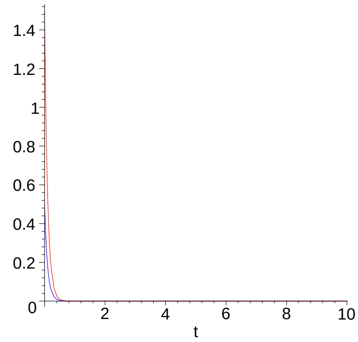

Fig. 4. δ, m_δ

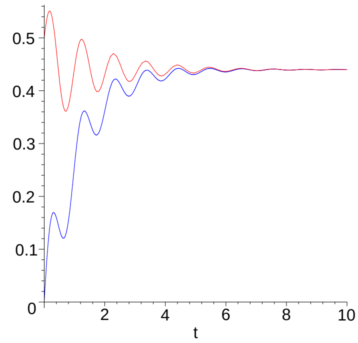

Fig. 5. θ, m_θ

For slow subsystem we notice that the second row of matrix $\tilde{C}_1$ is zero. We choose matrix W such that matrix

$$Q = \begin{pmatrix} \bar{C}_1 \\ W \end{pmatrix}, \text{ where } \bar{C}_1 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix},$$

is nonsingular. For example,

$$W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Write matrix Q^{-1} in the form $Q^{-1} = (R \;\; D)$, where

$$R = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The estimation of the vector v takes the form

$$m_v \;\; = \;\; D\alpha + (R + DV_1)y,$$

where

$$\dot{\alpha} = (W - V_1\bar{C}_1)(A_{11}D\alpha + A_{11}(DV_1 + R)y), \quad \alpha(0) = \alpha_0.$$

Let

$$V_1 = \begin{pmatrix} a \\ b \end{pmatrix}.$$

We have

$$(W - V_1\bar{C}_1)A_{11}D = \begin{pmatrix} -a & d_1 \\ -b & -d_3 \end{pmatrix}.$$

For example, let us put $a = 0$, $b > 0$.
As an estimation of the fast variable we can use any solution of the system

$$\varepsilon \dot{m}_z = -m_z.$$

For the estimation of the state vector of the initial system we have

$$m_1 = m_v + \varepsilon P m_2, \quad m_2 = m_z.$$

The Figures 6–9 demonstrate the dynamic of the state vector and its estimation.

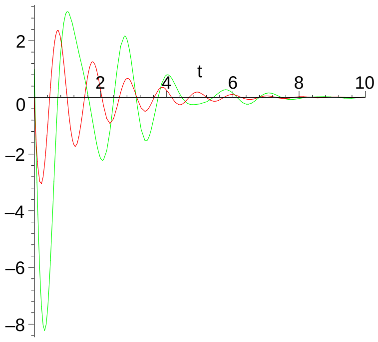

Fig. 6. $\dot{v}$, $m_{\dot{v}}$

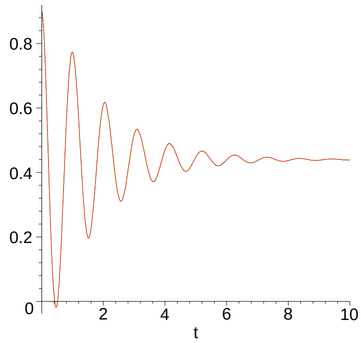

Fig. 7. v, m_v

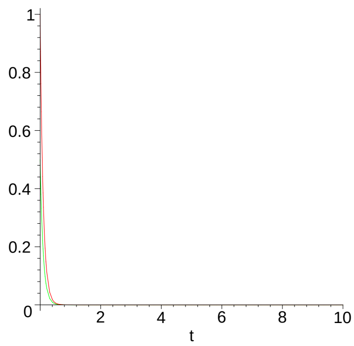

Fig. 8. δ, m_δ

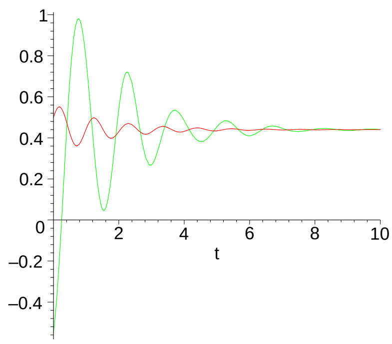

Fig. 9. θ, m_θ

Conclusion

The asymptotic decomposition method helped us to reduce the observation problems for the dynamic systems with slow and fast variables. This approach can be also used for solving the observation problems in a stochastic case.

Acknowledgements

The research has been supported by the Ministry of Education and Science of the Russian Federation (the Project 204).

References

1. Shchepakina E, Sobolev V, Mortell MP. Singular Perturbations: Introduction to System Order Reduction Methods with Applications. In: Springer Lecture Notes in Mathematics, Cham: Springer International Publishing, 2014; 2114.
2. Kononenko LI, Sobolev VA. Asymptotic expansion of slow integral manifolds. Sib. Math. J., 1994; 35: 1119-1132.
3. Voropaeva NV, Sobolev VA. Decomposition of a linear-quadratic optimal control problem with fast and slow variables. Automation and Remote Control, 2006; 67(8): 1185–1193.
4. Voropaeva NV. Decomposition of problems of optimal control and estimation for discrete systems with fast and slow variables. Automation and Remote Control, 2008; 69(6): 920–928.
5. Voropaeva NV, Sobolev VA. Geometric decomposition of singularly perturbed systems. Moscow: Fizmatlit, 2009. [in Russian]
6. Sobolev VA. Singular perturbations in linearly quadratic optimal control problems. Automation and Remote Control, 1991; 52(2): 180–189.
7. Mikheev Yu.V, Sobolev VA. Fridman EM. Asymptotic analysis of digital control systems. Automation and Remote Control, 1989; 49(9), 1175–1180.
8. Zharikova EN, Sobolev VA. Optimal periodic systems of control with singular perturbations. Automation and Remote Control, 1997; 58(7): 1188-1202.
9. Smetannikova E, Sobolev V. Regularization of cheap periodic control problems. Automation and Remote Control, 2005; 66(6): 903–916.
10. Afanasev VN, Kolmanovsky VB, Nosov VR. Mathematical theory of designing control systems. Moscow: Vysshaya shkola Publishers, 1998. [in Russian]