# Reasoning in a Rational Extension of $\mathcal{SROEL}$

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**Abstract.** In this work we define a rational extension $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ of the low complexity description logic $\mathcal{SROEL}(\sqcap, \times)$, which underlies the OWL EL ontology language. The logic is extended with a typicality operator $\mathbf{T}$, whose semantics is based on Lehmann and Magidor's ranked models and allows for the definition of defeasible inclusions. We consider both rational entailment and minimal entailment. We show that deciding instance checking under minimal entailment is a CONP-hard problem while, under rational entailment, instance checking can be computed in polynomial time. In particular, we develop a Datalog materialization calculus for instance checking under rational entailment.

## 1 Introduction

The need for extending Description Logics (DLs) with nonmonotonic features has led, in the last decade, to the development of several extensions of DLs, obtained by combining them with the most well-known formalisms for nonmonotonic reasoning [3, 36, 4, 14, 22, 16, 29, 11, 8, 13, 35, 6, 30, 12, 26, 5, 27] to deal with defeasible reasoning and inheritance, to allow for prototypical properties of concepts and to combine DLs with nonmonotonic rule-based languages under the answer set semantics [16], the well-founded semantics [15], the MKNF semantics [35, 30], as well as in Datalog +/- [28]. Systems integrating Answer Set Programming (ASP) [19, 18] and DLs have been developed, e.g., the DReW System for Nonmonotonic DL-Programs [37].

In this paper we study a preferential extension of the logic $\mathcal{SROEL}(\sqcap, \times)$, introduced by Krötzsch [32], which is a low-complexity description logic of the $\mathcal{EL}$ family [1] that includes local reflexivity, conjunction of roles and concept products and is at the basis of OWL 2 EL. Our extension is based on Kraus, Lehmann and Magidor (KLM) preferential semantics [31], and, specifically, on ranked models [34]. We call the logic $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ and define notions of rational and minimal entailment for it.

The semantics of ranked interpretations for DLs was first studied in [11], where a rational extension of $\mathcal{ALC}$ is developed allowing for defeasible concept inclusions of the form $C \tilde{\sqsubseteq} D$. In this work, following [23, 27], we extend the language of $\mathcal{SROEL}(\sqcap, \times)$ with typicality concepts of the form $\mathbf{T}(C)$, whose instances are intended to be the typical $C$ elements. Typicality concepts can be used to express defeasible inclusions of the form $\mathbf{T}(C) \sqsubseteq D$ ("the typical $C$ elements are $D$"). Here, however, as in [9, 21], we allow for typicality concepts to freely occur in concept inclusions. In this respect, the language with typicality that we consider is more general than the language with typicality in [27], where the typicality operator $\mathbf{T}(C)$ may only occur on the left hand side of inclusions as well as in assertions. For the language in [27], minimal ranked models have been shown to provide a semantic characterization to rational closure for the description logic $\mathcal{ALC}$, generalizing to DLs the rational closure by Lehmann and Magidor [34]. Alternative constructions of rational closure for $\mathcal{ALC}$ have been proposed in [13, 12]. All such constructions regard languages only containing strict or defeasible inclusions.

We show that, for general $\mathcal{SROEL}(\sqcap,\times)^{\mathbf{R}}\mathbf{T}$ KBs, deciding instance checking under minimal entailment is a CONP-hard problem. Furthermore, we define a Datalog translation for $\mathcal{SROEL}(\sqcap,\times)^{\mathbf{R}}\mathbf{T}$ which builds on the materialization calculus in [32], and, for typicality reasoning, is based on properties of ranked models, showing that instance checking for $\mathcal{SROEL}(\sqcap,\times)^{\mathbf{R}}\mathbf{T}$ can be computed in polynomial time under the rational entailment. This polynomial upper bound also extends to subsumption, with the consequence that a Rational Closure construction for $\mathcal{SROEL}(\sqcap,\times)^{\mathbf{R}}\mathbf{T}$, based on the definition in [27], can be computed in polynomial time. However, the minimal canonical model semantics does not provide a general semantic characterization of the rational closure for the logic $\mathcal{SROEL}(\sqcap,\times)$ with typicality, as a KB may have alternative minimal canonical models with incompatible rankings, or no canonical model at all. An extended abstract of this paper appeared in [20].

## 2 A rational extension of $\mathcal{SROEL}(\sqcap,\times)$

In this section we extend the notion of concept in $\mathcal{SROEL}(\sqcap,\times)$ adding typicality concepts (we refer to [32] for a detailed description of the syntax and semantics of $\mathcal{SROEL}(\sqcap,\times)$). We let $N_C$ be a set of concept names, $N_R$ a set of role names and $N_I$ a set of individual names. A concept in $\mathcal{SROEL}(\sqcap,\times)$ is defined as follows:

$$C := A \mid \top \mid \bot \mid C \sqcap C \mid \exists R.C \mid \exists S.Self \mid \{a\}$$

where $A \in N_C$ and $R, S \in N_R$. We introduce a notion of *extended concept* $C_E$ as follows:

$$C_E := C \mid \mathbf{T}(C) \mid C_E \sqcap C_E \mid \exists S.C_E$$

where $C$ is a $\mathcal{SROEL}(\sqcap,\times)$ concept. Hence, any concept of $\mathcal{SROEL}(\sqcap,\times)$ is also an extended concept; a typicality concept $\mathbf{T}(C)$ is an extended concept and can occur in conjunctions and existential restrictions, but it cannot be nested.

A KB is a triple $(TBox, RBox, ABox)$. $TBox$ contains a finite set of *general concept inclusions* (GCI) $C \sqsubseteq D$, where $C$ and $D$ are extended concepts; $RBox$ (as in [32]) contains a finite set of *role inclusions* of the form $S \sqsubseteq T$, $R \circ S \sqsubseteq T$, *role conjunctions* $S_1 \sqcap S_2 \sqsubseteq T$, *concept product axioms* and $C \times D \sqsubseteq T$ and $R \sqsubseteq C \times D$, where $C$ and $D$ are concepts, and $R, S, S_1, S_2, T$ are role names in $N_R$. $ABox$ contains *individual assertions* of the form $C(a)$ and $R(a, b)$, where $a, b \in N_I$, $R \in N_R$ and $C$ is an extended concept. Restrictions are imposed on the use of roles as in [32] (and, in particular, all the roles occurring in *Self* concepts and in role conjunctions must be *simple roles*, roughly speaking, roles which do not include the composition of other roles).

We define a semantics for $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ based on ranked models [34]. As done in [27] for $\mathcal{ALC}$, we define the semantics of $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ by adding to $\mathcal{SROEL}(\sqcap, \times)$ interpretations [32] a *preference relation* $<$ on the domain, which is intended to compare the "typicality" of domain elements. The typical instances of a concept $C$, i.e., the instances of $\mathbf{T}(C)$, are the instances of $C$ that are minimal with respect to $<$. The properties of the $<$ relation are defined in agreement with the properties of the preference relation in Lehmann and Magidor's *ranked models* in [34]. A semantics for DLs with defeasible inclusions based on ranked models was first proposed in [11].

**Definition 1.** *A $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ interpretation $\mathcal{M}$ is any structure $\langle \Delta, <, \cdot^I \rangle$ where:*

- *$\Delta$ is a domain; $\cdot^I$ is an interpretation function that maps each concept name $A \in N_C$ to a set $A^I \subseteq \Delta$, each role name $R \in N_R$ to a binary relation $R^I \subseteq \Delta \times \Delta$, and each individual name $a \in N_I$ to an element $a^I \in \Delta$. $\cdot^I$ is extended to complex concepts as usual:*
  $\top^I = \Delta$; $\quad \bot^I = \emptyset$; $\quad \{a\}^I = \{a^I\}$;
  $(C \sqcap D)^I = C^I \cap D^I$;
  $(\exists R.C)^I = \{x \in \Delta \mid \exists y \in C^I : (x, y) \in R^I\}$;
  $(\exists R.Self)^I = \{x \in \Delta \mid (x, x) \in R^I\}$;
  *and the composition of role interpretations is defined as follows:*
  $R_1^I \circ R_2^I = \{(x, z) \mid (x, y) \in R_1^I$ *and* $(y, z) \in R_2^I$*, for some* $y \in \Delta\}$
- *$<$ is an irreflexive, transitive, well-founded and modular relation over $\Delta$;*
- *the interpretation of concept $\mathbf{T}(C)$ is defined as follows:*

$$(\mathbf{T}(C))^I = Min_<(C^I)$$

*where $Min_<(S) = \{u : u \in S$ and $\nexists z \in S$ s.t. $z < u\}$.*

Furthermore, an irreflexive and transitive relation $<$ is *well-founded* if, for all $S \subseteq \Delta$, for all $x \in S$, either $x \in Min_<(S)$ or $\exists y \in Min_<(S)$ such that $y < x$. It is *modular* if, for all $x, y, z \in \Delta$, $x < y$ implies $x < z$ or $z < y$. The well-foundedness condition guarantees that if, for a non-extended concept $C$, there is a $C$ element in $\mathcal{M}$, then there is a minimal $C$ element in $\mathcal{M}$ (i.e., $C^I \neq \emptyset$ implies $(\mathbf{T}(C))^I \neq \emptyset$).

In the following, we will refer to $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ interpretations as *ranked interpretations*. Indeed, as in [34], modularity in preferential models can be equivalently defined by postulating the existence of a rank function $k_{\mathcal{M}} : \Delta \longmapsto \Omega$, where $\Omega$ is a totally ordered set. The preference relation $<$ can be defined from $k_{\mathcal{M}}$ as follows: $x < y$ if and only if $k_{\mathcal{M}}(x) < k_{\mathcal{M}}(y)$. Hence, in the following, we will assume that a rank function $k_{\mathcal{M}}$ is always associated with any model $\mathcal{M}$. We also define the *rank $k_{\mathcal{M}}(C)$ of a concept $C$ in the model $\mathcal{M}$* as $k_{\mathcal{M}}(C) = min\{k_{\mathcal{M}}(x) \mid x \in C^I\}$ (if $C^I = \emptyset$, then $C$ has no rank and we write $k_{\mathcal{M}}(C) = \infty$).

Observe that semantics of the typicality operator defined above is exactly the same as the one introduced in [27] for the typicality operator in $\mathcal{ALC} + \mathbf{T}_R$. Similarly to all other concept constructors, the typicality operator can be used in TBox and ABox with different restrictions, depending on the description logic. Differently from [27], where $\mathbf{T}(C)$ can only occur on the left-hand side of concept inclusions (namely, in typicality inclusions of the form $\mathbf{T}(C) \sqsubseteq D$) here, as in [9, 21], we do not put restrictions on the possible occurrences of typicality concepts $\mathbf{T}(C)$ in concept inclusions and in assertions. Instead, as in $\mathcal{SROEL}(\sqcap, \times)$, we do not allow negation, union and universal restriction which are allowed in $\mathcal{ALC}$. In the following, we call *simple* KBs the ones which only allow typicality concepts to occur on the left hand side of typicality inclusions. Given an interpretation $\mathcal{M}$ the notions of satisfiability and entailment are defined as usual.

**Definition 2 (Satisfiability and rational entailment).** *An interpretation* $\mathcal{M} = \langle \Delta, <, \cdot^I \rangle$ *satisfies:*
- *a concept inclusion* $C \sqsubseteq D$ *if* $C^I \subseteq D^I$*;*
- *a role inclusion* $S \sqsubseteq T$ *if* $S^I \subseteq T^I$*;*
- *a generalized role inclusion* $R \circ S \sqsubseteq T$ *if* $R^I \circ S^I \subseteq T^I$
- *a role conjunction* $S_1 \sqcap S_2 \sqsubseteq T$ *if* $S_1^I \cap S_2^I \subseteq T^I$*;*
- *a concept product axiom* $C \times D \sqsubseteq T$ *if* $C^I \times D^I \subseteq T^I$*;*
- *a concept product axiom* $R \sqsubseteq C \times D$ *if* $R^I \subseteq C^I \times D^I$*;*
- *an assertion* $C(a)$ *if* $a^I \in C^I$*;*
- *an assertion* $R(a, b)$ *if* $(a^I, b^I) \in R^I$.

*Given a KB* $K = (TBox, RBox, ABox)$, *an interpretation* $\mathcal{M} = \langle \Delta, <, \cdot^I \rangle$ satisfies $TBox$ *(resp.,* $RBox$, $ABox$*) if* $\mathcal{M}$ *satisfies all axioms in* $TBox$ *(resp.,* $RBox$, $ABox$*), and we write* $\mathcal{M} \models TBox$ *(resp.,* $RBox$, $ABox$*). An interpretation* $\mathcal{M} = \langle \Delta, <, \cdot^I \rangle$ *is a* model *of* $K$ *(and we write* $\mathcal{M} \models K$*) if* $\mathcal{M}$ *satisfies all the axioms in* $TBox$, $RBox$ *and* $ABox$.

*Let a query* $F$ *be either a concept inclusion* $C \sqsubseteq D$*, where* $C$ *and* $D$ *are extended concepts, or an individual assertion.* $F$ is rationally entailed by $K$, *written* $K \models_{sroelrt} F$, *if for all models* $\mathcal{M} = \langle \Delta, <, \cdot^I \rangle$ *of* $K$, $\mathcal{M}$ *satisfies* $F$. *In particular, the* instance checking *problem (under rational entailment) is the problem of deciding whether an assertion (*$C(a)$, $\mathbf{T}(C)(a)$ *or* $R(a, b)$*) is rationally entailed by* $K$.

Given the correspondence of typicality inclusions with conditional assertions $C \mid\sim D$, it can be easily seen that each ranked interpretation $\mathcal{M}$ satisfies the following semantic conditions, corresponding to Lehmann and Magidor's postulates of rational consequence relation [34] reformulated in terms of typicality, where, by $\mathbf{T}(A) \sqsubseteq B$ we mean that $\mathbf{T}(A) \sqsubseteq B$ is satisfied in $\mathcal{M}$, by $\mathbf{T}(A) \not\sqsubseteq \neg B$ we mean that $\mathbf{T}(A) \sqsubseteq \neg B$ is not satisfied in $\mathcal{M}$, and by $A \sqsubseteq B$ (or $A \equiv B$) we mean that $A \sqsubseteq B$ (or $A \equiv B$) is satisfied in $\mathcal{M}$ (a similar formulation of the semantic properties in terms of defeasible inclusions can be found in [11]):

$(LLE)$ If $A \equiv B$ and $\mathbf{T}(A) \sqsubseteq C$ then $\mathbf{T}(B) \sqsubseteq C$
$(RW)$ If $B \sqsubseteq C$ and $\mathbf{T}(A) \sqsubseteq B$ then $\mathbf{T}(A) \sqsubseteq C$
$(Refl)$ $\mathbf{T}(A) \sqsubseteq A$
$(And)$ If $\mathbf{T}(A) \sqsubseteq B$ and $\mathbf{T}(A) \sqsubseteq C$ then $\mathbf{T}(A) \sqsubseteq B \sqcap C$
$(Or)$ If $\mathbf{T}(A) \sqsubseteq C$ and $\mathbf{T}(B) \sqsubseteq C$ then $\mathbf{T}(A \sqcup B) \sqsubseteq C$
$(CM)$ If $\mathbf{T}(A) \sqsubseteq B$ and $\mathbf{T}(A) \sqsubseteq C$ then $\mathbf{T}(A \sqcap B) \sqsubseteq C$
$(RM)$ If $\mathbf{T}(A) \sqsubseteq C$ and $\mathbf{T}(A) \not\sqsubseteq \neg B$ then $\mathbf{T}(A \sqcap B) \sqsubseteq C$

It is easy to see that these semantic properties hold in all the ranked models. In particular, property (RM), can be reformulated as follows:

if $(\mathbf{T}(A) \sqcap B)^I \neq \emptyset$, then $(\mathbf{T}(A \sqcap B))^I \subseteq (\mathbf{T}(A))^I$

and, in this form, it is a rephrasing of property $(f_{\mathbf{T}} - R)$, in the semantics with selection function of the operator $\mathbf{T}$ studied in [27] (Appendix A) for $\mathcal{ALC} + \mathbf{T}_R$. This property has a syntactic counterpart in the axiom $\exists U.(\mathbf{T}(A) \sqcap B) \sqcap \mathbf{T}(A \sqcap B) \sqsubseteq \mathbf{T}(A)$, which holds in all the ranked models.

Consider the following example of knowledge base, stating that: typical Italians have black hair; typical students are young; they hate math, unless they are nerd (in which case they love math); all Mary's friends are typical students. We also have the assertions stating that Mary is a student, that Mario is an Italian student, and is a friend of Mary, Luigi is a typical Italian student, and Paul is a typical young student.

*Example 1.* $TBox$:

$(a)$ $\mathbf{T}(Italian) \sqsubseteq \exists hasHair.\{Black\}$
$(b)$ $\mathbf{T}(Student) \sqsubseteq Young$
$(c)$ $\mathbf{T}(Student) \sqsubseteq MathHater$
$(d)$ $\mathbf{T}(Student \sqcap Nerd) \sqsubseteq MathLover$
$(e)$ $\exists hasHair.\{Black\} \sqcap \exists hasHair.\{Blond\} \sqsubseteq \bot$
$(f)$ $MathLover \sqcap MathHater \sqsubseteq \bot$
$(g)$ $\exists friendOf.\{mary\} \sqsubseteq \mathbf{T}(Student)$

$ABox$:

$Student(mary)$, $friendOf(mario, mary)$, $(Student \sqcap Italian)(mario)$,
$\mathbf{T}(Student \sqcap Italian)(luigi)$, $\mathbf{T}(Student \sqcap Young)(paul)$, $\mathbf{T}(Student \sqcap Nerd)(tom)$

The fact that concepts $\mathbf{T}(C)$ can occur anywhere (apart from being nested in a $\mathbf{T}$ operator) can be used, e.g., to state that typical working students inherit properties of typical students ($\mathbf{T}(Student \sqcap Worker) \sqsubseteq \mathbf{T}(Student)$), in a situation in which typical students and typical workers have conflicting properties (e.g., as regards paying taxes). Also, we could state that there are typical students who are Italian: $\top \sqsubseteq \exists U.\mathbf{T}(Student \sqcap Italian)$, where $U$ is the universal role ($\top \times \top \sqsubseteq U$).

Standard DL inferences hold for $\mathbf{T}(C)$ concepts and $\mathbf{T}(C) \sqsubseteq D$ inclusions. For instance, we can conclude that Mario is a typical student (by (g)) and young (by (b)). However, by the properties of defeasible inclusions, Luigi, who is a typical Italian student, and Paul, who is a typical young student, both inherit the property of typical students of being math haters, respectively, by rational monotonicity (RM) and by cautious monotonicity (CM). Instead, as Tom is a typical nerd student, and typical nerd student are math lovers, this specific property of typical nerd students prevails over the less specific property of typical students of hating math. So we can consistently conclude that Tom is a $MathLover$.

A normal form for $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ knowledge bases can be defined. A KB in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ is in *normal form* if it admits all the axioms of a $\mathcal{SROEL}(\sqcap, \times)$ KB in normal form:

$$C(a) \qquad R(a,b) \qquad A \sqsubseteq \bot \qquad \top \sqsubseteq C \qquad A \sqsubseteq \{c\}$$
$$A \sqsubseteq C \qquad A \sqcap B \sqsubseteq C \qquad \exists R.A \sqsubseteq C \qquad A \sqsubseteq \exists R.B$$
$$\{a\} \sqsubseteq C \qquad \exists R.Self \sqsubseteq C \qquad A \sqsubseteq \exists R.Self$$

$$R \sqsubseteq T \qquad R \circ S \sqsubseteq T \qquad R \sqcap S \sqsubseteq T \qquad A \times B \sqsubseteq R \qquad R \sqsubseteq C \times D$$

(where $A, B, C, D \in N_C$, $R, S, T \in N_R$ and $a, b, c \in N_I$) and, in addition, it admits axioms of the form: $A \sqsubseteq T(B)$ and $T(B) \sqsubseteq C$ with $A, B, C \in N_C$. Extending the results in [1] and in [32], it is easy to see that, given a $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ KB, a semantically equivalent KB in normal form (over an extended signature) can be computed in linear time. In essence, for each concept $\mathbf{T}(C)$ occurring in the KB, we introduce two new concept names, $X_C$ and $Y_C$. A new KB is obtained by replacing all the occurrences of $\mathbf{T}(C)$ with $X_C$ in all the inclusions and assertions, and adding the following additional inclusion axioms:

$$X_C \sqsubseteq \mathbf{T}(Y_C), \;\; \mathbf{T}(Y_C) \sqsubseteq X_C, \;\; Y_C \sqsubseteq C, \;\; C \sqsubseteq Y_C$$

Then the new KB undergoes the normal form transformation for $\mathcal{SROEL}(\sqcap, \times)$ [32]. The resulting KB is linear in the size of the original one.

*Example 2.* Considering again the TBox in Example 1, inclusion $(a)$ $\mathbf{T}(Italian) \sqsubseteq \exists hasHair.\{Black\}$ is transformed in the following set of inclusions:
$(a_1)$ $X_I \sqsubseteq \exists hasHair.\{Black\}$
$(a_2)$ $X_I \sqsubseteq \mathbf{T}(Italian)$
$(a_3)$ $\mathbf{T}(Italian) \sqsubseteq X_I$
Inclusion $(d)$ $\mathbf{T}(Student \sqcap Nerd) \sqsubseteq MathLover$ is mapped to the set of inclusions:
$(d_1)$ $X_{SN} \sqsubseteq MathLover$
$(d_2)$ $X_{SN} \sqsubseteq \mathbf{T}(Y_{SN})$
$(d_3)$ $\mathbf{T}(Y_{SN}) \sqsubseteq X_{SN}$
$(d_4)$ $Student \sqcap Nerd \sqsubseteq Y_{SN}$
$(d_5)$ $Y_{SN} \sqsubseteq Student \sqcap Nerd$
Then $(a_1)$ is transformed further (the normal form transformation for $\mathcal{SROEL}(\sqcap, \times)$) into: $(a_1')$ $X_I \sqsubseteq \exists hasHair.B$ $\quad (a_1'')$ $B \sqsubseteq \{Black\}$

All the other axioms in the TBox, apart from (b) and (c), have to be transformed in normal form. Assertions are also subject to the normal form transformation. For instance, $\mathbf{T}(Student \sqcap Nerd)(tom)$ becomes $X_{SN}(tom)$, where $X_{SN}$ is one of the concept names introduced above.

## 3 Minimal entailment

In Example 1, we cannot conclude that all typical young Italians have black hair (and that Luigi has black hair) using rational monotonicity, as we do not know whether there is some typical Italian who is young. To supports such a stronger nonmonotonic inference, a minimal model semantics is needed to select those interpretations where individuals are as typical as possible. Among models of a KB, we select the minimal ones according to the following *preference relation* $\prec$ *over the set of ranked interpretations*. An interpretation $\mathcal{M} = \langle \Delta, <, I \rangle$ *is preferred to* $\mathcal{M}' = \langle \Delta', <', I' \rangle$ ($\mathcal{M} \prec \mathcal{M}'$) if: $\Delta = \Delta'$; $C^I = C^{I'}$ for all non-extended concepts $C$; for all $x \in \Delta$, $k_{\mathcal{M}}(x) \leq k_{\mathcal{M}'}(x)$, and there exists $y \in \Delta$ such that $k_{\mathcal{M}}(y) < k_{\mathcal{M}'}(y)$.

We can see that, in all the minimal models of the KB in Example 1 $luigi$ is an instance of the concept $\exists hasHair.\{Black\}$ and the inclusion $\mathbf{T}(Young \sqcap Italian) \sqsubseteq$

$\exists hasHair.\{Black\}$ is satisfied, as nothing prevents a $Young \sqcap Italian$ individual from having rank $0$.

In particular, we consider the notion of minimal canonical model defined in [27] to capture rational closure of an $\mathcal{ALC}$ KB extended with typicality. The requirement of a model to be canonical is used to guarantee that models contain enough individuals. Given a KB $K$ and a query $F$, let $\mathcal{S}$ be the set of all the (non-extended) concepts (and subconcepts) occurring in $K$ or $F$ together with their complements ($\mathcal{S}$ is finite). In the following, we will assume that all concepts occurring in the query $F$ are included in $K$.

**Definition 3 (Canonical models).** *A model $\mathcal{M} = \langle \Delta, <, I \rangle$ of $K$ is* canonical *if, for each set of $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ concepts $\{C_1, C_2, \ldots, C_n\} \subseteq \mathcal{S}$ consistent with $K$ (i.e., s.t. $K \not\models_{sroelrt} C_1 \sqcap C_2 \sqcap \ldots \sqcap C_n \sqsubseteq \bot$), there exists (at least) a domain element $x \in \Delta$ such that $x \in (C_1 \sqcap C_2 \sqcap \ldots \sqcap C_n)^I$.*

Among canonical models, we select the minimal ones.

**Definition 4.** *$\mathcal{M}$ is a* minimal canonical model *of $K$ if it is a canonical model of $K$ and it is minimal with respect to the preference relation $\prec$.*

**Definition 5 (Minimal entailment).** *Given a query $F$, $F$* is minimally entailed by *$K$, written $K \models_{min} F$ if, for all minimal canonical models $\mathcal{M}$ of $K$, $\mathcal{M}$ satisfies $F$.*

We can show that the problem of instance checking in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ under minimal entailment is CONP-hard. The proof is based on a reduction from tautology checking of propositional 3DNF formulae to instance checking in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ and its structure has similarities with the proof of CO-NP-hardness for $\mathcal{FL}$ subsumption in [2] (Chapter 3, Theorem 3.2). Given an alphabet of propositional variables $L = \{p_1, \ldots, p_k\}$, let $\gamma = G_1 \vee \ldots \vee G_n$ be a propositional formula where each disjunct $G_i = l_i^1 \wedge l_i^2 \wedge l_i^3$ ($i = 1, \ldots, n$) is the conjunction of three literals and each literal $l_i^j$ ($j = 1, \ldots, 3$) is either a variable $p \in L$ or its negation $\neg p$. The 3DNF tautology problem, i.e. the problem of deciding whether $\gamma$ is a tautology (in the propositional calculus), is known to be CONP-complete [17].

**Theorem 1.** *Instance checking in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ under minimal entailment is* CONP-*hard.*

*Proof. (sketch)* Given an alphabet of propositional variables $L = \{p_1, \ldots, p_k\}$ and a propositional formula in 3DNF $\gamma = G_1 \vee \ldots \vee G_n$ as defined above, we define a KB $K = (TBox, RBox, ABox)$ in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ as follows. We introduce in $N_C$ two concept names $P_h, \overline{P}_h$ for each variable $p_h \in L$, a concept name $D_\gamma$ associated with the formula $\gamma$ and a new concept name $E$. Let $R \in N_R$ be a role name and $a \in N_I$ be an individual name. We define $K$ as follows: $RBox = \{P_h \times \overline{P}_h \sqsubseteq R, h = 1, ..., k\}$, $ABox = \{\mathbf{T}(P_h \sqcap \overline{P}_h)(a), h = 1, ..., k\} \cup \{\mathbf{T}(E)(a)\}$, and $TBox$ contains the following inclusions:

(1) $\mathbf{T}(P_h) \sqcap \mathbf{T}(\overline{P}_h) \sqsubseteq \bot$,
(2) $\mathbf{T}(\top) \sqcap \exists R.\mathbf{T}(\top) \sqsubseteq \bot$
(3) $\mathbf{T}(E) \sqcap C_i^1 \sqcap C_i^2 \sqcap C_i^3 \sqsubseteq D_\gamma$, for each $G_i = l_i^1 \wedge l_i^2 \wedge l_i^3$

where $h = 1, \ldots, k$ and, for each $i = 1, \ldots, n$ and $j \in \{1, 2, 3\}$, $C_i^j$ is defined as follows:

$$C_i^j = \begin{cases} \mathbf{T}(P_h) & \text{if } l_i^j = p_h \\ \exists U.(\mathbf{T}(\top) \sqcap \mathbf{T}(P_h)) & \text{if } l_i^j = \neg p_h \end{cases}$$

Let us consider any model $\mathcal{M} = \langle \Delta, <, \cdot^I \rangle$ of $K$. Observe that, as $a^I \in P_h \sqcap \overline{P}_h$, $a^I$ cannot have rank 0, otherwise it would be both a typical $P_h$ and a typical $\overline{P}_h$, falsifying (1). By the role inclusions each $P_h$ element is in relation $R$ with any $\overline{P}_h$ element. Also, by (2), there cannot be a $P_h$ element $x$ and a $\overline{P}_h$ element $y$ both with rank 0, otherwise $x$ and $y$ would be related by $R$ and axiom (2) excludes that two $\mathbf{T}(\top)$ elements are in relation $R$. It is possible that, in a model of $K$, there are no $P_h$ elements with rank 0 and no $\overline{P}_h$ elements with rank 0. However, if we consider minimal canonical models of $K$, there must be either a $P_h$ element or a $\overline{P}_h$ element with rank 0.

Remember that $k_{\mathcal{M}}(C)$ is the rank of a concept $C$ in a ranked model $\mathcal{M}$. It can be seen that, in all the minimal canonical models of $K$, for all $h = 1, \ldots, k$, the following conditions hold:

(i) either $k_{\mathcal{M}}(P_h) = 0$ or $k_{\mathcal{M}}(\overline{P}_h) = 0$;
(ii) $k_{\mathcal{M}}(P_h \sqcap \overline{P}_h) = 1$ and $k_{\mathcal{M}}(a^I) = 1$.

As a consequence, $a^I$ is either a typical $P_h$ element (when the rank of $\overline{P}_h$ is 0) or a typical $\overline{P}_h$ (when the rank of $P_h$ is 0). So there are alternative minimal canonical models in which, for each $h$, $a^I$ is either a $\mathbf{T}(P_h)$, and in this case there exists a typical $\overline{P}_h$ element with rank 0; or $a$ is a $\mathbf{T}(\overline{P}_h)$, and in this case there exists a typical $P_h$ element with rank 0. Therefore, in any minimal canonical models $\mathcal{M}$ of $K$: either $a^I \in (\mathbf{T}(P_h))^I$ or $a^I \in (\exists U.(\mathbf{T}(\top) \sqcap \mathbf{T}(P_h)))^I$ (but not both). Then for $a^I$ the two concepts in the definition of $C_i^j$ are disjoint and complementary and the following can be proved:

$K \models_{min} D_\gamma(a)$ if and only if $\gamma$ is a tautology □

It is an open issue whether a similar proof can be done also for simple knowledge bases (i.e., for $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ knowledge bases where the typicality operator only occurs on the left hand side of concept inclusions $\mathbf{T}(C) \sqsubseteq D$). For simple KBs, it was proved for $\mathcal{ALC} + \mathbf{T}_R$ [27] that all minimal canonical models of the KB assign the same ranks to concepts, namely, the ranks determined by the rational closure construction. This is clearly true, in particular, for the fragment of $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ included in the language of $\mathcal{ALC}$ plus typicality (which however, does not contain nominals, role inclusions, and other constructs of $\mathcal{SROEL}(\sqcap, \times)$).

Note that $K$, in the proof above, has alternative minimal canonical models with incomparable rank assignments. The existence of alternative minimal models for a KB with free occurrences of typicality in the propositional case was observed in [9] for Propositional Typicality logic (PTL). As an example of a KB in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ with alternative minimal canonical models with incomparable rank assignments, consider $K' = (TBox', RBox', ABox')$, where $RBox' = \{P \times \overline{P} \sqsubseteq R\}$, $ABox' = \{\mathbf{T}(P \sqcap \overline{P})(a)\}$ and $TBox$ contains the inclusion $\mathbf{T}(P) \sqcap \mathbf{T}(\overline{P}) \sqsubseteq \bot$ and $\mathbf{T}(\top) \sqcap \exists R.\mathbf{T}(\top) \sqsubseteq \bot$ (meaning that two elements of rank 0 cannot be related by $R$). Consider the following two canonical models $\mathcal{M}_1$, $\mathcal{M}_2$ of $K'$, over the domain $\Delta = \{x, y, z, w\}$, where, for $i = 1, 2$, $P^{I_i} = \{y, z\}$, $\overline{P}^{I_i} = \{z, w\}$, $R^{I_i} = \{(z, z), (z, w), (y, z), (y, w)\}$ and

$a^{I_i} = z$. Furthermore, concerning the rankings, for $\mathcal{M}_1$, $k_{\mathcal{M}_1}(x) = k_{\mathcal{M}_1}(y) = 0$, $k_{\mathcal{M}_1}(z) = k_{\mathcal{M}_1}(w) = 1$; for $\mathcal{M}_2$, $k_{\mathcal{M}_2}(x) = k_{\mathcal{M}_2}(w) = 0$, $k_{\mathcal{M}_2}(z) = k_{\mathcal{M}_2}(y) = 1$. $\mathcal{M}_1$ and $\mathcal{M}_2$ are both minimal canonical models of $K'$ and have incomparable rankings, with $P$ having rank 0 in $\mathcal{M}_1$ and rank 1 in $\mathcal{M}_2$.

## 4 Deciding rational entailment in polynomial time

While instance checking in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ under minimal entailment is coNP-hard, in this section we prove that instance checking under rational entailment can be decided in polynomial time for normalized KBs, by defining a translation of a normalized KB into a set of Datalog rules, whose grounding is polynomial in the size of the KB. In particular, we extend the Datalog materialization calculus for $\mathcal{SROEL}(\sqcap, \times)$, proposed by Krötzsch [32], to deal with typicality concepts and with instance checking under rational entailment in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$.

The calculus in [32] uses predicates $inst(a, C)$ (whose meaning includes: the individual $a$ is an instance of concept name $C$, see [33] for details), $triple(a, R, b)$ ($a$ is in relation $R$ with $b$), $self(a, R)$ ($a$ is in relation $R$ with itself). To map a $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ KB to a Datalog program, we add predicates to represent that: an individual $a$ is a typical instance of a concept name ($typ(a, C)$); the ranks of two individuals $a$ and $b$ are the same ($sameRank(a, b)$); the rank of $a$ is less or equal than the one of $b$ ($leqRank(a, b)$).

Besides the constants for individuals in $N_I$ (which are assumed to be finitely many), the calculus in [32] exploits auxiliary constants $aux^{A \sqsubseteq \exists R.C}$ (one for each inclusion of the form $A \sqsubseteq \exists R.C$) to deal with existential restriction. We also need to introduce an auxiliary constant $aux_C$ for any concept $\mathbf{T}(C)$ occurring in the KB or in the query, used as a representative typical $C$, in case $C$ is non-empty.

Given a normalized KB $K = (TBox, RBox, ABox)$ and query $Q$ of the form $C(a)$ or $\mathbf{T}(C)(a)$, where $C$ is a concept name in the normalized KB, the Datalog program for instance checking in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$, i.e. for querying whether $K \models_{sroelrt} Q$, is a program $\Pi(K)$, the union of:

1. $\Pi_K$, the representation of $K$ as a set of Datalog facts, based on the input translation in [32];
2. $\Pi_{IR}$, the inference rules of the basic calculus in [32];
3. $\Pi_{RT}$, containing the additional rules for reasoning with typicality in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$.

A query $Q$ of the form $\mathbf{T}(C)(a)$, or $C(a)$, is mapped to a goal $G_Q$ of the form $typ(a, C)$, or $inst(a, C)$. Observe that restricting queries to concept names is not a severe restriction as an arbitrary query $C(b)$ can be replaced by a query $A(b)$ with $A$ new concept name, by adding $C \sqsubseteq A$ to the TBox [1] and, of course, this inclusion is normalized when normalizing TBox.

We define $\Pi(K)$ in such a way that $G_Q$ is derivable in Datalog from $\Pi(K)$ (written $\Pi(K) \vdash G_Q$) if and only if $K \models_{sroelrt} Q$.

$\Pi_K$ includes the result of the input translation in section 3 in [32] where $nom(a)$, $cls(A)$, $rol(R)$ are used for $a \in N_I$ , $A \in N_C$, $R \in N_R$, and, for example:

- $subClass(a, C)$, $subClass(A, c)$, $subClass(A, C)$ are used for $C(a)$, $A \sqsubseteq \{c\}$, $A \sqsubseteq C$;
- $subEx(R, A, C)$ is used for $\exists R.A \sqsubseteq C$ ;

and similar statements represent other axioms in the normalized KB.

The following is the additional mapping for the extended syntax of the $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ normal form (note that no mapping is needed for assertions $\mathbf{T}(C)(a)$, as they do not occur in a normalized KB):

$$\begin{array}{ll} A \sqsubseteq \mathbf{T}(B) & \mapsto supTyp(A, B) \\ \mathbf{T}(B) \sqsubseteq C & \mapsto subTyp(B, C) \end{array}$$

Also, we need to add $top(\top)$ to the input specification.

$\Pi_{IR}$ contains all the inference rules from [32][1]:

(1) $inst(x, x) \leftarrow nom(x)$
(2) $self(x, v) \leftarrow nom(x), triple(x, v, x)$
(3) $inst(x, z) \leftarrow top(z), inst(x, z')$
(4) $inst(x, y) \leftarrow bot(z), inst(u, z), inst(x, z'), cls(y)$
(5) $inst(x, z) \leftarrow subClass(y, z), inst(x, y)$
(6) $inst(x, z) \leftarrow subConj(y1, y2, z), inst(x, y1), inst(x, y2)$
(7) $inst(x, z) \leftarrow subEx(v, y, z), triple(x, v, x'), inst(x', y)$
(8) $inst(x, z) \leftarrow subEx(v, y, z), self(x, v), inst(x, y)$
(9) $triple(x, v, x') \leftarrow supEx(y, v, z, x'), inst(x, y)$
(10) $inst(x', z) \leftarrow supEx(y, v, z, x'), inst(x, y)$
(11) $inst(x, z) \leftarrow subSelf(v, z), self(x, v)$
(12) $self(x, v) \leftarrow supSelf(y, v), inst(x, y)$
(13) $triple(x, w, x') \leftarrow subRole(v, w), triple(x, v, x')$
(14) $self(x, w) \leftarrow subRole(v, w), self(x, v)$
(15) $triple(x, w, x'') \leftarrow subRChain(u, v, w), triple(x, u, x'), triple(x', v, x'')$
(16) $triple(x, w, x') \leftarrow subRChain(u, v, w), self(x, u), triple(x, v, x')$
(17) $triple(x, w, x') \leftarrow subRChain(u, v, w), triple(x, u, x'), self(x', v)$
(18) $triple(x, w, x) \leftarrow subRChain(u, v, w), self(x, u), self(x, v)$
(19) $triple(x, w, x') \leftarrow subRConj(v1, v2, w), triple(x, v1, x'), triple(x, v2, x')$
(20) $self(x, w) \leftarrow subRConj(v1, v2, w), self(x, v1), self(x, v2)$
(21) $triple(x, w, x') \leftarrow subProd(y1, y2, w), inst(x, y1), inst(x', y2)$
(22) $self(x, w) \leftarrow subProd(y1, y2, w), inst(x, y1), inst(x, y2)$
(23) $inst(x, z1) \leftarrow supProd(v, z1, z2), triple(x, v, x')$
(24) $inst(x, z1) \leftarrow supProd(v, z1, z2), self(x, v)$
(25) $inst(x', z2) \leftarrow supProd(v, z1, z2), triple(x, v, x')$
(26) $inst(x, z2) \leftarrow supProd(v, z1, z2), self(x, v)$
(27) $inst(y, z) \leftarrow inst(x, y), nom(y), inst(x, z)$
(28) $inst(x, z) \leftarrow inst(x, y), nom(y), inst(y, z)$
(29) $triple(z, u, y) \leftarrow inst(x, y), nom(y), triple(z, u, x)$

[1] Here, $u, v, x, y, z, w$, possibly with suffixes, are variables.

Note that "statements $inst(a, b)$, with $a$ and $b$ individuals, encode equality of $a$ and $b$" [33].

$\Pi_{RT}$, i.e. the set of rules to deal with typicality, is as follows; it contains rules for $supTyp$ and $subTyp$ axioms, and rules that deal with the rank of domain elements. In the rules, $x, y, z, A, B, C$ are all Datalog variables.

$(SupTyp)$ $typ(x, z) \leftarrow supTyp(y, z), inst(x, y)$
$(SubTyp)$ $inst(x, z) \leftarrow subTyp(y, z), typ(x, y)$
$(Refl)$ $inst(x, y) \leftarrow typ(x, y)$
$(A0)$ $typ(aux_C, C) \leftarrow inst(x, C)$
$(A1)$ $leqRank(x, y) \leftarrow typ(x, B), inst(y, B)$
$(A2)$ $sameRank(x, y) \leftarrow typ(x, A), typ(y, A)$
$(A3)$ $typ(x, B) \leftarrow sameRank(x, y), inst(x, B), typ(y, B)$
$(A4)$ $typ(x, B) \leftarrow inst(x, A), supTyp(A, B)$
$(B1)$ $sameRank(x, z) \leftarrow sameRank(x, y), sameRank(y, z)$
$(B2)$ $sameRank(x, y) \leftarrow sameRank(y, x)$
$(B3)$ $sameRank(x, x) \leftarrow inst(x, T)$
$(B4)$ $leqRank(x, y) \leftarrow sameRank(y, x)$
$(B5)$ $leqRank(x, z) \leftarrow leqRank(x, y), leqRank(y, z)$
$(B6)$ $sameRank(x, y) \leftarrow leqRank(x, y), leqRank(y, x)$
$(B7)$ $sameRank(x, y) \leftarrow nom(y), inst(x, y)$

Rule $(Refl)$ corresponds to the reflexivity property (see Section 2). Rules $(A0) - (A4)$ encode properties of ranked models: if there is a $C$ element, there must be a typical $C$ element $(A0)$; a typical $B$ element has a rank less or equal to the rank of any $B$ element $(A1)$; two elements which are both typical $A$ elements have the same rank $(A2)$; if $x$ is a B element and has the same rank as a typical $B$ element, $x$ is also a typical $B$ element $(A3)$; if $x$ is an $A$ element and all $A$'s are typical $B$'s, then $x$ is a typical $A$ $(A4)$. $(B1) - (B7)$ define properties of rank order. In particular, by $(B7)$, two constants that correspond to the same domain element have the same rank.

The semantic properties of rational consequence relation introduced in Section 2 are enforced by the specification above. Consider, for instance, $(CM)$. Suppose that $subTyp(A, B)$ and $subTyp(A, C)$ are in $\Pi_K$ (as $\mathbf{T}(A) \sqsubseteq B$, $\mathbf{T}(A) \sqsubseteq C$ are in $K$) and that $D$ is a concept name defined to be equivalent to $A \sqcap B$ in $K$. Suppose that $typ(a, D)$ holds. One can infer $typ(a, A)$ and hence $inst(a, C)$, i.e., typical $A \sqcap B$'s inherit from typical $A$'s the property of being $C$'s (the inference for $Paul$ in Example 1). In fact, $typ(a, A)$ is inferred showing that $a$ (who is a typical $D$ and an $A$, as it is a $D$) and $aux_A$ (who is a typical $A$, by $(A1)$, and a $D$, since all the typical $A$'s are also $B$'s and hence $A \sqcap B$'s) have the same rank. In fact, using$(A1)$ twice, one can conclude both $leqRank(a, aux_A)$ and $leqRank(aux_A, a)$ so that, by $(B6)$, $sameRank(a, aux_A)$. Then, by $(A3)$, we infer $typ(a, A)$. With rule $(subTyp)$, from $typ(a, A)$ and $subTyp(A, C)$, we conclude $inst(a, C)$.

Reasoning in a similar way, one can see that also the properties $(RM)$ and $(LLE)$ are enforced by the rules above. In particular, for (RM), we can show that: from the fact that there is a domain element $a$ who is a $\mathbf{T}(A)$ and a $C$ element (i.e. $typ(a, A)$ and $inst(a, C)$ hold), and from the fact that there is a $b$ who is a typical $A \sqcap C$ element (i.e. that $typ(b, D)$ holds, for some concept $D$ equivalent to $A \sqcap C$), we can conclude that $b$ is also a typical $A$ element (i.e. $typ(b, A)$ holds). Inference in $\mathcal{SROEL}(\sqcap, \times)$ already takes care of the semantic properties of conjunctive consequences $(And)$ and right weakening $(RW)$.

**Theorem 2.** *For a $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ KB in normal form $K$, and a query $Q$ of the form $\mathbf{T}(C)(a)$ or $C(a)$, $K \models_{sroelrt} Q$ if and only if $\Pi(K) \vdash G_Q$.*

*Proof.* ***(sketch)*** For completeness, we procede by contraposition, similarly to [33]. Assume that $inst(a, C)$ (respectively, $typ(a, C)$) is not derivable from $\Pi(K)$. Let $J$ be the minimal Herbrand model of the Datalog program $\Pi(K)$; then $inst(a, C) \notin J$ (resp. $typ(a, C) \notin J$). From $J$ we build a ranked model $\mathcal{M}$ for $K$ such that $C(a)$ (respectively, $\mathbf{T}(C)(a)$) is not satisfied in $\mathcal{M}$. As in [33], we can build the domain $\Delta$ of $\mathcal{M}$ from the set $Const$ including all the name constants $c \in N_I$ occurring in the ASP program $\Pi(K)$ as well as all the auxiliary constants, then defining an equivalence relation $\approx$ over constants and the domain $\Delta$ including the equivalence classes and, possibly, additional domain elements for auxiliary constants, as in the proof of Lemma 3 in [33]. $J$ contains all the details about the interpretation of concepts and roles, from which an interpretation $\mathcal{M}$ can be defined (for instance, for $c \in N_I$, $[c] \in A^I$ iff $inst(c, A) \in J$, and similarly for other domain elements and for roles). However, predicates $sameRank$ and $leqRank$ only provide partial information about the ranks of the domain elements. We define a relation $<$ over constants, letting $x < y$ iff there is a concept name $C$, s.t. $typ(x, C), inst(y, C) \in J$ and $typ(y, C) \notin J$ and we show that its transitive closure is a strict partial order. Also, we show that $<$ is compatible with the $sameRank$ predicate in $J$ and with the $\approx$ equivalence relation between constants so that $<$ can be extended to a modular partial order over the domain $\Delta$. First, a partial ordering over elements in $\Delta$ is defined, letting $[c] < [d]$ iff $c < d$ (where the definition does not depend on the choice of the representative element in a class) and similarly for domain elements corresponding to auxiliary constants. Then the elements in $\Delta$ are partitioned into the sets $Rank_0, \ldots, Rank_n$, where $Rank_i$ (the set of domain elements of rank $i$) is defined by induction on $i$, as follows: $Rank_0$ contains all the elements $x \in \Delta$ such that there is no $y \in \Delta$ with $y < x$; $Rank_i$ contains all the elements $x \in \Delta - (Rank_0 \cup \ldots \cup Rank_{i-1})$ such that there is no $y \in \Delta - (Rank_0 \cup \ldots \cup Rank_{i-1})$ with $y < x$. We let $n$ be the least integer such that $\Delta - (Rank_0 \cup \ldots \cup Rank_n) = \emptyset$. It can be shown that $\mathcal{M}$ is a model of $K$ and it does not satisfy $C(a)$ (respectively, $\mathbf{T}(C)(a)$).

Proving the soundness of the Datalog encoding, requires showing that, if $\Pi(K) \vdash G_Q$, for a query $Q$ of the form $\mathbf{T}(C)(a)$ or $C(a)$, then, $Q$ is a logical consequence of $K$. The proof is similar to the proof of Lemma 1 in [33]. First we associate to each constant $c$ of the Datalog program $\Pi(K)$ a concept expression $\kappa(c)$ a follows:

if $c \in N_I$ then $\kappa(c) = \{c\}$;
if $c = aux^{\alpha}$, for $\alpha = A \sqsubseteq \exists R.B$, then $\kappa(c) = B \sqcap \exists R^{-}.A$;
if $c = aux_C$, then $\kappa(c) = \mathbf{T}(C)$.

The following statements:

- if $\Pi(K) \vdash inst(c, A)$, for $A \in N_C$, then $K \models_{sroelrt} \kappa(c) \sqsubseteq A$;
- if $\Pi(K) \vdash inst(c, d)$, for $d \in N_I$, then $K \models_{sroelrt} \kappa(c) \sqsubseteq \{d\}$;
- if $\Pi(K) \vdash typ(a, A)$, then $K \models_{sroelrt} \kappa(c) \sqsubseteq \mathbf{T}(A)$;

- if $\Pi(K) \vdash triple(c, R, d)$, then $K \models_{sroelrt} \kappa(c) \sqsubseteq \exists R.\kappa(d)$;
- if $\Pi(K) \vdash self(c, R)$, for $A \in N_C$, then $K \models_{sroelrt} \kappa(c) \sqsubseteq \exists R.Self$;
- if $\Pi(K) \vdash sameRank(c, d)$ then for all models $\mathcal{M}$ of $K$, $k_{\mathcal{M}}(c^I) = k_{\mathcal{M}}(d^I)$;
- if $\Pi(K) \vdash leqRank(c, d)$ then, for all models $\mathcal{M}$ of $K$, $k_{\mathcal{M}}(c^I) \leq k_{\mathcal{M}}(d^I)$.

can be proved by induction on the height of the derivation tree of each atom from the program $\Pi(K)$. □

$\Pi(K)$ contains a polynomial number of rules and exploits a polynomial number of concepts in the size of $K$, hence instance checking in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ can be decided in polynomial time using the calculus in Datalog. The encoding can be processed, e.g., in an ASP solver such as Clingo or DLV (with the proper capitalization of variables); computation of the (unique, in this case) answer set takes a negligible time for KBs with a hundred assertions (half of them with $\mathbf{T}$).

Exploiting the approach presented in [32], a version of the Datalog specification where predicates $inst$, $typ$, $triple$ and $self$ have an additional parameter (and is therefore less efficient than the previous one, although polynomial) can be used to check subsumption for $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$.

For simple $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ knowledge bases, i.e., for KBs where the typicality operator only occurs on the left hand side of inclusions, the materialization calculus for subsumption can be used to construct the rational closure of TBox, adopting the construction in [27] (Definitions 21 and 23). Such construction can be rephrased replacing the exceptionality check in $\mathcal{ALC} + \mathbf{T}_R$ with the exceptionality check in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ and the entailment in $\mathcal{ALC} + \mathbf{T}_R$ with the entailment in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$. In particular, in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ one can define, for a simple KB $K$, the notion of exceptionality as follows: *$C$ is exceptional wrt $K$* iff $K \models_{sroelrt} \mathbf{T}(\top) \sqcap C \sqsubseteq \bot$. This subsumption is not in the language of normalized KBs, but it can be replaced by the subsumption $A \sqsubseteq \bot$, adding $\mathbf{T}(\top) \sqsubseteq X$ and $X \sqcap C \sqsubseteq A$ to $K$. The construction requires a quadratic number of subsumption checks (in the number of typicality inclusions in the KB, and, hence, in the size of the KB), each one requiring polynomial time, using the above mentioned polynomial calculus for subsumption.

The correspondence between the rational closure construction and the canonical minimal model semantics in [27], does not extend to all the constructs in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ and, specifically, the canonical model semantics is not adequate for dealing with nominals. In particular, there are knowledge bases with no canonical model and knowledge bases with more than one minimal canonical model (as the knowledge base $K'$ at the end of Section 3). However, in many cases, the rational closure of a KB with no canonical model is still meaningful. What has to be devised is, on the one hand, a less restrictive semantic requirement to give meaning also to KBs containing nominals; on the other hand, a syntactic condition to identify the KBs for which the rational closure is by itself meaningful and corresponds to the semantics. In this paper, we do not address these issues and we leave them for further work.

## 5 Related Work

Among the recent nonmonotonic extensions of DLs are the formalisms for combining DLs with logic programming rules, such as for instance, [16, 15], [35], [30] and Datalog +/- [28]. DL-programs in [16, 15] support a loose coupling of DL ontologies and rule-based reasoning under the answer set semantics and under the well-founded semantics, where rules may contain DL-atoms in their bodies, corresponding to queries to a DL ontology, which can be modified according to an input list of updates. In [30] a general DL language is introduced, which extends $\mathcal{SROIQ}$ with nominal schemas and epistemic operators according to the MKNF semantics [35], which encompasses some of the most prominent nonmonotonic rule languages, including ASP. In [5] a non monotonic extension of DLs is proposed based on a notion of overriding, supporting normality concepts and enjoying good computational properties. In particular, it preserves the tractability of low complexity DLs, including $\mathcal{EL}^{++}$ and $DL\text{-}lite$. In [10], the CKR framework is presented, which is based on *SROIQ-RL*, allows for defeasible axioms with local exceptions and a translation to Datalog with negation. It is shown that instance checking over a CKR reduces to (cautious) inference under the answer sets semantics.

Preferential extensions of low complexity DLs in the $\mathcal{EL}$ and DL-lite families have been studied In [24, 25], based on preferential interpretations which are not required to be modular, and tableaux-based proof methods have been developed for them. In [25], for a preferential extension of $\mathcal{EL}^{\bot}$ based on a minimal model semantics different from the one in this paper, it is shown that minimal entailment is EXPTIME-hard already for simple KBs, similarly to what happens for circumscriptive KBs [6].

## 6 Conclusions

In this paper we defined a rational extension $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\,\mathbf{T}$ of the low complexity description logic $\mathcal{SROEL}(\sqcap, \times)$, which underlies the OWL EL ontology language, introducing a typicality operator. For general KBs, we have shown that minimal entailment in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ is CONP-hard. When free occurrences of typicality concepts in concept inclusions are allowed, alternative minimal models may exist with different rank assignments to concepts. In [9] this phenomenon has been analyzed in the context of PTL, considering alternative preference relations over ranked interpretations which coincide over simple KBs but, for general ones, define different notions of entailment satisfying alternative and possibly incompatible postulates.

Building on the materialization calculus for $\mathcal{SROEL}(\sqcap, \times)$ in Datalog presented in [32], a calculus for instance checking and subsumption under rational entailment is defined, showing that these problems can be decided in polynomial time.

This result also provides a polynomial upper bound for the construction of the rational closure of a knowledge base in $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$. Although for the fragment of $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ which is also included in the language of $\mathcal{ALC} + \mathbf{T}_R$ in [27] the rational closure is semantically characterized by the minimal canonical models of the KB, a general semantic characterization of the rational closure for the logic $\mathcal{SROEL}(\sqcap, \times)$ is still missing.

Future work may also include optimizations, based on modularity as in [7], of the calculus for rational entailment, and the development of rule based inference methods for $\mathcal{SROEL}(\sqcap, \times)^{\mathbf{R}}\mathbf{T}$ minimal entailment based on model generation in ASP. An upper bound on the complexity of minimal entailment for general KBs has to be established. A further issue to understand is whether a materialization calculus can be defined also for the preferential extensions of DLs in the $\mathcal{EL}$ family in [24, 25], whose interpretations are not required to be modular.

Apart from providing a complexity upper bound, the Datalog encoding presented in this paper is intended to provide a way to integrate the use of $\mathcal{SROEL}(\sqcap, \times)$ KBs under rational entailment with other kinds of reasoning that can be performed in ASP, and, by extending the encoding to deal with alternative models of the KB, also to allow the experimentation of alternative notions of minimal entailment, as advocated in [9]. The approach can be possibly integrated with systems like DReW [37], that already exploits the mapping by Krötzsch for OWL 2 EL.

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## References


1. F. Baader, S. Brandt, and C. Lutz. Pushing the $\mathcal{EL}$ envelope. In *Proc IJCAI 2005*, pages 364–369, 2005.
2. F. Baader, D. Calvanese, D. L. McGuinness, D. Nardi, and P. F. Patel-Schneider, editors. *The Description Logic Handbook: Theory, Implementation, and Applications*. Cambridge University Press, 2003.
3. F. Baader and B. Hollunder. Embedding defaults into terminological knowledge representation formalisms. In *Proceedings of the 3rd International Conference on Principles of Knowledge Representation and Reasoning (KR'92). Cambridge, MA, October 25-29, 1992.*, pages 306–317, 1992.
4. F. Baader and B. Hollunder. Priorities on defaults with prerequisites, and their application in treating specificity in terminological default logic. *J. of Automated Reasoning*, 15(1):41–68, 1995.
5. P. A. Bonatti, M. Faella, I. Petrova, and L. Sauro. A new semantics for overriding in description logics. *Artif. Intell.*, 222:1–48, 2015.
6. P. A. Bonatti, M. Faella, and L. Sauro. Defeasible inclusions in low-complexity dls. *J. Artif. Intell. Res. (JAIR)*, 42:719–764, 2011.
7. P. A. Bonatti, I. M. Petrova, and L. Sauro. Optimizing the computation of overriding. In *Proc. ISWC 2015*, pages 356–372, 2015.
8. Piero A. Bonatti, Carsten Lutz, and Frank Wolter. The Complexity of Circumscription in DLs. *J. of Artificial Intelligence Research*, 35:717–773, 2009.
9. R. Booth, G. Casini, T. Meyer, and I. J. Varzinczak. On the entailment problem for a logic of typicality. In *Proc. IJCAI 2015*, pages 2805–2811, 2015.
10. L. Bozzato, T. Eiter, and L. Serafini. Contextualized knowledge repositories with justifiable exceptions. In *DL 2014*, pages 112–123, 2014.
11. K. Britz, J. Heidema, and T. Meyer. Semantic preferential subsumption. In G. Brewka and J. Lang, editors, *Proc. KR 2008*, pages 476–484, 2008.
12. G. Casini, T. Meyer, I. J. Varzinczak, , and K. Moodley. Nonmonotonic Reasoning in Description Logics: Rational Closure for the ABox. In *Proc. DL 2013*, pages 600–615, 2013.
13. G. Casini and U. Straccia. Rational Closure for Defeasible Description Logics. In *Proc. JELIA 2010*, LNAI 6341, pages 77–90. Springer, 2010.
14. F. M. Donini, D. Nardi, and R. Rosati. Description logics of minimal knowledge and negation as failure. *ACM Transactions on Computational Logic (ToCL)*, 3(2):177–225, 2002.

15. T. Eiter, G. Ianni, T. Lukasiewicz, and R. Schindlauer. Well-founded semantics for description logic programs in the semantic web. *ACM Trans. Comput. Log.*, 12(2):11, 2011.
16. T. Eiter, G. Ianni, T. Lukasiewicz, R. Schindlauer, and H. Tompits. Combining answer set programming with description logics for the semantic web. *Artif. Intell.*, 172(12-13):1495–1539, 2008.
17. M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, 1979.
18. M. Gelfond. *Handbook of Knowledge Representation, chapter 7, Answer Sets*. Elsevier, 2007.
19. M. Gelfond and N. Leone. Logic programming and knowledge representation - the A-Prolog perspective. *Artif. Intell.*, 138(1-2):3–38, 2002.
20. L. Giordano and Theseider Dupré D. Reasoning in a Rational Extension of SROEL. In *DL2016*, volume 1577 of *CEUR Workshop Proceedings*, 2016.
21. L. Giordano and V. Gliozzi. Encoding a preferential extension of the description logic *SROIQ* into *SROIQ*. In *Proc. ISMIS 2015*, volume 9384 of *LNCS*, pages 248–258. Springer, 2015.
22. L. Giordano, V. Gliozzi, N. Olivetti, and G. L. Pozzato. Preferential Description Logics. In *Proceedings of LPAR 2007*, volume 4790 of *LNAI*, pages 257–272. Springer-Verlag, 2007.
23. L. Giordano, V. Gliozzi, N. Olivetti, and G. L. Pozzato. ALC+T: a preferential extension of Description Logics. *Fundamenta Informaticae*, 96:1–32, 2009.
24. L. Giordano, V. Gliozzi, N. Olivetti, and G. L. Pozzato. Prototypical reasoning with low complexity description logics: Preliminary results. In *Proc. LPNMR 2009*, pages 430–436, 2009.
25. L. Giordano, V. Gliozzi, N. Olivetti, and G. L. Pozzato. Reasoning about typicality in low complexity DLs: the logics $\mathcal{EL}^{\perp}\mathbf{T}_{min}$ and *DL-Lite*$_c\mathbf{T}_{min}$. In Toby Walsh, editor, *Proceedings of the 22nd International Joint Conference on Artificial Intelligence (IJCAI 2011)*, pages 894–899, Barcelona, Spain, July 2011. Morgan Kaufmann.
26. L. Giordano, V. Gliozzi, N. Olivetti, and G. L. Pozzato. A NonMonotonic Description Logic for Reasoning About Typicality. *Artificial Intelligence*, 195:165–202, 2013.
27. L. Giordano, V. Gliozzi, N. Olivetti, and G. L. Pozzato. Semantic characterization of rational closure: From propositional logic to description logics. *Artif. Intell.*, 226:1–33, 2015.
28. G. Gottlob, A. Hernich, C. Kupke, and T. Lukasiewicz. Stable model semantics for guarded existential rules and description logics. In *Proc. KR 2014*, 2014.
29. P. Ke and U. Sattler. Next Steps for Description Logics of Minimal Knowledge and Negation as Failure. In *Proc. DL 2008*, volume 353 of *CEUR Workshop Proceedings*, 2008.
30. M. Knorr, P. Hitzler, and F. Maier. Reconciling OWL and non-monotonic rules for the semantic web. In *ECAI 2012*, page 474479, 2012.
31. S. Kraus, D. Lehmann, and M. Magidor. Nonmonotonic reasoning, preferential models and cumulative logics. *Artif. Intell.*, 44(1-2):167–207, 1990.
32. M. Krötzsch. Efficient inferencing for OWL EL. In *Proc. JELIA 2010*, pages 234–246, 2010.
33. M. Krötzsch. Efficient inferencing for the description logic underlying OWL EL. *Tech. Rep. 3005, Institute AIFB, Karlsruhe Institute of Technology*, 2010.
34. Daniel Lehmann and Menachem Magidor. What does a conditional knowledge base entail? *Artificial Intelligence*, 55(1):1–60, 1992.
35. Boris Motik and Riccardo Rosati. Reconciling Description Logics and rules. *J. ACM*, 57(5), 2010.
36. U. Straccia. Default inheritance reasoning in hybrid KL-ONE-style logics. In *Proc. IJCAI 1993*, pages 676–681, 1993.
37. G. Xiao, T. Eiter, and S. Heymans. The DReW system for nonmonotonic dl-programs. In *Proc. CSWS 2012*, pages 383–390, 2012.