The Fuzzy Description Logic f-$\mathcal{SHIN}$

Giorgos Stoilos[1], Giorgos Stamou[1], Vassilis Tzouvaras[1], Jeff Z. Pan[2] and Ian Horrocks[2]

[1] Department of Electrical and Computer Engineering, National Technical University of Athens, Zographou 15780, Greece

[2] School of Computer Science, The University of Manchester Manchester, M13 9PL, UK

Abstract. In the Semantic Web information would be retrieved, processed, combined, shared and reused in the maximum automatic way possible. Obviously, such procedures involve a high degree of uncertainty and imprecision. For example ontology alignment or information retrieval are rarely true or false procedures but usually involve confidence degrees or provide rankings. Furthermore, it is often the case that information itself is imprecise and vague like the concept of a "tall" person, a "hot" place and many more. In order to be able to represent and reason with such type of information in the Semantic Web (SW), as well as, enhance SW applications we present an extension of the Description Logic $\mathcal{SHIN}$ with fuzzy set theory. We present the semantics as well as detailed reasoning algorithms for the extended language.

1 Introduction

Uncertainty, like imprecision and vagueness, is a factor that can cause the degradation of the performance of a system. To this end, many applications and domains have incorporated mathematical frameworks that deal with such type of information, resulting in the improvement of their effectiveness. Applications like robotics [1], computer vision [2] and many more have embraced frameworks like fuzzy set theory [3] in order to improve their performance. On the other hand, in the Semantic Web context, little work has been carried out towards this direction. Apart from the fact that uncertainty is many times a feature of information itself, as for example the concepts of a "tall" man, a "fast" car, a "blue" sky and many more, applications like information retrieval, automatic information sharing and reuse are hardly true or false procedures but rather a matter of a degree. The need for covering vagueness in the Semantic Web has been stressed many times the past years [4–6]. It has been pointed out that dealing with such information would improve many Semantic Web applications [7–9].

Knowledge in the SW is usually structured in the form of ontologies [10]. This has led to considerable efforts to develop a suitable ontology language, culminating in the design of the OWL Web Ontology Language [11], which is now a W3C recommendation. The OWL recommendation actually consists of three languages of increasing expressive power, namely OWL Lite, OWL DL and OWL Full. *OWL Lite* and *OWL DL* are, basically very expressive description logics; they are almost[3] equivalent to the $\mathcal{SHIF}(\mathbf{D}^+)$ and $\mathcal{SHOIN}(\mathbf{D}^+)$ DLs. OWL Full is clearly undecidable because it does not impose restrictions on the use of transitive properties. Although the above DL languages are very expressive, they feature expressive limitations regarding their ability to represent vague and imprecise knowledge. As obvious, in order to make applications that use DLs able to cope with vague and uncertain information we have to extend them with a theory capable of representing this kind of information. One such important theory is fuzzy set theory.

In the current paper we extend the results obtained in [9] for fuzzy $\mathcal{SI}$ (f-$\mathcal{SI}$) to the language $\mathcal{SHIN}$, thus creating f-$\mathcal{SHIN}$. $\mathcal{SHIN}$ extends $\mathcal{SI}$ [12] with number restrictions and role hierarchies [13]. Number restrictions give us the ability to restrict the number of objects that a certain object is related to by a specific relation. For example we can state that a car has exactly four wheels, writing $\mathsf{Car} \equiv \mathsf{Vehicle} \sqcap \geq 4\mathsf{hasWheel} \sqcap \leq 4\mathsf{hasWheel}$. But though this definition is correct, it faces many limitations, for example, in the context of image processing where several wheels of a car in an image might be hidden. Hence a detected object can belong to a concept like, $\geq 4\mathsf{hasWheel}$, only to a certain degree. On the other hand role hierarchies allow us to state sub-role/super-role relations, as for example the relation that holds between the hasChild and hasOffspring roles. Regarding expressive power, $\mathcal{SHIN}$ is more expressive than OWL-Lite, ignoring data-types. In the following we will introduce the syntax of f-$\mathcal{SHIN}$ and present a detailed procedure to reason with the extended language.

2 Syntax and Semantics of f-$\mathcal{SHIN}$

In this section we introduce the DL f-$\mathcal{SHIN}$. As pointed out in the fuzzy DL literature [9,14], fuzzy extensions of DLs involve only the *assertion* of individuals to concepts and the semantics of the new language. Hence, as usual we have an alphabet of distinct concept names ($\mathbf{C}$), role names ($\mathbf{R}$) and individual names ($\mathbf{I}$). f-$\mathcal{SHIN}$-roles and f-$\mathcal{SHIN}$-concepts are defined as follows:

Definition 1. *Let $RN \in \mathbf{R}$ be a role name, R an f-$\mathcal{SHIN}$-role, C, D f-$\mathcal{SHIN}$-concepts. Valid f-$\mathcal{SHIN}$-roles are defined by the abstract syntax: $R ::= RN \mid R^-$. The inverse relation of roles is symmetric, and to avoid considering roles such as R^{--}, we define a function* Inv*, which returns the inverse of a role, more precisely $\mathsf{Inv}(R) := RN^-$ if $R = RN$ and $\mathsf{Inv}(R) := RN$ if $R = RN^-$.*

The set of f-$\mathcal{SHIN}$ concepts is the smallest set such that:

1. *every concept name $C \in CN$ is an f-$\mathcal{SHIN}$-concept,*
2. *if C and D are f-$\mathcal{SHIN}$-concepts, R is an f-$\mathcal{SHIN}$-role, S a* simple *f-$\mathcal{SHIN}$-role [15] and $p \in \mathbb{N}$, then $(C \sqcup D)$, $(C \sqcap D)$, $(\neg C)$, $(\forall R.C)$, $(\exists R.C)$, $(\geq pS)$ and $(\leq pS)$ are also f-$\mathcal{SHIN}$ concepts.*

[3] They also provide annotation properties, which Description Logics don't.

Table 1. Semantics of f-$\mathcal{SHIN}$-concepts

$$
\begin{array}{rcl}
\top^{\mathcal{I}}(a) & = & 1 \\
\bot^{\mathcal{I}}(a) & = & 0 \\
(\neg C)^{\mathcal{I}}(a) & = & 1 - C^{\mathcal{I}}(a) \\
(C \sqcup D)^{\mathcal{I}}(a) & = & max(C^{\mathcal{I}}(a), D^{\mathcal{I}}(a)) \\
(C \sqcap D)^{\mathcal{I}}(a) & = & min(C^{\mathcal{I}}(a), D^{\mathcal{I}}(a)) \\
(\forall R.C)^{\mathcal{I}}(a) & = & \inf_{b \in \Delta^{\mathcal{I}}} \{max(1 - R^{\mathcal{I}}(a,b), C^{\mathcal{I}}(b))\} \\
(\exists R.C)^{\mathcal{I}}(a) & = & \sup_{b \in \Delta^{\mathcal{I}}} \{min(R^{\mathcal{I}}(a,b), C^{\mathcal{I}}(b))\} \\
(\geq pR)^{\mathcal{I}}(a) & = & \sup_{b_1,\ldots,b_p \in \Delta^{\mathcal{I}}} \min_{i=1}^{p} R^{\mathcal{I}}(a, b_i) \\
(\leq pR)^{\mathcal{I}}(a) & = & \inf_{b_1,\ldots,b_{p+1} \in \Delta^{\mathcal{I}}} \max_{i=1}^{p+1} \{1 - R^{\mathcal{I}}(a, b_i)\} \\
(R^{-})^{\mathcal{I}}(b,a) & = & R^{\mathcal{I}}(a,b)
\end{array}
$$

A fuzzy $TBox$ is a finite set of fuzzy concept axioms. Let A be a concept name, C a f-$\mathcal{SHIN}$-concept. Fuzzy concept axioms of the form $A \sqsubseteq C$ are called *fuzzy inclusion introductions*; fuzzy concept axioms of the form $A \equiv C$ are called *fuzzy equivalence introductions*. Note that how to deal with *general fuzzy concept inclusions* [12] still remains an open problem in fuzzy concept languages. A fuzzy $RBox$ is a finite set of fuzzy role axioms. Fuzzy role axioms of the form $\mathsf{Trans}(RN)$, where RN is a role name, are called *fuzzy transitive role axioms*; fuzzy role axioms of the form $R \sqsubseteq S$ are called *fuzzy role inclusion* axioms. We use the notation $\sqsubseteq^{*}$ to denote the transitive-reflexive closure of $\sqsubseteq$. A role R is called sub-role (super-role) of a role S if $R \sqsubseteq^{*} S$ ($S \sqsubseteq^{*} R$). A fuzzy $ABox$ is a finite set of fuzzy assertions. A *fuzzy assertion* [14] is of the form $\langle a : C \bowtie n \rangle$, $\langle (a,b) : R \bowtie n \rangle$, where $\bowtie$ stands for $\geq, >, \leq$ or $<$ or $a \not= b$, for $a, b \in \mathbf{I}$. Intuitively, a fuzzy assertion of the form $\langle a : C \geq n \rangle$ means that the membership degree of a to the concept C is at least equal to n. We call assertions defined by $\geq, >$ *positive* assertions, while those defined by $\leq, <$ *negative* assertions [9]. A fuzzy knowledge base Σ is a triple $\langle \mathcal{T}, \mathcal{R}, \mathcal{A} \rangle$, where $\mathcal{T}$ is a fuzzy $TBox$, $\mathcal{R}$ is a fuzzy $RBox$ and $\mathcal{A}$ is a fuzzy $ABox$. A pair of assertions are called *conjugated* if they impose contradicting restrictions. For example, the pair of assertions $\langle \phi \geq n \rangle$ and $\langle \phi < m \rangle$, with $n \geq m$ contradict to each other. In the presence of role hierarchies one should also take into consideration possible sub- or super-roles when checking for such contradictions. For example the assertions $\langle (a,b) : R \geq 0.7 \rangle$ and $\langle (a,b) : P \leq 0.4 \rangle$, with $P \sqsubseteq^{*} R$ are conjugated. For a detailed description of the possible conjugated pairs the reader is referred to [14].

The semantics of fuzzy DLs are provided by a *fuzzy interpretation* [9,14]. A fuzzy interpretation is a pair $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$ where the domain $\Delta^{\mathcal{I}}$ is a non-empty set of objects and $\cdot^{\mathcal{I}}$ is a fuzzy interpretation function, which maps an individual name a to elements of $\mathsf{a}^{\mathcal{I}} \in \Delta^{\mathcal{I}}$ and a concept name A (role name R) to a membership function $\mathsf{A}^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1]$ ($R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$). Moreover, fuzzy interpretations are extended to interpret arbitrary f-$\mathcal{SHIN}$-concepts and roles. The complete set of semantics is depicted in Table 1, where inf stands for the infimum and sup for the supremum of a set. Note that apart from the fuzzy number restrictions, the interpretation of fuzzy concepts and concept constructors is the usual one found in the DL literature [9, 14, 16], where the Gödel conjunction (t(a,b)=min(a,b)), the Gödel disjunction (u(a,b)=max(a,b)) and the Kleen-Dienes fuzzy implication ($\mathcal{J}$ (a,b)=max(1-a,b)) are used for performing the fuzzy set theoretic operations. The semantics of fuzzy number restrictions were first presented in [17]. We chose to follow these semantics because, as pointed out in [17], they are derived by the First-Order formulae of classical number restrictions [17]. In [9] the naming f$_{KD}$-$\mathcal{SI}$ was used due to the usage of the Kleen-Dienes fuzzy implication. Since we also use the same implication here, from now on, we will refer to the extended language as f$_{KD}$-$\mathcal{SHIN}$.

An f$_{KD}$-$\mathcal{SHIN}$-concept C is *satisfiable* iff there exists some fuzzy interpretation $\mathcal{I}$ for which there is some $a \in \Delta^{\mathcal{I}}$ such that $C^{\mathcal{I}}(a) = n$, and $n \in (0, 1]$. A fuzzy interpretation $\mathcal{I}$ satisfies a fuzzy *TBox* $\mathcal{T}$ iff $\forall a \in \Delta^{\mathcal{I}}$, $A^{\mathcal{I}}(a) \leq C^{\mathcal{I}}(a)$ for each $A \sqsubseteq C$ in $\mathcal{T}$ and $A^{\mathcal{I}}(a) = C^{\mathcal{I}}(a)$ for each $A \equiv C$ in $\mathcal{T}$. The semantics of fuzzy inclusion axioms is the usual one found in fuzzy set theory [3]. A fuzzy interpretation $\mathcal{I}$ satisfies a fuzzy *RBox* $\mathcal{R}$ iff $\forall a, b, c \in \Delta^{\mathcal{I}}$, $R^{\mathcal{I}}(a, c) \geq sup_{b \in \Delta^{\mathcal{I}}}\{min(R^{\mathcal{I}}(a, b), R^{\mathcal{I}}(b, c))\}$ for each Trans (R) in $\mathcal{R}$, and $\forall \langle a, b\rangle \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, $R^{\mathcal{I}}(a, b) \leq S^{\mathcal{I}}(a, b)$ for each $R \sqsubseteq S$. Note that the semantics of role inclusion axioms $R \sqsubseteq S$ imply $\mathsf{Inv}(R) \sqsubseteq \mathsf{Inv}(S)$. A fuzzy relation R, defined over the domain $X \times X$, is called *sup-min transitive* iff $R(x, z) \geq \sup_{y \in X} min(R(x, y), R(y, z))$. Given a fuzzy interpretation $\mathcal{I}$, $\mathcal{I}$ satisfies $\langle a : C \geq n\rangle$ if $C^{\mathcal{I}}(a^{\mathcal{I}}) \geq n$, $\mathcal{I}$ satisfies $\langle (a, b) : R \geq n\rangle$ if $R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \geq n$, while $\mathcal{I}$ satisfies $a \neq b$ if $a^{\mathcal{I}} \neq b^{\mathcal{I}}$. The satisfiability of fuzzy assertions with $\leq, >$ and $<$ is defined analogously. A fuzzy interpretation satisfies a fuzzy *ABox* $\mathcal{A}$ if it satisfies all fuzzy assertions in $\mathcal{A}$. In this case, we say $\mathcal{I}$ is a *model* of $\mathcal{A}$. If $\mathcal{A}$ has a model then we say that it is *consistent*. Finally, a fuzzy knowledge base Σ is satisfiable iff there exists a fuzzy interpretation $\mathcal{I}$ which satisfies all axioms in Σ. Moreover, Σ *entails* an assertion $\langle \phi \bowtie n\rangle$ or a fuzzy concept inclusion axiom $C \sqsubseteq D$, written $\Sigma \models \langle \phi \bowtie n\rangle$ or $\Sigma \models C \sqsubseteq D$, iff any model of Σ also satisfies the fuzzy assertion or fuzzy concept inclusion axiom, respectively. The problems of entailment and subsumption can be reduced to fuzzy knowledge base satisfiability as is shown in [14].

Since a fuzzy *ABox* $\mathcal{A}$ might contain many positive assertions for the same individual (pair of individuals), without forming a contradiction, it is in our interest to compute what is the best lower and upper truth-value bounds of a fuzzy assertion. In [14] the concept of *greatest lower bound* of a fuzzy assertion w.r.t. Σ was defined as $glb(\Sigma, \phi) = \sup\{n : \Sigma \models \langle \phi \geq n\rangle\}$, and that of a *least upper bound* as, $lub(\Sigma, \phi) = \inf\{n : \Sigma \models \langle \phi \leq n\rangle\}$, where ϕ represents a crisp assertion of the form $a : C$ or $(a, b) : R$. Observe that $\sup \emptyset = 0$ and $\inf \emptyset = 1$. A procedure to solve the best truth-value bound was provided in [14]. Such a procedure can also be used in our framework.

3 A fuzzy tableau for f$_{KD}$-$\mathcal{SHIN}$ *ABoxes*

Most of the inference services of fuzzy DLs, can be reduced to the problem of consistency checking for *ABoxes* [14]. Consistency is usually checked with tableaux algorithms that try to construct a fuzzy tableau for a fuzzy *ABox* $\mathcal{A}$ [9], which is an abstraction of a model of $\mathcal{A}$ [13]. The tableau has a forest-like structure with nodes representing the individuals that appear in $\mathcal{A}$, and edges between nodes, which represent the relations that hold between two individuals. Each node is labelled with a set of triples of the form $\langle D, \bowtie, n\rangle$, which denote the concept, the type of inequality and the membership degree that the individual of the node has been asserted to belong to D. We call such triples *membership triples*. For triples of a single node, the concepts of conjugated, positive and negative triples can be defined in the obvious way. Since the expansion rules decompose the initial concept, the concepts that appear in triples are sub-concepts of the initial concept. Sub-concepts of a concept D are denoted by $sub(D)$. The set of all sub-concepts that appear within an *ABox* is denoted by $sub(\mathcal{A})$.

Since the De'Morgan laws are satisfied by the operations we use in the current paper [3] all concepts are assumed to be in their *negation normal form* (NNF) [18]. In the following we use the symbols $\rhd$ and $\lhd$ as a placeholder for the inequalities $\geq, >$ and $\leq, <$ and the symbol $\bowtie$ as a placeholder for all types of inequations. Furthermore we use the symbols $\bowtie^-$, $\rhd^-$ and $\lhd^-$ to denote their *reflections*. For example the reflection of $\leq$ is $\geq$ and that of $>$ is $<$.

Definition 2. *Let $\mathcal{A}$ be an f_{KD}-$\mathcal{SHIN}$ ABox, $\boldsymbol{R}_A$ the set of roles occurring in $\mathcal{A}$ together with their inverses, $\mathbf{I}_{\mathcal{A}}$ the set of individuals in $\mathcal{A}$, $\mathcal{X}$ the set $\{\geq, >, \leq, <\}$ and $\mathcal{R}$ a fuzzy RBox. A fuzzy tableau T for $\mathcal{A}$ w.r.t. $\mathcal{R}$ is a quadruple ($\boldsymbol{S}$, $\mathcal{L}$, $\mathcal{E}$, $\mathcal{V}$) such that:*

- *$\boldsymbol{S}$ is a non-empty set of individuals (nodes),*
- *$\mathcal{L} : \boldsymbol{S} \to 2^{sub(A)} \times \mathcal{X} \times [0,1]$ maps each element of $\boldsymbol{S}$ to membership triples,*
- *$\mathcal{E} : \boldsymbol{R}_A \to 2^{\boldsymbol{S}\times\boldsymbol{S}} \times \mathcal{X} \times [0,1]$ maps each role to membership triples,*
- *$\mathcal{V} : \mathbf{I}_{\mathcal{A}} \to \boldsymbol{S}$ maps individuals occurring in $\mathcal{A}$ to elements in $\boldsymbol{S}$.*

For all $s, t \in \boldsymbol{S}$, $C, E \in sub(\mathcal{A})$, and $R \in \boldsymbol{R}_A$, T satisfies:

1. *If $\langle \neg C, \bowtie, n\rangle \in \mathcal{L}(s)$, then $\langle C, \bowtie^-, 1-n\rangle \in \mathcal{L}(s)$,*
2. *If $\langle C \sqcap E, \rhd, n\rangle \in \mathcal{L}(s)$, then $\langle C, \rhd, n\rangle \in \mathcal{L}(s)$ and $\langle E, \rhd, n\rangle \in \mathcal{L}(s)$,*
3. *If $\langle C \sqcup E, \lhd, n\rangle \in \mathcal{L}(s)$, then $\langle C, \lhd, n\rangle \in \mathcal{L}(s)$ and $\langle E, \lhd, n\rangle \in \mathcal{L}(s)$,*
4. *If $\langle C \sqcup E, \rhd, n\rangle \in \mathcal{L}(s)$, then $\langle C, \rhd, n\rangle \in \mathcal{L}(s)$ or $\langle E, \rhd, n\rangle \in \mathcal{L}(s)$,*
5. *If $\langle C \sqcap E, \lhd, n\rangle \in \mathcal{L}(s)$, then $\langle C, \lhd, n\rangle \in \mathcal{L}(s)$ or $\langle E, \lhd, n\rangle \in \mathcal{L}(s)$,*
6. *If $\langle \forall R.C, \rhd, n\rangle \in \mathcal{L}(s)$ and $\langle\langle s,t\rangle, \rhd', n_1\rangle \in \mathcal{E}(R)$ is conjugated with $\langle\langle s,t\rangle, \rhd^-, 1-n\rangle$, then $\langle C, \rhd, n\rangle \in \mathcal{L}(t)$,*
7. *If $\langle \exists R.C, \lhd, n\rangle \in \mathcal{L}(s)$ and $\langle\langle s,t\rangle, \rhd, n_1\rangle \in \mathcal{E}(R)$ is conjugated with $\langle\langle s,t\rangle, \lhd, n\rangle$, then $\langle C, \lhd, n\rangle \in \mathcal{L}(t)$,*
8. *If $\langle \exists R.C, \rhd, n\rangle \in \mathcal{L}(s)$, then there exists $t \in \boldsymbol{S}$ such that $\langle\langle s,t\rangle, \rhd, n\rangle \in \mathcal{E}(R)$ and $\langle C, \rhd, n\rangle \in \mathcal{L}(t)$,*
9. *If $\langle \forall R.C, \lhd, n\rangle \in \mathcal{L}(s)$, then there exists $t \in \boldsymbol{S}$ such that $\langle\langle s,t\rangle, \lhd^-, 1-n\rangle \in \mathcal{E}(R)$ and $\langle C, \lhd, n\rangle \in \mathcal{L}(t)$,*
10. *If $\langle \exists S.C, \lhd, n\rangle \in \mathcal{L}(s)$, and $\langle\langle s,t\rangle, \rhd, n_1\rangle \in \mathcal{E}(R)$ is conjugated with $\langle\langle s,t\rangle, \lhd, n\rangle$, for some $R \sqsubseteq^* S$ with $\mathsf{Trans}(R)$, then $\langle \exists R.C, \lhd, n\rangle \in \mathcal{L}(t)$,*

11. *If* $\langle \forall S.C, \rhd, n\rangle \in \mathcal{L}(s)$ *and* $\langle\langle s,t\rangle, \rhd', n_1\rangle \in \mathcal{E}(R)$ *is conjugated with* $\langle\langle s,t\rangle, \rhd^-, 1-n\rangle$*, for some* $R \sqsubseteq^* S$ *with* $\mathsf{Trans}(R)$*, then* $\langle \forall R.C, \rhd, n\rangle \in \mathcal{L}(t)$*,*
12. $\langle\langle s,t\rangle, \bowtie, n\rangle \in \mathcal{E}(R)$ *iff* $\langle\langle t,s\rangle, \bowtie, n\rangle \in \mathcal{E}(\mathsf{Inv}(R))$*,*
13. *If* $\langle\langle s,t\rangle, \rhd, n\rangle \in \mathcal{E}(R)$ *and* $R \sqsubseteq^* S$ *then,* $\langle\langle s,t\rangle, \rhd, n\rangle \in \mathcal{E}(S)$*,*
14. *If* $\langle \geq pR, \rhd, n\rangle \in \mathcal{L}(x)$*, then* $|\{t \in \boldsymbol{S} \mid \langle\langle s,t\rangle, \rhd, n\rangle \in \mathcal{E}(R)\}| \geq p$*,*
15. *If* $\langle \leq pR, \lhd, n\rangle \in \mathcal{L}(x)$*, then* $|\{t \in \boldsymbol{S} \mid \langle\langle s,t\rangle, \lhd^-, 1-n\rangle \in \mathcal{E}(R)\}| \geq p+1$*,*
16. *If* $\langle \geq pR, \lhd, n\rangle \in \mathcal{L}(x)$*, then* $|\{t \in \boldsymbol{S} \mid \langle\langle s,t\rangle, \rhd, n_i\rangle \in \mathcal{E}(R)\}| \leq p-1$*, conjugated with* $\langle\langle s,t\rangle, \lhd, n\rangle$*,*
17. *If* $\langle \leq pR, \rhd, n\rangle \in \mathcal{L}(x)$*, then* $|\{t \in \boldsymbol{S} \mid \langle\langle s,t\rangle, \rhd', n_i\rangle \in \mathcal{E}(R)\}| \leq p$ *conjugated with* $\langle\langle s,t\rangle, \rhd^-, 1-n\rangle$*,*
18. *There do not exist two conjugated triples in any label of any individual* $x \in \boldsymbol{S}$*,*
19. *If* $\langle a : C \bowtie n\rangle \in \mathcal{A}$*, then* $\langle C, \bowtie, n\rangle \in \mathcal{L}(\mathcal{V}(a))$*,*
20. *If* $\langle (a,b) : R \bowtie n\rangle \in \mathcal{A}$*, then* $\langle\langle \mathcal{V}(a), \mathcal{V}(b)\rangle, \bowtie, n\rangle \in \mathcal{E}(R)$*,*
21. *If* $a \not\doteq b \in \mathcal{A}$*, then* $\mathcal{V}(a) \neq \mathcal{V}(b)$

Properties 10 and 11 are a consequence of the fact that the supremum and infimum restrictions have to be preserved, when relations that have transitive sub-roles participate in negative existential and positive value restrictions. The membership degrees that the concepts are being propagated, in Properties 10 and 11, is the same as in the nodes that cause propagation. The proof of this property is quite technical and omitted here. Properties 14-17 are a direct consequence of the semantics of fuzzy number restrictions and the fact that from the De' Morgan laws we can establish equivalences between negative and positive triples.

Lemma 1. *An* f_{KD}*-*$\mathcal{SHIN}$*-ABox* $\mathcal{A}$ *is consistent w.r.t.* $\mathcal{R}$ *iff there exists a fuzzy tableau for* $\mathcal{A}$ *w.r.t.* $\mathcal{R}$*.*

3.1 The Tableaux Algorithm

In order to decide *ABox* consistency a procedure that constructs a fuzzy tableau for an f_{KD}-$\mathcal{SHIN}$ *ABox* has to be determined. In the current section we will provide the technical details for constructing a correct tableaux algorithm. As pointed out in [13] algorithms that decide consistency of an *ABox* work on *completion-forests* rather than on *completion-trees*. This is because an *ABox* might contain several individuals with arbitrary roles connecting them. Such a forest is a collection of trees that correspond to the individuals in the *ABox*.

Nodes in the completion-forest are labelled with a set of triples $\mathcal{L}(x)$ (*node triples*), which contain membership triples. More precisely we define $\mathcal{L}(x)\{\langle C, \bowtie, n\rangle\}$, where $C \in sub(\mathcal{A})$ and $n \in [0,1]$. Furthermore, edges $\langle x,y\rangle$ are labelled with a set $\mathcal{L}(\langle x,y\rangle)$ (*edge triples*) defined as, $\mathcal{L}(\langle x,y\rangle) = \{\langle R, \bowtie, n\rangle\}$, where $R \in \mathbf{R}_A$. The algorithm expands the tree either by expanding the set $\mathcal{L}(x)$, of a node x with new triples, or by adding new leaf nodes.

If nodes x and y are connected by an edge $\langle x,y\rangle$, then y is called a *successor* of x and x is called a *predecessor* of y, *ancestor* is the transitive closure of *predecessor*. A node x is called an $S-$*neighbour* of a node x if for some R with

Table 2. Tableaux expansion rules

Rule	Description
$(\neg)$	if 1. $\langle \neg C, \bowtie, n\rangle \in \mathcal{L}(x)$ 2. and $\langle C, \bowtie^{-}, 1-n\rangle \notin \mathcal{L}(x)$ then $\mathcal{L}(x) \rightarrow \mathcal{L}(x) \cup \{\langle C, \bowtie^{-}, 1-n\rangle\}$
$(\sqcap_{\rhd})$	if 1. $\langle C_1 \sqcap C_2, \rhd, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked, and 2. $\{\langle C_1, \rhd, n\rangle, \langle C_2, \rhd, n\rangle\} \not\subseteq \mathcal{L}(x)$ then $\mathcal{L}(x) \rightarrow \mathcal{L}(x) \cup \{\langle C_1, \rhd, n\rangle, \langle C_2, \rhd, n\rangle\}$
$(\sqcup_{\lhd})$	if 1. $\langle C_1 \sqcup C_2, \lhd, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked, and 2. $\{\langle C_1, \lhd, n\rangle, \langle C_2, \lhd, n\rangle\} \not\subseteq \mathcal{L}(x)$ then $\mathcal{L}(x) \rightarrow \mathcal{L}(x) \cup \{\langle C_1, \lhd, n\rangle, \langle C_2, \lhd, n\rangle\}$
$(\sqcup_{\rhd})$	if 1. $\langle C_1 \sqcup C_2, \rhd, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked, and 2. $\{\langle C_1, \rhd, n\rangle, \langle C_2, \rhd, n\rangle\} \cap \mathcal{L}(x) = \emptyset$ then $\mathcal{L}(x) \rightarrow \mathcal{L}(x) \cup \{C\}$ for some $C \in \{\langle C_1, \rhd, n\rangle, \langle C_2, \rhd, n\rangle\}$
$(\sqcap_{\lhd})$	if 1. $\langle C_1 \sqcap C_2, \lhd, n\rangle \mathcal{L}(x)$, x is not indirectly blocked, and 2. $\{\langle C_1, \lhd, n\rangle, \langle C_2, \lhd, n\rangle\} \cap \mathcal{L}(x) = \emptyset$ then $\mathcal{L}(x) \rightarrow \mathcal{L}(x) \cup \{C\}$ for some $C \in \{\langle C_1, \lhd, n\rangle, \langle C_2, \lhd, n\rangle\}$
$(\exists_{\rhd})$	if 1. $\langle \exists R.C, \rhd, n\rangle \in \mathcal{L}(x)$, x is not blocked, 2. x has no R-neighbour y connected with a triple $\langle P^{*}, \rhd, n\rangle$, $P \overset{*}{\sqsubseteq} R$ and $\langle C, \rhd, n\rangle \in \mathcal{L}(y)$ then create a new node y with $\mathcal{L}(\langle x, y\rangle) = \{\langle R, \rhd, n\rangle\}$, $\mathcal{L}(y) = \{\langle C, \rhd, n\rangle\}$,
$(\forall_{\lhd})$	if 1. $\langle \forall R.C, \lhd, n\rangle \in \mathcal{L}(x)$, x is not blocked, 2. x has no R-neighbour y connected with a triple $\langle P^{*}, \lhd^{-}, 1-n\rangle$, $P \overset{*}{\sqsubseteq} R$ and $\langle C, \lhd, n\rangle \in \mathcal{L}(y)$ then create a new node y with $\mathcal{L}(\langle x, y\rangle) = \{\langle R, \lhd^{-}, 1-n\rangle\}$, $\mathcal{L}(y) = \{\langle C, \lhd, n\rangle\}$,
$(\forall_{\rhd})$	if 1. $\langle \forall R.C, \rhd, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked, and 2. x has an R-neighbour y with $\langle C, \rhd, n\rangle \notin \mathcal{L}(y)$ and 3. $\langle *, \rhd^{-}, 1-n\rangle$ is conjugated with the positive triple that connects x and y then $\mathcal{L}(y) \rightarrow \mathcal{L}(y) \cup \{\langle C, \rhd, n\rangle\}$,
$(\exists_{\lhd})$	if 1. $\langle \exists R.C, \lhd, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked and 2. x has an R-neighbour y with $\langle C, \lhd, n\rangle \notin \mathcal{L}(y)$ and 3. $\langle *, \lhd, n\rangle$ is conjugated with the positive triple that connects x and y then $\mathcal{L}(y) \rightarrow \mathcal{L}(y) \cup \{\langle C, \lhd, n\rangle\}$,
$(\forall_{+})$	if 1. $\langle \forall R.C, \rhd, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked, and 2. there is some P, with $\mathsf{Trans}(P)$, and $P \overset{*}{\sqsubseteq} R$, x has a P-neighbour y with, $\langle \forall P.C, \rhd, n\rangle \notin \mathcal{L}(y)$, and 3. $\langle *, \rhd^{-}, 1-n\rangle$ is conjugated with the positive triple that connects x and y then $\mathcal{L}(y) \rightarrow \mathcal{L}(y) \cup \{\langle \forall P.C, \rhd, n\rangle\}$,
$(\exists_{+})$	if 1. $\langle \exists R.C, \lhd, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked and 2. there is some P, with $\mathsf{Trans}(P)$, and $P \overset{*}{\sqsubseteq} R$, x has a P-neighbour y with, $\langle \exists P.C, \lhd, n\rangle \notin \mathcal{L}(y)$, and 3. $\langle *, \lhd, n\rangle$ is conjugated with the positive triple that connects x and y then $\mathcal{L}(y) \rightarrow \mathcal{L}(y) \cup \{\langle \exists P.C, \lhd, n\rangle\}$,
$(\geq_{\rhd})$	if 1. $\langle \geq pR, \rhd, n\rangle \in \mathcal{L}(x)$, x is not blocked, 2. there are no p R-neighbours $y_1, ..., y_p$ connected to x with a triple $\langle P^{*}, \rhd, n\rangle$, $P \overset{*}{\sqsubseteq} R$, 3. and $y_i \neq y_j$ for $1 \leq i < j \leq p$ then create p new nodes $y_1, ..., y_p$, with $\mathcal{L}(\langle x, y_i\rangle) = \{\langle R, \rhd, n\rangle\}$ and $y_i \neq y_j$ for $1 \leq i < j \leq p$
$(\leq_{\lhd})$	if 1. $\langle \leq pR, \lhd, n\rangle \in \mathcal{L}(x)$, x is not blocked, then apply $(\geq_{\rhd})$-rule for the triple $\langle \geq (p+1)R, \lhd^{-}, 1-n\rangle$
$(\leq_{\rhd})$	if 1. $\langle \leq pR, \rhd, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked, 2. there are $p+1$ R-neighbours $y_1, ..., y_{p+1}$ connected to x with a triple $\langle P^{*}, \rhd', n_i\rangle$, $P \overset{*}{\sqsubseteq} R$, 3. which is conjugated with $\langle P^{*}, \rhd^{-}, 1-n\rangle$, and there are two of them y, z, with no $y \neq z$ and 4. y is neither a root node nor an ancestor of z then 1. $\mathcal{L}(z) \rightarrow \mathcal{L}(z) \cup \mathcal{L}(y)$ and 2. if z is an ancestor of x then $\mathcal{L}(\langle z, x\rangle) \longrightarrow \mathcal{L}(\langle z, x\rangle) \cup \mathsf{Inv}(\mathcal{L}(\langle x, y\rangle))$ else $\mathcal{L}(\langle x, z\rangle) \longrightarrow \mathcal{L}(\langle x, z\rangle) \cup \mathcal{L}(\langle x, y\rangle)$ 3. $\mathcal{L}(\langle x, y\rangle) \longrightarrow \emptyset$ 4. Set $u \neq z$ for all u with $u \neq y$
$(\geq_{\lhd})$	if 1. $\langle \geq pR, \lhd, n\rangle \in \mathcal{L}(x)$, x is not indirectly blocked, then apply $(\leq_{\rhd})$-rule for the triple $\langle \leq (p-1)R, \lhd^{-}, 1-n\rangle$
$(\leq_{r_{\rhd}})$	if 1. $\langle \leq pR, \rhd, n\rangle \in \mathcal{L}(x)$, 2. there are $p+1$ R-neighbours $y_1, ..., y_{p+1}$ connected to x with a triple $\langle P^{*}, \rhd', n_i\rangle$, $P \overset{*}{\sqsubseteq} R$, 3. conjugated with $\langle P^{*}, \rhd^{-}, 1-n\rangle$, and there are two of them y, z, both root nodes, with no $y \neq z$ then 1. $\mathcal{L}(z) \rightarrow \mathcal{L}(z) \cup \mathcal{L}(y)$ and 2. For all edges $\langle y, w\rangle$: i. if the edge $\langle z, w\rangle$ does not exist, create it with $\mathcal{L}(\langle z, w\rangle) = \emptyset$ ii. $\mathcal{L}(\langle z, w\rangle) \longrightarrow \mathcal{L}(\langle z, w\rangle) \cup \mathcal{L}(\langle y, w\rangle)$ 3. For all edges $\langle w, y\rangle$: i. if the edge $\langle w, z\rangle$ does not exist, create it with $\mathcal{L}(\langle w, z\rangle) = \emptyset$ ii. $\mathcal{L}(\langle w, z\rangle) \longrightarrow \mathcal{L}(\langle w, z\rangle) \cup \mathcal{L}(\langle w, y\rangle)$ 4. Set $\mathcal{L}(y) = \emptyset$ and remove all edges to/from y 5. Set $u \neq z$ for all u with $u \neq y$ and set $y \doteq z$
$(\geq_{r_{\lhd}})$	if 1. $\langle \geq pR, \lhd, n\rangle \in \mathcal{L}(x)$, then apply $(\leq_{r_{\rhd}})$-rule for the triple $\langle \leq (p-1)R, \lhd^{-}, 1-n\rangle$

$R \sqsubseteq^* S$ either y is a successor of x and $\mathcal{L}(\langle x, y\rangle) = \langle R, \bowtie, n\rangle$ or y is a predecessor of x and $\mathcal{L}(\langle y, x\rangle) = \langle \mathsf{Inv}(R), \bowtie, n\rangle$. We then say that the edge triple *connects* x and y to a degree of n.

A node x is *blocked* iff it is not a root node and it is either directly or indirectly blocked. A node x is directly blocked iff none of its ancestors is blocked, and it has ancestors x', y and y' such that: (i) y is not a root node, (ii) x is a successor of x' and y a successor of y', (iii) $\mathcal{L}(x) = \mathcal{L}(y)$ and $\mathcal{L}(x') = \mathcal{L}(y')$ and, (iv) $\mathcal{L}(\langle x', x\rangle) = \mathcal{L}(\langle y', y\rangle)$. In this case we say that y blocks x. A node y is indirectly blocked iff none of its ancestors is blocked, or it is a successor of a node x and $\mathcal{L}(\langle x, y\rangle) = \emptyset$.

The algorithm initializes a forest $\mathcal{F}_\mathcal{A}$ to contain a root node x_0^i, for each individual $a_i \in \mathbf{I}_\mathcal{A}$ occurring in the *ABox* $\mathcal{A}$ and additionally $\{\langle C_i, \bowtie, n\rangle\} \cup \mathcal{L}(x_0^i)$, for each assertion of the form $\langle a_i : C_i \bowtie n\rangle$ in $\mathcal{A}$, and an edge $\langle x_0^i, x_0^j\rangle$ if $\mathcal{A}$ contains an assertion $\langle (a_i, a_j) : R_i \bowtie n\rangle$, with $\{\langle R_i, \bowtie, n\rangle\} \cup \mathcal{L}(\langle x_0^i, x_0^j\rangle)$ for each assertion of the form $\langle (a_i, a_j) : R_i \bowtie n\rangle$ in $\mathcal{A}$. At last we initialize the relation $\not\doteq$ as $x_0^i \not\doteq x_0^j$ if $a_i \not\doteq a_j \in \mathcal{A}$ and the relation $\doteq$ to be empty. $\mathcal{F}_\mathcal{A}$ is then expanded by repeatedly applying the rules from Table 2. We use the notation R^* to denote either the role R or the role returned by $\mathsf{Inv}(R)$, and the notation $\langle *, \bowtie, n\rangle$, to denote any role that participates in such a triple.

For a node x, $\mathcal{L}(x)$ is said to contain a clash if it contains one of the following: (a) two conjugated pairs of triples, (b) one of the triples $\langle \bot, \geq, n\rangle$, $\langle \top, \leq, n\rangle$, with $n > 0$, $n < 1$, $\langle \bot, >, n\rangle$, $\langle \top, <, n\rangle$ $\langle C, <, 0\rangle$ or $\langle C, >, 1\rangle$, (c) some triple $\langle \leq pR, \rhd, n\rangle \in \mathcal{L}(x)$ and x has $p+1$ R-neighbours $y_0, ..., y_p$, connected to x with a triple $\langle P^*, \rhd', n_i\rangle$, $P \sqsubseteq^* R$, which is conjugated with $\langle P^*, \rhd^-, 1-n\rangle$, and $y_i \not\doteq y_j$, for all $0 \leq i < j \leq p$, or (d) some triple $\langle \geq pR, \lhd, n\rangle \in \mathcal{L}(x)$ and x has p R-neighbours $y_0, ..., y_{p-1}$, connected to x with a triple $\langle P^*, \rhd, n_i\rangle$, $P \sqsubseteq^* R$, which is conjugated with $\langle P^*, \lhd, n\rangle$, and $y_i \not\doteq y_j$, for all $0 \leq i < j \leq p$. A completion-forest is *clash-free* if none of its nodes contains a clash, and it is *complete* if none of the expansion rules is applicable.

Lemma 2. *Let $\mathcal{A}$ be an f_{KD}-$\mathcal{SHIN}$ ABox and $\mathcal{R}$ a fuzzy RBox. Then*

1. *when started for $\mathcal{A}$ and $\mathcal{R}$ the tableaux algorithm terminates*
2. *$\mathcal{A}$ has a fuzzy tableau w.r.t. $\mathcal{R}$ if and only if the expansion rules can be applied to $\mathcal{A}$ and $\mathcal{R}$ such that they yield a complete and clash-free completion forest.*

4 Related Work

Much work has been carried out towards combining fuzzy logic and description logics during the last decade. The initial idea was presented by Yen in [19], where a *structural subsumption* algorithm was provided in order to perform reasoning. The DL language used was a sub-language of the basic DL $\mathcal{ALC}$. Reasoning in fuzzy $\mathcal{ALC}$ was latter presented in [14], as well as in other approaches [20, 21], where an additional concept constructor, called membership manipulator was included in the extended language. In all these approaches tableaux decision procedures were presented for performing reasoning services. The operations used to interpret the concept constructors in all these approaches were the same ones as in our context. Approaches towards more expressive DLs, are presented in [16], where the DL is $\mathcal{ALCQ}$, and in [17], where the language is $\mathcal{SHOIN}(\mathbf{D}^+)$. The former one also included fuzzy quantifiers, which is a new novel idea for fuzzy DLs. Unfortunately, in both these approaches only the semantics of the extended languages were provided and no reasoning algorithms. As far as we know the most expressive fuzzy DL presented till now, which also covers reasoning, is f_{KD}-$\mathcal{SI}$, appeared in [9]. The present work provides an extension of the latter one to an even more expressive DL, namely $\mathcal{SHIN}$.

5 Conclusions

The importance and role that uncertainty, like vagueness (fuzziness) and imprecision, plays in the Semantic web context, as well as to many applications that use DLs to capture, represent and perform reasoning with domain knowledge has been stressed many times in the literature [4–8]. To this extent we have presented an extension of the very expressive description logic $\mathcal{SHIN}$ with fuzzy set theory. Description logics are very powerful and expressive logical formalisms, which are used by ontology creation languages in the Semantic Web context. Moreover, fuzzy set theory is one very important theory for capturing and dealing with vagueness. Additionally, we have presented a detailed reasoning algorithm for deciding fuzzy ABox consistency. In order to achieve this goal we have provided an investigation of the properties of fuzzy cardinalities, in order to provide sound rules for such types of concept constructors. As far as future directions are concerned, these will include the extension of the $\mathcal{SHOIN}(\mathcal{G})$ description with fuzzy set theory. $\mathcal{SHOIN}(\mathcal{G})$ extends $\mathcal{SHIN}$ with nominals [22] and datatype groups [23].

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