A version of rough mereology suitable for rough sets

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Abstract. We address in principle the notion of a boundary and we propose a version of mereology better adapted to rough set theory than the original version. We discuss the motivation and differences between the original and proposed now versions of rough mereology and we show that the Pawlak notion of a boundary in rough set theory is a particular case of the more general notion of a boundary in the rough mereological theory proposed in this work.

Keywords: rough set theory, rough mereology, truly rough inclusion, boundary

1 Introduction: the idea of a boundary, in general and in Pawlak's theory

It is evident to all who study rough set idea that the most important notion is the notion of a boundary and most important things that conform to that notion are boundaries of concepts as they witness the uncertainty of the concept. The notion of a boundary has been the subject of investigation by philosophers, logicians, topologists from ancient times to now. The basic problem with the notion of a boundary stems of course from the fact that our perception of the world is continuous whereas the world has a discrete structure. Whence follow the philosophical dilemmas like Bolzano's paradox of the two touching balls A and B. The question is where A ends and B begins? If q is the last point on A which must exists by closedness of A, then if p is the first point on B and p is not q then A and B do not touch and there is no boundary between them but we know they touch as we are not able to push A or B any further. Similar are problems by Leonardo of air and water: what is the boundary between water in the river and air? Those and other similar questions have occupied philosophers since times of antiquity. One needs not to reach for real physical phenomena in order to find problems and difficulties with boundaries, e.g., time imagined continuity have caused similar problems: what is the last moment an event occurs? See Varzi [9] and Smith [8] for a discussion of philosophical aspects of the notion of a boundary.

Mathematicians resolved the problem by postulating the completeness of the real line which cannot be dissected into two open disjoint non–empty sets; returning to philosophical aspects of the notion of a boundary, this implies that when an event has the last moment p in which it happens, we are not able to point to the first moment in which it does not happen. They also approached the problem of a boundary from a local point of view with the idea of a neighborhood and closeness: the boundary of a set is the set of points 'infinitely close' to the set in the sense that each neighborhood of a boundary point must needs intersect the set and its complement, see Engelking [1].

This point of view prevails in Pawlak's idea of a boundary see Pawlak [3] but the implementation of it proceeds in a distinct way. In the first place, one needs the data about reality in the form of an *information system* i.e. a tuple (U, A, val, V) where U is a set of *objects* representing physical entities, processes, moments of time etc., A is a set of *attributes* each of which maps objects in U into the *set of values* V by means of a mapping $val : U \times A \rightarrow V$. The value $val(u, a)$ is often written down in a simpler form of $a(u)$.

Objects are then coded as *information sets* of the form $Inf(u) = \{a(u) : a \in A\}$. The crux of the rough set approach is in identification of objects having the same information sets: $Ind(u, v)$ if and only if $Inf(u) = Inf(v)$, where $Ind(u, v)$ is the *indiscernibility relation* . From that moment on, objects loose their real names and become visible only by their information sets which allows for making some of them indiscernible. The equivalence relation of indiscernibility partitions the universe U into classes which are also regarded as primitive *granules of knowledge* $[u]$ or *black boxes*.

One addresses the problem of *concepts* understood as subsets of the universe U and distinguishes among them *certain* ones as those which can be assembled from granules by the union of sets operator i.e. a concept $C \subseteq U$ is certain if and only if $C = \bigcup\{[u] : [u] \subseteq C\}$. Other concepts are not certain, commonly called *rough*. A rough concept R then must have an object u such that the class $[u]$ is not contained in it but it does intersect the complement $U \setminus R$. Such objects in R constitute the *boundary* of R, BdR, i.e. $BdR = \{u \in U : [u] \cap (U \setminus R) \neq \emptyset \neq [u] \cap R\}$. One may say that the boundary of R consists of objects which have their copies both in R and in $U \setminus R$. This is clearly a topological approach as the class $[u]$ is the least neighborhood of u in the partition topology induced by the indiscernibility relation Ind.

Let us point to advantages of this approach: first it is objective as the shape of the boundary follows by the data without any intervention of subjective factors which are so essential in e.g. fuzzy set theory see Zadeh [12]; next, it relies solely on data without resorting to e.g. real numbers or other auxiliary external to data factors.

2 Mereology and its extensions into domain of uncertainty

Mereology due to Stanisław Leśniewski, see Leśniewski [2], Sinisi [7], is based on the notion of a part relation, $part(x, y)$ (*'x is a part to y'*) which satisfies over a universe U conditions:

M1 For each $x \in U$ it is not true that $part(x, x)$;

M2 For each triple x, y, z of things in U if $part(x, y)$ and $part(y, z)$, then $part(x, z)$.

The notion of an *element* is defined as the relation $el(x, y)$ which holds true if $part(x, y)$ or $x = y$. It follows that the relation of being an element is a partial order on U and $el(x, y)$ and $el(y, x)$ are true simultaneously if and only if $x = y$. Clearly, $part(x, y)$ if and only if $el(x, y)$ and $x \neq y$.

The last important notion is that of the *class* understood as the object which represents a collective entity i.e. a *property*: for a property Ψ on U which is not void, the class pf Ψ, $Cls\Psi$, is the object such that:

C1 If $\Psi(u)$ holds true, then $el(u, Cls\Psi)$;

C2 For each u, if $el(u, Cls\Psi)$ then there are objects p, q such that $el(p, u)$, $el(p, q)$, $\Psi(q)$ hold true.

2.1 Our rough mereology as up to now

We proposed a scheme which extended mereology based on the part relation to the version based on the *'part to a degree'* relation, see Polkowski [6], written down as the relation $\mu(x, y, r)$ (*x is a part of y to a degree of at least of r*) on the universe U endowed with the mereological notion of the element el. The assumptions abut μ reflected the basic true properties of the partial containment:

RM1 $\mu(x, x, 1)$ for each $x \in U$;

RM2 $\mu(x, y, 1)$ if and only if $el(x, y)$ holds true;

RM3 If $\mu(x, y, 1)$ and $\mu(z, x, r)$ hold true then $\mu(z, y, r)$ holds true;

RM4 If $\mu(x, y, r)$ and $s < r$ hold true then $\mu(x, y, s)$ holds true.

This approach has required the a priori mereology on the universe U of which the relation μ was a diffusion or fuzzification, cf. Varzi [10]. Assume that $f : [0, 1]^2 \to [0, 1]$ is a continuous in each coordinate and symmetric function such that $f(1, r) = r$ for each $r \in [0, 1]$. We will call any such f a *pre–norm*.

We say for a pre–norm f that relation μ is f–transitive if and only if from true conditions $\mu(x, y, r)$ and $\mu(y, z, s)$ the true condition $\mu(x, z, f(r, s))$ follows.

Proposition 1. *If the relation μ on the universe U is f–transitive then the relation $\pi(x, y)$ which holds true if and only if $\mu(x, y, 1)$ and $\mu(y, x, 1)$ hold true is an equivalence relation on U.*

Indeed, $\pi(x, x1)$ holds true by RM1; symmetry is evident by definition; transitivity follows by $f-$transitivity of μ.

An attempt to apply this version of μ in the rough set context is handicapped by the fact that by RM2, the relation π should be the identity which does not capture the full scope of rough set cases. We need something more flexible to accomodate equivalence relations of indiscernibility.

2.2 A rough mereology for rough set theory

As stated below, we need a mereology which may account for rough set theoretic contexts. We assume an information system (U, A, val, V) as our playground.

We define a *truly rough inclusion*, $\mu_{tr}(x, y, r)$ as a relation which satisfies on U the following assumptions:

TRM1 $\mu_{tr}(x, x, 1)$;

TRM2 There is a partition P on U such that $\mu_{tr}(x, y, 1)$ if and only if x and y are in the same partition class $[x]_P$;

TRM3 If $\mu_{tr}(x, y, 1)$ and $\mu_{tr}(z, x, r)$ then $\mu_{tr}(z, y, r)$;

TRM4 If $\mu_{tr}(x, y, r)$ and $s < r$ then $\mu_{tr}(x, y, s)$.

TRM5 The truly rough inclusion $\mu_{tr}(x, y, r)$ is f–transitive for some pre–norm f.

The predicate $el(x, y)$ if $\mu_{tr}(x, y, 1)$ defines x as an *element of* y.

We sum up basic consequences of our assumptions.

Proposition 2. *The following are true by conditions TRM1-TRM5.*

1. $\mu_{tr}(x, y, 1)$ *implies* $\mu_{tr}(y, x, 1)$ *i.e.* μ_{tr} *is symmetric.*

2. The relation $el(x, y)$ *is symmetric and* $el(x, y)$ *and* $el(y, x)$ *imply that* $[x]_P = [y]_P$.

3. $\{y : \mu_{tr}(y, x, 1)\}=[x]_P$.

4. The relation $\mu_{tr}(x, y, 1)$ *is transitive in the sense that* $\mu_{tr}(z, x, 1)$ *and* $\mu_{tr}(x, y, 1)$ *imply* $\mu_{tr}(z, y, 1)$.

3 Boundaries in rough mereology for rough sets

We use the language of predicates on the universe U in definitions of boundaries by means of a truly rough inclusions μ_{tr}.

3.1 A general scheme for boundaries

For a truly rough inclusion μ_{tr}, and $x \in U$, $r \in [0, 1]$, we define a new predicate $N(x, r)(z)$ if there exists an $s \geq r$ such that $\mu(z, x, s)$. $N(x, r)$ is the *neighborhood granular predicate about* x *of radius* r.

Consider a predicate Ψ on U having a non–empty meaning $[\Psi]$. The *complement to* Ψ is the predicate $-\Psi$ such that $-\Psi(x)$ if and only if not $\Psi(x)$. We define the *upper extension of* Ψ *of radius* r, denoted Ψ_r^+ by letting $\Psi_r^+(x)$ if there exists z such that $\Psi(z)$ and $N(x,r)(z)$. Similarly, we define the *lower restriction of* Ψ *of radius* r, denoted Ψ_r^- by letting $\Psi_r^-(x)$ if and only if not $(-\Psi)_r^+(x)$.

Proposition 3. *1. Predicates Ψ_r^+ and Ψ_r^- are disjoint in the sense that there is no $z \in U$ such that $\Psi_r^+(z)$ and $\Psi_r^-(z)$ hold true. 2. If $\Psi_r^+(x)$ holds true then $\Psi_r^+(y)$ holds true for each y such that $\mu_{tr}(y,x,1)$. 3. If $\Psi_r^-(x)$ holds true then $\Psi_r^-(y)$ holds true for each y such that $\mu_{tr}(y,x,1)$.*

Proof. Claim 1 follows by definitions of the two predicates. For Claim 2, consider x,y such that $\Psi_r^+(x)$ and $\mu_{tr}(y,x,1)$. There exists z such that $\Psi(z)$, $N(x,s)(z)$ hold true with some $s \geq r$ so $\mu_{tr}(z,x,s)$ holds true. By symmetry of μ_{tr}, we have $\mu_{tr}(x,y,1)$ true and f–transitivity of μ_{tr} for an adequate pre–norm f implies that $\mu_{tr}(z,y,f(1,s))$ holds true i.e. $\mu_{tr}(z,y,s)$ holds true which means that $N(y,r)(z)$ holds true and finally $\Psi_r^+(y)$ holds true. For Claim 3, assume that $\Psi_r^-(x)$ and $\mu_{tr}(y,x,1)$ hold true i.e.

$$\neg\exists z, s \geq r.\mu_{tr}(z,x,s) \wedge \neg\Psi(z) \tag{1}$$

which is equvalent to

$$\mu_{tr}(z,x,s) \rightarrow \Psi(z). \tag{2}$$

As $\mu_{tr}(y,x,1)$ is equivalent to $\mu_{tr}(x,y,1)$, we have by f–transitivity of μ_{tr} that

$$\mu_{tr}(z,y,s) \rightarrow \Psi(z), \tag{3}$$

which is equivalent to the thesis $\Psi_r^-(y)$.

We will say that a predicate Ψ is *el–saturated* if and only if true formulas $\Psi(x)$ and $el(y,x)$ imply that $\Psi(y)$. A corollary to Claim 3 in Proposition 3 says that for each $r \in [0,1]$, predicates Ψ_r^+ and Ψ_r^- are el–saturated.

A global and local definition of the boundary For a predicate Ψ, we define the predicate *boundary of* Ψ *with respect to a truly rough inclusion* μ_{tr}, denoted $Bd_{\mu_{tr}}\Psi$ as follows:

$$Bd_{\mu_{tr}}\Psi \leftrightarrow (\neg\Psi_1^+) \wedge (\neg\Psi_1^-). \tag{4}$$

Arguing like in proof of Proposition 3, we prove the following

Proposition 4. *1. $Bd_{\mu_{tr}}\Psi$ is el–saturated 2. For no $z \in U$, $Bd_{\mu_{tr}}\Psi(z) \wedge \Psi_1^+(z)$ is true and for no $z \in U$, $Bd_{\mu_{tr}}\Psi(z) \wedge \Psi_1^-(z)$ is true.*

Proposition 5. *For each $x \in U$, $Bd_{\mu_{tr}}\Psi(x)$ holds true if and only if there exist $z,y \in U$ such that $\Psi(z)$, $-\Psi(y)$, $\mu_{tr}(z,x,1)$, $\mu_{tr}(y,x,1)$.*

A predicate *Open* is defined on predicates on U and a predicate Φ on U is *open*, $Open(\Phi)$ in symbols if and only if it is el–saturated.

Corollary 1. *$Open(\Psi_r^+)$ and $Open(\Psi_r^-)$ hold true for each $r \in [0,1]$.*
$Open(Bd_{\mu_{tr}}\Psi)$ holds true.

Proposition 6. *For a finite collection of predicates $\{\Psi_1, \Psi_2, \ldots, \Psi_k\}$ if $Open(\Psi_i)$ holds true for each $i \leq k$, then $Open(\bigvee_i \Psi_i)$ holds true.*

A predicate *Closed* holds true for a predicate Ψ if and only if $Open(-\Psi)$ holds true.

Corollary 2. *$Closed(\Psi_r^+)$ and $Closed(\Psi_r^-)$ hold true for each $r \in [0,1]$.*
$Closed(Bd_{\mu_{tr}}\Psi)$ holds true.

For the mereotopological notion of boundary see also Polkowski and Semeniuk–Polkowska [4], [5] and Varzi [11].

3.2 The Pawlak notion of a boundary is a special case of truly rough mereological notion of a boundary

We return to an information system (U, A, val, V). We derive a truly rough inclusion from any Archimedean t–norm. There exist two non–isomorphic Archimedean t–norms:

- the Łukasiewicz t–norm $L(x,y) = max\{0, x+y-1\}$;
- the product t–norm $P(x,y) = x \cdot y$.

Both these t–norms admit a Hilbert–style representation

$$t(x,y) = g(f(x) + f(y)),$$

where $f : [0,1] \to [0,1]$ is a continuous decreasing function with $f(0) = 1$, and $g : [0,1] \to [0,1]$ is the inverse to f. In case of the t–norm L, $f(x) = 1 - x$ and $g(y) = 1 - y$. We let for an Archimedean t–norm t:

$$\mu^t(x,y,r) \text{ if and only if } g(\frac{card(Dis(x,y))}{card(A)}) \geq r, \tag{5}$$

where $Dis(x,y) = \{a \in A : a(x) \neq a(y)\}$.

In particular, as for the Łukasiewicz t–norm we have $g(y) = 1 - y$, the *Łukasiewicz truly rough inclusion* can be defined as

$$\mu^L_{tr}(x,y,r) \text{ if and only if } \frac{card(Ind(x,y))}{card)A)} \geq r, \tag{6}$$

where $Ind(x,y) = A \setminus Dis(x,y)$. In particular, μ^L is L–transitive.

The predicate of element $el(x,y)$ holds true if and only if $\mu^L_{tr}(x,y,1)$ holds true if and only if $Ind(x,y)$ i.e. x, y are indiscernible. Hence, a predicate is el–saturated if and only if its meaning is the union of a family of indiscernibility classes and rough mereological notions of Ψ_1^+ and Ψ_1^- become, respectively, the notions of the upper and the lower approximations of the meaning of Ψ and the meaning of the boundary predicate $Bd_{\mu^L}\Psi$ is the boundary of the meaning of Ψ.

4 Conclusions

We have proposed a new version of rough mereology suitable for rough set theory and we show that the rough set theory is a particular case of an abstract rough mereotopological theory.

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