Logics for Representation of Propositions with Fuzzy Modalities

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Abstract. In the paper we introduce logical calculi for representation of propositions with modal operators indexed by fuzzy values. There calculi are called Heyting-valued modal logics. We introduce the concept of a Heyting-valued Kripke model and consider a semantics of Heyting-valued modal logics at the class of Heyting-valued Kripke models.

1 Introduction

The formalism of propositional modal logic and its proof technique is one of the most powerful approaches for knowledge representation and reasoning about dynamic systems, databases, etc.

In the present paper we introduce more general modal propositional formalism, which allows to express propositions with modalities indexed by elements of a complete Heyting algebra. In this formalism any proposition A can be augmented with a modal operator of the form $\Box_a$, which can be interpreted, for example, as a value of necessity of A (or a value of confidence of A, or a value of plausibility of A, or a probability of A, or something else). This formalism can be considered as a logical foundation for

- reasoning about objects that are incomplete and inconsistent, such as databases with incomplete and unclear information,
- model checking for discrete models which are rough approximations of analyzed systems.

Mathematical approaches to representation of knowledges with taking into account an uncertainty and incompleteness of knowledges were considered in several papers, in particular, in [3]–[13]. The most of them are related to quantitative evaluation of uncertainty.

Uncertainty of information can appear by several causes.

1. An information under processing can be unclear, approximate, and not verified, and for correct processing of such information it is necessary to have a formalism for taking into account a value of reliability of information under processing.
2. If we investigate a complex system, such that its detail and exact representation is impossible, then we construct a rough model of this system, which has small complexity, and instead of this system we investigate its rough model.

But because the original system and its model are essentially not identical, then their properties can differ. Thus, for correct investigation of the system on the base of such model it is necessary to have an approach to evaluation the difference between properties of a rough model and properties of original system. Values of the difference can be not only quantitative, but also qualitative. For example, the set of such values can be a boolean algebra of subsets of some set of situations (i.e. states of an environment), in which the analyzed system does work. A value of equivalence between the system and its model (with respect to the properties under checking) can be defined for example as a set of situations in which these properties are equivalent for the original system and for its model. A value of truth of the properties under checking can be defined as a subset of this set, which consists of situations, in which the analysed properties does hold. These situations can be augmented by quantitative parameters (their weights, probabilities, etc.), and the set of such values can be more complex (if the sets of the parameters are totally ordered sets, then the set of values of truth is a Heyting algebra).

The main goal of the present paper is to construct a logical framework, which can serve as a logical foundation for representation of such uncertain information. The proposed formalism can be used also for design of specification languages of a behavior of dynamic systems with uncertain information about their structure and behavior, by analogy with the specification languages based on temporal logic for description of properties of program systems and electronic circuits ([2]). Some recent approaches to logic representation of propositions with fuzziness can be found in [16], [17].

The paper is organized as follows. In section 2 we introduce the syntax of Heyting-valued modal logics and define a minimal Heyting-valued modal logic HVK. In section 3 we introduce the concept of a Heyting-valued Kripke model and define the semantics of Heyting-valued modal formulas at the class of Heyting-valued Kripke models. We also consider an example of a Heyting-valued Kripke model related to description logics. In section 4 we introduce a concept of a canonical model of a Heyting-valued modal logic, and in section 5 we use the concept of a canonical model for the proof of completeness for minimal Heyting-valued modal logic HVK at the class of Heyting-valued Kripke models. In the conclusion we summarize the results of the paper and describe problems for future research.

2 Heyting-valued modal logics

2.1 Complete Heyting algebras

We shall assume that a set of fuzzy values which can occur in formulas of Heyting-valued modal logics has some algebraic properties, namely, it is a complete Heyting algebra. In this section we remind a definition of this concept.

A **complete lattice** is a partially ordered set $\mathcal{H}$, such that for every subset $Q \subseteq \mathcal{H}$ there are elements $\inf(Q)$ and $\sup(Q)$ of $\mathcal{H}$ such that for every $b \in \mathcal{H}$

$$(\forall q \in Q \quad b \leq q) \quad \Leftrightarrow \quad b \leq \inf(Q),$$

$$(\forall q \in Q \quad q \leq b) \quad \Leftrightarrow \quad \sup(Q) \leq b.$$

The elements $\inf(\mathcal{H})$ and $\sup(\mathcal{H})$ will be denoted by the symbols 0 and 1 respectively.

For every finite subset

$$Q = \{a_1, \ldots, a_n\} \subseteq \mathcal{H}$$

the elements $\inf(Q)$ and $\sup(Q)$ will be denoted by the symbols

$$a_1 \wedge \ldots \wedge a_n \quad \text{and} \quad a_1 \vee \ldots \vee a_n$$

respectively.

These elements will be denoted also by the symbols

$$\left\{\begin{array}{c} a_1 \\ \ldots \\ a_n \end{array}\right\} \quad \text{and} \quad \left[\begin{array}{c} a_1 \\ \ldots \\ a_n \end{array}\right]$$

respectively.

A **complete Heyting algebra** can be defined as a complete lattice $\mathcal{H}$, with a binary operation

$$\rightarrow: \ \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{H},$$

such that for every $a, b, c \in \mathcal{H}$

$$a \wedge b \leq c \quad \Leftrightarrow \quad a \leq b \rightarrow c \tag{1}$$

Below the symbol $\mathcal{H}$ denotes some fixed complete Heyting algebra.

For every $a, b \in \mathcal{H}$ the symbol $a \leftrightarrow b$ denotes the element $\left\{\begin{array}{c} a \rightarrow b \\ b \rightarrow a \end{array}\right\}$.

One of the most important examples of a complete Heyting algebra is a set of n–tuples

$$\{(a_1, \ldots, a_n) \mid a_1 \in M_1, \ldots, a_n \in M_n\}$$

where $M_1, \ldots, M_n$ are complete totally ordered sets (for example, every M_i is a segment $[0, 1]$), and $(a_1, \ldots, a_n) \leq (b_1, \ldots, b_n)$ iff for every $i = 1, \ldots, n \quad a_i \leq b_i$. For every pair $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n)$

$$a \rightarrow b = (c_1, \ldots, c_n), \quad \text{where } c_i = \begin{cases} 1, & \text{if } a_i \leq b_i \\ b_i, & \text{otherwise} \end{cases}$$

2.2 Heyting-valued modal formulas

Let PV be a countable set, elements of which will be called **propositional variables**.

The set Fm of **Heyting-valued modal formulas (HVMFs)** is defined inductively as follows.

- Every $p \in PV$ is a HVMF.
- Every $a \in \mathcal{H}$ is a HVMF.
- If A and B are HVMFs, then the strings $A \wedge B$, $A \vee B$, and $A \to B$ are HVMFs.
- If A is a HVMF, and $a \in \mathcal{H}$, then $\Box_a A$ if a HVMF.

The symbols $\Box_a$ are called **Heyting-valued modal operators**.

A HVMF $\Box_a A$ can be interpreted as the proposition

"the plausibility value of A is equal to a".

For every list $A_1, \dots, A_n$ of HVMFs the strings

$$A_1 \wedge A_2 \wedge \ldots \wedge A_n \ \text{ and } \ A_1 \vee A_2 \vee \ldots \vee A_n$$

are the restricted notations of the HVMFs

$$A_1 \wedge (A_2 \wedge (\ldots \wedge A_n) \ldots) \ \text{ and } \ A_1 \vee (A_2 \vee (\ldots \vee A_n) \ldots)$$

respectively.

These HVMFs will be denoted also by the symbols

$$\left\{ \begin{array}{c} A_1 \\ \ldots \\ A_n \end{array} \right\} \quad \text{and} \quad \left[\begin{array}{c} A_1 \\ \ldots \\ A_n \end{array} \right]$$

respectively.

For every pair A, B of HVMFs the string $A \leftrightarrow B$ is a restricted notation of the HVMF $\left\{ \begin{array}{c} A \to B \\ B \to A \end{array} \right\}$.

2.3 Substitutions

A **substitution** is a pair

$$\theta \ = \ ((p_1, \dots, p_n), \ (A_1, \dots, A_n)) \tag{2}$$

where $p_1, \dots, p_n$ are distinct variables, and $A_1, \dots, A_n$ are HVMFs.

For every substitution (2) and every HVMF A the symbol $\theta(A)$ denotes a result of substitution for every $i = 1, \dots, n$ the HVMF A_i instead of all occurrences of p_i in A.

2.4 Tautologies

Let A and B be HVMFs. We shall say that B is obtained from A by an equivalent transformation, if

- there is a subformula of A of the form $a \wedge b, a \vee b$, or $a \to b$, where $a, b \in \mathcal{H}$,
- B is a result of a substitution in A the corresponded element of $\mathcal{H}$ instead of this subformula.

We shall consider HVMFs A and B as equal (and write $A = B$) iff the pair (A, B) belongs to the least equivalency relation generated by pairs of the form (C, D), where D can be obtained from C by an equivalent transformation.

Let A be a HVMF without modal operators, and the list of variables of A has the form $(p_1, \ldots, p_n)$. A is said to be a **tautology**, if $\theta(A) = 1$ for every substitution (2), such that $\forall i \in \{1, \ldots, n\} \quad A_i = a_i \in \mathcal{H}$.

2.5 Heyting-valued modal logics

A **Heyting-valued modal logic (HVML)** is a set L of HVMFs such that

- every tautology belongs to L,
- for every A, B of HVMFs and every $a \in \mathcal{H}$

$$\Box_a \left\{ \begin{matrix} A \\ B \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \Box_a A \\ \Box_a B \end{matrix} \right\} \quad \in L, \tag{3}$$

- for every $a \in \mathcal{H}$

$$a \to \Box_a 1 \quad \in L, \tag{4}$$

- for every HVMF A and every $a \in \mathcal{H}$

$$\Box_a A \to a \quad \in L, \tag{5}$$

- for every HVMFs A, B

$$\begin{array}{ll} \textbf{if} & A \;\in L \textbf{ and } A \to B \;\in L \\ \textbf{then} & B \;\in L \end{array} \tag{6}$$

- for every HVMF A and every substitution θ

$$\begin{array}{ll} \textbf{if} & A \in L \\ \textbf{then} & \theta(A) \in L \end{array} \tag{7}$$

- for every HVMFs A, B and every $a, b \in \mathcal{H}$

$$\begin{array}{ll} \textbf{if} & a \to (A \to B) \;\in L \\ \textbf{then} & a \to (\Box_b A \to \Box_b B) \;\in L \end{array} \tag{8}$$

– for every HVMF A and every subset $\{a_i \mid i \in \Im\} \subseteq \mathcal{H}$

$$\begin{array}{ll} \textbf{if} & \forall i \in \Im \quad a_i \to A \quad \in L \\ \textbf{then} & (\sup\limits_{i \in \Im} a_i) \to A \quad \in L. \end{array} \tag{9}$$

This definition implies that there is a minimal (with respect to the inclusion) HVML, which we shall denote by the symbol HVK.

It is not so difficult that the inference rule

$$\begin{array}{ll} \textbf{if} & a_1 \to A_1 \quad \in L, \\ & \dots \\ & a_n \to A_n \quad \in L \\ & \left(\begin{array}{r} \text{where } a_1, \dots, a_n \in \mathcal{H}, \quad \text{and} \\ A_1, \dots, A_n \quad \text{are HVMFs} \end{array} \right) \\ \textbf{then} & \left\{ \begin{array}{c} a_1 \\ \dots \\ a_n \end{array} \right\} \to \left\{ \begin{array}{c} A_1 \\ \dots \\ A_n \end{array} \right\} \quad \in L \end{array} \tag{10}$$

is admissible for every HVML.

For every HVMF A and every HVML L the symbol

$$[\![A]\!]_L$$

denotes a supremum of the set

$$\{a \in \mathcal{H} \mid a \to A \quad \in L\}. \tag{11}$$

This definition and (9) imply

$$\forall a \in \mathcal{H} \qquad a \to A \quad \in L \quad \Leftrightarrow \quad a \leq [\![A]\!]_L.$$

3 Heyting-valued Kripke models

3.1 Heyting–valued sets

Remind ([1]) that a **Heyting–valued set (HS)** (over a complete Heyting algebra $\mathcal{H}$) is a pair

$$W = (X, \mu) \tag{12}$$

where

– X is a set (which is called a **support** of W), and
– μ is a mapping of the form

$$\mu : X \times X \to \mathcal{H}$$

such that

$$\forall x, y \in X \qquad \mu(x, y) = \mu(y, x) \tag{13}$$

$$\forall x, y, z \in X \qquad \left\{ \begin{array}{c} \mu(x, y) \\ \mu(y, z) \end{array} \right\} \leq \mu(x, z) \tag{14}$$

For every pair $x, y \in X$ the element $\mu(x, y)$ is called a **similarity value between x and y**.

For example, let

- X be a set of humans,
- $\{a_1, \ldots, a_n\}$ be a list of some their characteristics (age, sex, salary, reputation, health, etc.),
- $M_1, \ldots, M_n$ are complete totally ordered sets of similarity values related to the characteristics $a_1, \ldots, a_n$ respectively,
- a Heyting algebra $\mathcal{H}$ has the form

$$M_1 \times \ldots \times M_n \tag{15}$$

We can consider X as a Heyting–valued set over (15), where for every pair $x, y \in X$ their similarity $\mu(x, y)$ is a n–tuple $c_1, \ldots, c_n$, such that for every $i \in \{1, \ldots n\}$ if x and y are similar with respect to the characteristics a_i, then c_i is in proximity to the maximal element of M_i.

For every $x \in X$ the element $\mu(x, x)$ is called a **membership value** of x at the HS (12).

Let $W = (X, \mu)$ be a HS. A **Heyting–valued binary relation (HR)** on W is a mapping R of the form $R : X \times X \to \mathcal{H}$, such that

$$\forall x, y, x', y' \in X \qquad \left\{ \begin{array}{l} R(x, y) \\ \mu(x, x') \\ \mu(y, y') \end{array} \right\} \leq R(x', y'), \tag{16}$$

$$\forall x, y \in X \qquad R(x, y) \leq \left\{ \begin{array}{l} \mu(x, x) \\ \mu(y, y) \end{array} \right\}. \tag{17}$$

For every pair $(x, y) \in X \times X$ the element $R(x, y)$ can be interpreted as a **belonging value** of this pair to the HR R.

A **Heyting–valued subset (HSS)** of a HS (12) is a mapping s of the form

$$s : X \to \mathcal{H} \tag{18}$$

such that

$$\forall x, x' \in X \qquad \left\{ \begin{array}{l} s(x) \\ \mu(x, x') \end{array} \right\} \leq s(x'), \tag{19}$$

$$\forall x \in X \qquad s(x) \leq \mu(x, x). \tag{20}$$

For every $x \in X$ the element $s(x)$ can be interpreted as a **membership value** of x at the HSS (18).

The set of all HSSs of a HS (12) will be denoted by the symbol $Sub(W)$.

Below

- for every HS W its support will be denoted by the same symbol W,
- for every pair of elements of the support the similarity value between x and y will be denoted by the symbol $W(x, y)$, and
- for every $x \in W$ the membership value of x at the HS W will be denoted by the symbol $W(x)$.

3.2 Definition of a Heyting-valued Kripke model

A **Heyting-valued Kripke model (HVKM)** is a triple M of the form

$$M = (W, \{R_a \mid a \in \mathcal{H}\}, \xi) \tag{21}$$

where

- W is a HS, elements of which are called **objects** (or **worlds**),
- $\{R_a \mid a \in \mathcal{H}\}$ is a $\mathcal{H}$–tuple of HRs on W, which are called **transition relations**,
- ξ is a mapping of the form

$$\xi : PV \to Sub(W) \tag{22}$$

  which is called an **evaluation of variables**.

3.3 Evaluation of HVMFs at HVKMs

For every HVMF A and every HVKM (21) an **evaluation of** A **at** M is the mapping

$$[\![A]\!]_M : W \to \mathcal{H},$$

which maps every $x \in W$ to the element $[\![A]\!]_x \in \mathcal{H}$, which is defined as follows:

- if $A = p \in PV$, then $[\![A]\!]_x \stackrel{\text{def}}{=} \xi(p)(x)$,
- if $A = a \in \mathcal{H}$, then $[\![A]\!]_x \stackrel{\text{def}}{=} \left\{ \begin{array}{l} a \\ W(x) \end{array} \right\}$,
- if $A = B \wedge C$, then $[\![A]\!]_x \stackrel{\text{def}}{=} [\![B]\!]_x \wedge [\![C]\!]_x$,
- if $A = B \vee C$, then $[\![A]\!]_x \stackrel{\text{def}}{=} [\![B]\!]_x \vee [\![C]\!]_x$,
- if $A = B \to C$, then $[\![A]\!]_x \stackrel{\text{def}}{=} \left\{ \begin{array}{l} [\![B]\!]_x \to [\![C]\!]_x \\ W(x) \end{array} \right\}$,
- if $A = \Box_a B$, then $[\![A]\!]_x \stackrel{\text{def}}{=} \left\{ \begin{array}{l} a \\ \inf\limits_{y \in W} (R_a(x, y) \to [\![B]\!]_y) \\ W(x) \end{array} \right\}$.

It is not so difficult to prove that $[\![A]\!]_M$ is a HSS of the HS W.

3.4 An example of a HVKM

In this section we give an example of a HVKM related to description logic ([14]).

Description Logic is a language for formal description of complex concepts on the base of *atomic concepts* and binary relations, called *atomic roles*. Assume that there are given

- a set $\mathcal{I}$ of **individuals**,
- a set $\mathcal{C}$ of **atomic concepts**, and every atomic concept $c \in \mathcal{C}$ represents a subset $[\![c]\!] \subset \mathcal{I}$

– a set $\mathcal{R}$ of **atomic roles**, and every atomic role $r \in \mathcal{R}$ represents a binary relation $[\![r]\!] \subseteq \mathcal{I} \times \mathcal{I}$.

Description Logic allows to represent complex notions by **concept terms**, i.e. expressions that are built from atomic concepts and atomic roles with use of the concept constructors:

– boolean operations (conjunction ($\sqcap$), etc.), and
– quantifier operations of the form $\forall r$, where $r \in \mathcal{R}$.

Every concept term represents a subset $[\![t]\!] \subseteq \mathcal{I}$, which is defined by induction as follows:

– $[\![t_1 \sqcap t_2]\!] \stackrel{\text{def}}{=} [\![t_1]\!] \cap [\![t_2]\!]$,
– $[\![\forall r.t]\!] \stackrel{\text{def}}{=} \{a \in \mathcal{I} \mid \text{for every} \;\; b \in \mathcal{I} \quad (a,b) \in [\![r]\!] \Rightarrow b \in [\![t]\!]\}$.

For example (the example is borrowed from [15]), if

– $\mathcal{I}$ consists of all humans,
– the atomic concept `Woman` is interpreted as the set of all women, and
– the atomic role `child` is interpreted as the set of all pairs (a, b) of humans, such that b is a child of a

then the concept of all women having only daughters can be represented by the concept term

$$\texttt{Woman} \sqcap \forall\texttt{child.Woman}$$

Let $\mathcal{R}^*$ be the set of all finite sequences of elements of $\mathcal{R}$.
Every sequence $r = (r_1, \ldots, r_n) \in \mathcal{R}^*$ represents a binary relation

$$[\![r]\!] \stackrel{\text{def}}{=} [\![r_1]\!] \circ \ldots \circ [\![r_n]\!] \quad \subseteq \mathcal{I} \times \mathcal{I}$$

Elements of $\mathcal{R}^*$ can be interpreted as *derivative roles*, and will be referred briefly as *roles*.

Let $\mathcal{H}$ be the set $\mathcal{P}(\mathcal{R}^*)$ of all subsets of the set $\mathcal{R}^*$. $\mathcal{H}$ is a complete Heyting algebra, because it is a complete boolean algebra.

We can consider the set $\mathcal{I}$ of humans as a Heyting-valued set (over $\mathcal{H} = \mathcal{P}(\mathcal{R}^*)$), where for every pair $(x, y) \in \mathcal{I} \times \mathcal{I}$ the similarity value $\mathcal{I}(x, y)$ consists of all roles $r \in \mathcal{R}^*$ such that x and y are equal with respect to r (we do not clarify the concepts of equality of humans with respect to a role, because it seems to be intuitively clear, but the precise definition of this notion requires a strong linguistic foundation).

A HVKM related to this example has the following components.

– Objects of this HVKM are humans, and similarity value between them was described above.
– For every $\rho \in \mathcal{H}$ and every pair x, y of humans the set $R_\rho(x, y)$ consists of all roles $r \in \rho$ such that $(x, y) \in [\![r]\!]$.

- The set PV is equal to the set $\mathcal{C}$ of atomic concepts, and for every $c \in \mathcal{C}$ the evaluation
$$\xi(c) : \mathcal{I} \to \mathcal{P}(\mathcal{R}^*)$$
is defined as follows: for every human $x \in \mathcal{I}$
$$\xi(c)(x) \stackrel{\text{def}}{=} \begin{cases} \mathcal{I}(x), & \text{if } x \in [\![c]\!] \\ \emptyset, & \text{otherwise.} \end{cases}$$

3.5 Truth of HVMFs at HVKMs

A HVMF A is said to be **true at an object** x of a HVKM (21), if
$$[\![A]\!]_x = W(x). \tag{23}$$

A HVMF A is said to be **true at a HVKM** (21), if A is true at every object of (21).

It is not so difficult to prove that every HVMF $A \in HVK$ is true at every HVKM, because

- every tautology is true at every HVKM,
- HVMFs from (3), (4) and (5) are true at every HVKM, and
- inference rules (6), (7), (8) and (9) preserve the truth property at every HVKM.

Below we prove the inverse statement: if a HVMF A is true at every HVKM, then $A \in HVK$.

4 Canonical models of HVMLs

4.1 Consistent HVMLs

A HVML L is **consistent**, if for every $a \in \mathcal{H}$ $\quad a \in L \quad \Rightarrow \quad a = 1$.

It is not so difficult to prove that HVK is consistent.

Below every HVML under consideration is assumed to be consistent.

4.2 *L*–consistent sets of HVMFs

Let

- L be a consistent HVML, and
- u be a set of HVMFs.

The set u is said to be L–**consistent**, if for

- every finite subset of the set u, which has the form
$$\{a_1 \to A_1, \ldots, a_n \to A_n\} \tag{24}$$
(where $a_1, \ldots, a_n \in \mathcal{H}$, $\quad A_1, \ldots, A_n$ are HVMFs, and

– every $b \in \mathcal{H}$

the statement

$$\left\{\begin{matrix} A_1 \\ \dots \\ A_n \end{matrix}\right\} \to b \quad \in L \tag{25}$$

implies the inequality

$$\left\{\begin{matrix} a_1 \\ \dots \\ a_n \end{matrix}\right\} \leq b. \tag{26}$$

4.3 Properties of *L*–consistent sets

For every pair u_1, u_2 of sets of HVMFs the inequality

$$u_1 \leq u_2 \tag{27}$$

means that

$$\text{for every HVMF of the form } \; a \to A \; \in u_1$$
$$a = 0 \;\text{ or }\; \exists\, b \geq a : \; b \to A \; \in u_2.$$

Theorem 1. For every pair u_1, u_2 of sets of HVMFs the inequality (27) implies that

$$u_2 \text{ is } L\text{–consistent} \;\Rightarrow\; u_1 \text{ is } L\text{–consistent}.$$

Theorem 2. Every consistent HVML is a L–consistent set.

Below the symbol L denotes some fixed consistent HVML.

Theorem 3. Let

– u be a L–consistent set,
– A be a HVMF, and
– Q be the set of all elements $a \in \mathcal{H}$ such that

$$u \cup \{a \to A\} \quad \text{is } L\text{–consistent}. \tag{28}$$

Then for every $a \in \mathcal{H}$

$$a \leq \sup(Q) \quad \Leftrightarrow \quad a \in Q.$$

The element $\sup(Q)$, which corresponds to A and u, will be denoted by the symbol

$$[\![A]\!]_u \tag{29}$$

The definition of the element $[\![A]\!]_u$ implies that for every set u of HVMFs the following implication holds:

$$\begin{array}{l} u \quad \text{is } L\text{–consistent} \quad \Rightarrow \quad \forall\, A \in Fm \\ u \cup \{[\![A]\!]_u \to A\} \quad \text{is } L\text{–consistent} \end{array} \tag{30}$$

Theorem 4. Let u_1 and u_2 be L–consistent sets, such that

$$u_1 \leq u_2.$$

Then for every HVMF A

$$[\![A]\!]_{u_2} \leq [\![A]\!]_{u_1}. \tag{31}$$

Theorem 5. Let

– u be a L–consistent set of HVMFs, and
– A, B be a pair of HVMFs, such that

$$A \to B \quad \in L \tag{32}$$

Then

$$[\![A]\!]_u \leq [\![B]\!]_u. \tag{33}$$

Theorem 6. For

– every L–consistent set u, and
– every HVMF A

the following inequality holds:

$$[\![A]\!]_L \leq [\![A]\!]_u. \tag{34}$$

4.4 *L*–complete sets of HVMFs

Let x be a set of HVMFs.

The set x is said to be L–**complete**, if

– x is L–consistent, and
– for every HVMF A

$$[\![A]\!]_x \to A \quad \in x. \tag{35}$$

4.5 Completion of *L*–consistent sets

Let

– u be a L–consistent set, and
– x be a L–complete set.

x is said to be a **completion of** u, if

$$u \leq x \tag{36}$$

Theorem 7. For every L–consistent set u there is its completion x.

Below we shall assume that $\mathcal{H}$ satisfies the additional condition:

$$\forall\, a \in \mathcal{H} \quad (a \to 0) \to 0 \;=\; a. \tag{37}$$

This condition is equivalent to the condition that $\mathcal{H}$ is a boolean algebra with respect to the operations $\wedge, \vee, \neg$, where $\forall a \in \mathcal{H} \quad \neg a \stackrel{\text{def}}{=} a \to 0$.

4.6 Canonical models of HVMLs

A **canonical model** of a HVML L is a HVKM

$$M_L \stackrel{\text{def}}{=} (W_L, \{R_{L,a} \mid a \in \mathcal{H}\}, \xi_L)$$

the components of which are defined as follows.

- W_L consists of all L–complete sets.
  For every pair $x, y \in W_L$

$$W_L(x,y) \stackrel{\text{def}}{=} \inf_{A \in Fm} ([\![A]\!]_x \leftrightarrow [\![A]\!]_y) \tag{38}$$

  Note that this definition implies that

$$\forall x \in W_L \qquad W_L(x) = 1. \tag{39}$$

- For every $a \in \mathcal{H}$ $\quad R_{L,a}$ is a HR on W_L, $R_{L,a} : W_L \times W_L \to \mathcal{H}$, where

$$\begin{array}{c} \forall x, y \in W_L \\ R_{L,a}(x,y) \stackrel{\text{def}}{=} \inf_{A \in Fm} ([\![\Box_a A]\!]_x \to [\![A]\!]_y) \end{array} \tag{40}$$

- ξ_L is a mapping of the form $\xi_L : PV \to Sub(W_L)$, where for every $p \in PV$ the HSS $\xi_L(p) : W_L \to \mathcal{H}$ is defined as follows:

$$\forall x \in W_L \qquad \xi_L(p)(x) \stackrel{\text{def}}{=} [\![p]\!]_x. \tag{41}$$

It is not so difficult to prove that

- W_L satisfies (13) and (14),
- $R_{L,a}$ satisfies (16) and (17), and
- $\xi_L(p)$ satisfies (19) and (20).

4.7 Main property of canonical models

Theorem 8. For every HVMF A and every $x \in W_L$

$$[\![A]\!](x) = [\![A]\!]_x. \tag{42}$$

5 Completeness of HVK

Theorem 9. If a HVMF A is true at every HVKM, then $A \in HVK$.
Proof.
Assume that $A \notin HVK$. Prove that A is not true at a certain object of the canonical model of HVK.
Note that the set

$$\{([\![A]\!]_{HVK} \to 0) \to (A \to 0)\} \tag{43}$$

is HVK–consistent, because for every $b \in \mathcal{H}$ the statement

$$(A \to 0) \to b \quad \in HVK \tag{44}$$

implies the inequality

$$[\![A]\!]_{HVK} \to 0 \leq b \tag{45}$$

Indeed, (44) implies that

$$\begin{array}{llll} (b \to 0) \to A & \in HVK & \Rightarrow & \\ b \to 0 \ \leq \ [\![A]\!]_{HVK} & & \Rightarrow & (45) \end{array}$$

Theorem 7 implies that HVK–consistency of the set (43) implies that

$$\exists x \in W_{HVK} : \quad [\![A]\!]_{HVK} \to 0 \ \leq \ [\![A \to 0]\!]_x \tag{46}$$

Since the set x is HVK–complete, then (46) implies that

$$[\![A \to 0]\!]_x = [\![A]\!]_x \to [\![0]\!]_x = [\![A]\!]_x \to 0 \tag{47}$$

(42), (46) and (47) imply the inequality

$$[\![A]\!]_{HVK} \to 0 \ \leq \ [\![A]\!](x) \to 0 \tag{48}$$

which is equivalent to the inequality

$$[\![A]\!](x) \ \leq \ [\![A]\!]_{HVK} \tag{49}$$

Prove that A is not true at the object x.
If A is true at x, then (23) and (39) imply that

$$[\![A]\!](x) = 1 \tag{50}$$

(49) and (50) imply the equality $[\![A]\!]_{HVK} = 1$, which implies $A \in HVK$.
This contradicts to the assumption that $A \notin HVK$.

6 Conclusion

In the paper we have introduced a new framework for representation of propositions which can contain fuzzy modalities. We have defined the concept of a Heyting-valued modal logic and have proved the completeness theorem for the minimal Heyting-valued modal logic. The directions of further research related to the introduces concepts and results can be the following.

1. Prove the completeness theorem without the condition $(a \to 0) \to 0 = a$ for every $a \in \mathcal{H}$.
2. Investigate the problems of finite model property and decidability of minimal HVML.
3. Define the concept of a Heyting-valued proof for first-order logics, and introduce a Heyting-valued provability logics related to the concept of a Heyting-valued proof, investigate properties of Heyting-valued provability logics.
4. Design a specification language and model checking algorithms for Heyting-valued dynamic systems based on the proposed framework.

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